University of Oradea

Department of Mathematics and Computer Science

HABILITATION THESIS

Title:

SUCCESSIVE APPROXIMATIONS IN CRISP CONTEXT, FUZZY CONTEXT AND SPLINES WITH OPTIMAL PROPERTIES

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Contents

Introduction

This habilitation thesis presents the research results obtained by the author in the last nine years, after the rendering of the PhD degree and it is focused on three research directions: (1) the development and extending of the method of successive approximations for neutral type differential and integro-differential equations (applying the Perov's fixed point theorem) and for functional differential and integral equations (obtaining the recent method of successive interpolations created in [83], [63], [56], [57], and [69]-[72]); (2) the study of optimal properties for cubic splines arising in fitting data; (3) the algebraic properties of the set of fuzzy numbers and numerical iterative methods for fuzzy integral equations based on successive approximations. The method of successive approximations was initiated by Picard and Lindelöf (see [203] and [167]) for differential equations and was combined with quadrature rules for Cauchy problems by Ionescu (see [149]). Afterwards, this method was improved by using the idea from the Gauss-Seidel iterations and obtaining the waveform relaxation method (see [166]) and the dynamic iteration method (developed by Bjørhus in [89] and used for functional differential equations with mixed argument in [210]).

The first chapter extends the results of the PhD thesis concerning the applications of the Perov's fixed point theorem to integro-differential equations and to neutral type differential equations. The Perov's fixed point theorem (see [201]) generalizes the Banach's fixed point theorem from the frame of metric spaces to those of generalized metric spaces with the metric taking vectorial values and offers an effective tool to investigate the existence and uniqueness of the solution for systems of integral equations. Moreover, the convergence of the corresponding sequence of successive approximations can be proved similarly extending the fixed point technique and obtaining apriori and a posteriori error estimates. In this context, are studied in the first chapter Volterra integro-differential equations with constant delay (see [60], [51], [52]) and similar equations containing non-integral term (see [65]), presenting sufficient conditions for the existence and uniqueness of the solution and for the convergence of the sequence of successive approximations to this solution, together with the apriori error estimate in this approximation. The study is completed with Fredholm integro-differential equations on Banach spaces (see [48] and [63]) and as particular case, the two-point boundary value problem for second order differential equations is investigated (see [50], [58], [53]). The approach of first and second order initial value problems for neutral type differential equations with constant delay (see [44] and [55]) illustrates the efficiency of the fixed point technique generated by the Perov's fixed point theorem.

In the second chapter we present the notion of quadratic oscillation in average (QOA), firstly introduced in [62], which is a measure of the deviations of a spline of interpolation curve from the data polygon. The global QOA is defined for the whole set of subintervals generated by a partition (see [73]), besides the partial QOA is defined only for some specified subintervals, usually a few first and last subintervals (see [81]). Hence, on the one hand, the derivatives in the Hermite type cubic spline with deficiency 2 are estimated in order to minimize the global QOA (see [73]) being uniquely determined. The approximation properties of the obtained cubic spline are pointed out. On the other hand, the free parameters of the quadratic, cubic, and quartic spline generated by initial conditions are obtained such that the partial QOA is minimized on the first subinterval (see [84]). Since the failure of the Akima's interpolation procedure in fitting data is the suggestion of four artificial slopes near the end-points (see [9]) generating considerable oscillations in these marginal regions, we propose an alternative to estimate the first two and the last two derivatives such that the partial QOA to be minimized on the first two and on the last two subintervals (see [81]).

The third chapter is devoted to the recently introduced, by the author, method of successive interpolations created for functional differential and integral equations. The results are obtained before now for pantograph type argument, but the method could be extended for equations with generally variable retarded or advanced argument. The first ideas of this method appears in [83]. The method is based on the insertion of a proper interpolation procedure at the terms of the sequence of successive approximations and it is considered to show the possibility to efficiently extend the method of successive approximations to equations with deviating argument (see [63]). Using the complete-natural cubic spline, the solution of initial value problems associated to first, second and third order functional differential equations is approximated by successive interpolations (see [85]). First order initial value problems with deviating argument are solved by successive interpolations by using natural cubic splines (see [79]) and a kind of Hermite type cubic spline (see [82]), besides second order initial value problems with deviating argument are approached using the cubic spline from [145] (see [69]) and Birkhoff type cubic splines (see [78]). The solution of two-point boundary value problems associated to second order differential equations with deviating argument is approximated by successive interpolations using Birkhoff type cubic splines (see [70]) and natural cubic splines (see [71]). The fourth order two-point boundary value problem with deviating argument generalizes the beam equation and its solution is approximated using natural cubic splines (see [86]). The use of the natural cubic spline generated by initial conditions for Hammerstein-Fredholm type functional integral equations is illustrated in [57] and [74]. Fredholm and Volterra functional integral equations are approached by successive interpolations by using natural cubic splines in [56] and [87].

The algebraic properties of the set of fuzzy numbers and some numerical methods for fuzzy integral equations form the object of the fourth chapter. The categorial structure for the binary operations with fuzzy numbers and the isomorphism between the additive and multiplicative structures on the set of fuzzy numbers are obtained in [67]. Remarks on the distributivity properties of the product of fuzzy numbers with crisp numbers are pointed out in [23]. Recently, the algebraic structure of the set of one-sided fuzzy numbers and some applications to a mathematical model from epidemiology are obtained (see [77]). Iterative numerical methods for Hammerstein-Fredholm (see [80]) and Urysohn type Fredholm (see [68]) nonlinear fuzzy integral equations are constructed and presented, proving the convergence and establishing a practical stopping criterion of the algorithm. A numerical method for nonlinear Volterra fuzzy integral equations with constant delay arising in epidemiology is developed in [75] proving the convergence and the numerical stability with respect to the first iteration of the method, and showing its accuracy on some numerical examples.

In the last chapter we present some new directions of the research development as continuation of the obtained results. Hence, from the computer aided geometric design point of view will be interesting to extend the concept of quadratic oscillation in average to the context of planar and spatial parametric curves and to parametric surfaces, which is still an open problem.

Since the method of successive interpolations was developed before now only for pantograph type and vanishing delays, will be useful to see how could be extended this method to the case of variable delay and variable advance, with a special attention to boundary value problems for second and fourth order functional differential equations. Moreover, it is an open problem to investigate the possibility to apply the technique of successive interpolations to functional differential equations of neutral type, for instance to the neutral pantograph equation. Another open problem is to choose the proper fuzzy interpolation procedure in order to apply the technique of successive interpolations to fuzzy functional integral equations.

Guided by our recent investigations and by the results obtained in [144] we intend to try the finding of the solution of an open problem from the topic of fuzzy numbers: the construction of operations with fuzzy numbers with more properties than the existing operations. For instance, in our opinion, the distributivity properties can be enriched by choosing appropriate operations and by using an appropriate representation of fuzzy numbers.

Chapter 1

Applications of the Perov's fixed point theorem

1.1 Introduction

In this chapter we present the results obtained by the author concerning the applications of the Perov's fixed point theorem. Some of the results improve those existing in the literature (see for instance, the approach of the two-point boundary value problem for neutral type second order differential equations), and others illustrate for the first time the effectiveness and the applicability of the fixed point technique generated by the Perov's theorem. These results were obtained in [44], [48], [50]-[55], [58], [60], and [65]. The Perov's fixed point theorem is a powerful tool to approach the existence and uniqueness of the solution for systems of equations (algebraic, differential, integral, integro-differential and so on). The frame of this theorem is a complete generalized metric space with the metric taking values in the positive cone of the Euclidean space $\mathbb{R}^n$. More precisely, let X be a nonempty set, $n \in \mathbb{N}$ and $d : X \times X \to \mathbb{R}^n_+$ where,

$$\mathbb{R}^n_+ = \{x = (x_1, ..., x_n) \in \mathbb{R}^n : x_i \geq 0, \quad \forall i = \overline{1, n}\}.$$

Definition 1 *(see [211]) The pair (X, d) is generalized metric space iff the function d has the following properties:*
(gm1) $d(x, y) \geq 0$, $\forall x, y \in X$ and $d(x, y) = 0 \Longleftrightarrow x = y$
(gm2) $d(y, x) = d(x, y)$, $\forall x, y \in X$
(gm3) $d(x, y) \leq d(x, z) + d(z, y)$, $\forall x, y, z \in X$.
The function d is called generalized metric.

The Euclidean space $\mathbb{R}^n$ is ordered by the relation:

$$x \leq y \Longleftrightarrow x_i \leq y_i, \quad \forall i = \overline{1, n},$$

for $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n) \in \mathbb{R}^n$.

A generalized metric space is complete if any fundamental sequence in X is convergent.

Definition 2 *(see [211]) Let $M_n(\mathbb{R}_+)$ be the set of matrices with positive elements. Let (X, d) be a generalized metric space. A map $T : X \to X$ satisfies a generalized Lipschitz inequality if there exists a matrix $A \in M_n(\mathbb{R}_+)$ such that:*

$$d(T(x), T(y)) \leq Ad(x, y), \quad \forall x, y \in X. \tag{1.1}$$

Theorem 3 *(Perov, see [201], [211]) Let (X, d) be a generalized metric space and $A : X \to X$ a mapping which has the generalized Lipschitz inequality property with a matrix $Q \in M_n(\mathbb{R}_+)$. If all eigenvalues of Q lies in the open unit ball from the complex plane, then:*
(i) the operator A has a unique fixed point $x^* \in X$
(ii) for any $x_0 \in X$, the sequence $(x_m)_{m\in\mathbb{N}} \subset X$ defined by $x_m = A(x_{m-1})$, $\forall m \in \mathbb{N}^*$, is convergent to x^*
(iii) the following apriori error estimate:

$$d(x_m, x^*) \leq Q^m \cdot (I_n - Q)^{-1} \cdot d(x_0, x_1), \quad \forall m \in \mathbb{N}^* \tag{1.2}$$

and a posteriori error estimate

$$d(x_m, x^*) \leq Q \cdot (I_n - Q)^{-1} \cdot d(x_m, x_{m-1}), \quad \forall m \in \mathbb{N}^* \tag{1.3}$$

hold.

In the matrices calculus, it is known the following result:

Proposition 4 *Let $A \in \mathcal{M}_n(\mathbb{R}_+)$. The following statements are equivalent*
(i) $A^k \to 0$ for $k \to \infty$
(ii) All eigenvalues of A belong to the open unit disc from $\mathbb{R}^2$
(iii) The matrix $(I_n - A)^{-1}$ is invertible and

$$(I_n - A)^{-1} = I + A + A^2 + ... + A^n + ...$$

In [201], Theorem 3 is applied to the system

$$\begin{cases} x_i'(t) = f(t, x_1, ..., x_n) \\ x_i(t_i) = x_i^0 \end{cases}, \quad i = \overline{1, n}$$

written in the equivalent form

$$x_i(t) = x_i^0 + \int_{t_i}^{t} f(s, x_1(s), ..., x_n(s))\, ds, \quad i = \overline{1, n}$$

and obtaining sufficient conditions for the existence and uniqueness of the solution. As a distinct case, it is approached the two-point boundary value problem

$$\begin{cases} x'' = f(t, x, x') \\ x(a) = x(b) = 0 \end{cases}, \quad t \in [a, b]$$

written in the equivalent form

$$\begin{cases} x(t) = \displaystyle\int_a^b G(t, s) \cdot f(s, x(s), x'(s))\, ds \\ x'(t) = \displaystyle\int_a^b \frac{\partial G(t,s)}{\partial t} \cdot f(s, x(s), x'(s))\, ds \end{cases}, \quad t \in [a, b]$$

and proving the existence and uniqueness of the solution, where G is the well-known Green function.

As an application of the Perov's fixed point theorem to smooth dependence by parameters of the solution, I. A. Rus obtained the following "Fiber generalized contraction theorem":

Theorem 5 *(Rus, see [212], [214], [215])Let* (X, d) *be a metric space (generalized or not) and* (Y, ρ) *be a complete generalized metric space* ($\rho(x, y) \in \mathbb{R}^n_+$). *Let* $A : X \times Y \to X \times Y$ *be a continuous operator and* $C : X \times Y \to Y$ *an operator. Suppose that :*
(i) $B : X \to X$ *is a weakly Picard operator*
(ii) $A(x, y) = (B(x), C(x, y))$, *for all* $x \in X$, $y \in Y$
(iii) there exists a matrix $Q \in M_n(\mathbb{R}_+)$, *with* $Q^m \to 0$ *as* $m \to \infty$, *such that*

$$\rho(C(x, y_1), C(x, y_2)) \leq Q \cdot \rho(y_1, y_2),$$

for all $x \in X$, y_1 *and* $y_2 \in Y$.
Then, the operator A is weakly Picard operator. Moreover, if B is Picard operator, then A is Picard operator.

The notions of Picard and weakly Picard operator are defined by Rus, as follows:

Definition 6 *(Rus, see [213], [215])Let* (X, d) *be a metric space. An operator* $A : X \to X$ *is Picard operator if there exists* $x^* \in X$ *such that :*
(a) $F_A = \{x^*\}$,
(b) the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ *converges to* x^*, *for all* $x_0 \in X$, *where* $A^0 = 1_X$, $A^1 = A$, $A^n = A \circ A^{n-1}$, $\forall n \in \mathbb{N}^*$.
An operator $A : X \to X$ *is weakly Picard operator if the sequence* $(A^n(x_0))_{n\in\mathbb{N}}$ *converges for all* $x_0 \in X$ *and the limit (which may depend on* x_0*) is a fixed point of A.*

In [212], Theorem 5 is applied to the study of smooth dependence by parameters of the solution of a system of delay integral equations arising in population dynamics (see [99]).

In the following sections we present the applications of the Theorems 3 and 5 to integro-differential equations and to neutral type differential equations.

1.2 Trapezoidal type quadrature rules

1.2.1 Trapezoidal and perturbed trapezoidal quadrature rules

In [42]-[49], [54]-[58], and [60]-[63], in the approximation of the solution of integro-differential equations or neutral type differential equations, the numerical method is obtained by combining the method of successive approximations with a suitable quadrature rule. The quadrature rules involved are trapezoidal and perturbed-trapezoidal, the last one being a variant of the Euler-Mac Laurin quadrature formula.

For a continuous function $f : [a, b] \longrightarrow \mathbb{R}$, the well-known trapezoidal quadrature formula is

$$\int_a^b f(x)dx = \frac{b-a}{2} \cdot [f(a) + f(b)] + R(f)$$

where for the remainder the following estimate holds

$$|R(f)| \leq \frac{b-a}{2} \cdot \varpi(f; \frac{b-a}{2}) \tag{1.4}$$

using the modulus of continuity. In the case $f \in C^2[a, b]$, the classical error estimate is

$$|R(f)| \leq \frac{(b-a)^3 \|f''\|_\infty}{12} \tag{1.5}$$

where $\|f''\|_\infty = \max\{|f''(x)| : x \in [a,b]\}$. Recently, in [103], for Lipschitzian functions and for functions with continuous first derivative, the following estimates were obtained:

$$|R(f)| \leq \begin{cases} \frac{L(b-a)^2}{4}, & \text{if } f \in Lip[a,b] \\ \frac{(b-a)^2\|f'\|_\infty}{4}, & \text{if } f \in C^1[a,b]. \end{cases} \tag{1.6}$$

where $L > 0$ is the Lipschitz constant and $\|f'\|_\infty = \max\{|f'(x)| : x \in [a,b]\}$. According to [118], the constant $\frac{1}{4}$ in (1.6) is best possible in the sense that it cannot be replaced by a smaller quantity. When $f \in C^1[a,b]$ with Lipschitzian first derivative, the following error estimate was obtained in [46]:

$$|R(f)| \leq \frac{(b-a)^3 L'}{12} \tag{1.7}$$

where $L' > 0$ is the Lipschitz constant of f'.

Considering a partition of the interval $[a,b]$,

$$\Delta : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$$

and denoting $h_i = x_{i+1} - x_i$, $\forall i = \overline{0, n-1}$, and

$$I_n(f,\Delta) = \frac{1}{2}\sum_{i=0}^{n-1} h_i[f(x_i) + f(x_{i+1})],$$

based on the inequality (1.6) the following error estimates are obtained in [103]:

$$|R_n(f,\Delta)| \leq \begin{cases} \frac{L}{4}\sum\limits_{i=0}^{n-1} h_i^2, & \text{if } f \in Lip[a,b] \\ \frac{\|f'\|}{4}\sum\limits_{i=0}^{n-1} h_i^2, & \text{if } f \in C^1[a,b], \end{cases} \tag{1.8}$$

where

$$R_n(f,\Delta) = \int_a^b f(x)dx - I_n(f,\Delta).$$

In the case $f \in C[a,b]$ it obtains

$$|R_n(f,\Delta)| \leq \frac{1}{2}\sum_{i=0}^{n-1} h_i \varpi(f; \frac{1}{2}h_i) \tag{1.9}$$

and for $f \in C^2[a,b]$, the classical error estimate is

$$|R_n(f,\Delta)| \leq \frac{\|f''\|}{12}\sum_{i=0}^{n-1} h_i^3. \tag{1.10}$$

When $f \in C^1[a,b]$ with Lipschitzian f' the following error estimate is obtained

$$|R_n(f,\Delta)| \leq \frac{L'}{12}\sum_{i=0}^{n-1} h_i^3. \tag{1.11}$$

For equidistant partition the inequalities (1.8-1.11) become

$$|R_n(f,\Delta)| \le \frac{b-a}{2}\cdot \varpi(f;\frac{b-a}{2n}), \qquad \text{if } f\in C[a,b] \tag{1.12}$$

and

$$|R_n(f,\Delta)| \le \begin{cases} \frac{(b-a)^2 L}{4n}, & \text{if } f\in Lip[a,b] \\ \frac{(b-a)^2\cdot\|f'\|}{4n}, & \text{if } f\in C^1[a,b] \\ \frac{(b-a)^3\|f''\|}{12n^2}, & \text{if } f\in C^2[a,b] \\ \frac{(b-a)^3 L'}{12n^2}, & \text{if } f'\in Lip[a,b]. \end{cases} \tag{1.13}$$

Sharp error bounds of the trapezoidal rule for Hölder continuous functions, Lipschitzian functions and differentiable functions with $f'\in L^p[a,b]$ and $f''\in L^p[a,b]$ were obtained in [110]. The rate of convergence of the trapezoidal rule for absolutely continuous even functions was established in [205]. It is proved in [156] and [228] that the trapezoidal rule could be exponentially convergent for periodic and analytic functions over an interval equal to the period.

The remainder estimates (1.13) were used in [42] and [43] for the method of successive approximations combined with the trapezoidal rule in the numerical method for first order differential equations with constant delay, and for ordinary first order initial value problems, respectively.

The perturbed trapezoidal formula is the following quadrature formula of Euler-Mac Laurin type:

$$\int_a^b f(x)dx = \frac{b-a}{2}[f(a)+f(b)] - \frac{(b-a)^2}{12}[f'(b)-f'(a)] + R(f).$$

The error estimate for the case $f\in C^4[a,b]$ was obtained by Petr in [202]:

$$|R(f)| \le \frac{(b-a)^5\cdot\left\|f^{IV}\right\|_\infty}{720}$$

Recently, Barnett and Dragomir had obtained in [25] the estimate for functions $f\in C^3[a,b]$,

$$|R(f)| \le \frac{(b-a)^4\cdot\|f'''\|_\infty}{160}$$

and in [26], for functions $f\in C^3[a,b]$ with Lipschitzian third derivative:

$$|R(f)| \le \frac{(b-a)^5\cdot L'''}{720}$$

where $L'''>0$ is the Lipschitz constant of f'''. Here, $\left\|f^{IV}\right\|_\infty = \max\{\left|f^{IV}(x)\right| : x\in[a,b]\}$ and $\|f'''\|_\infty = \max\{|f'''(x)| : x\in[a,b]\}$.

Considering a partition of the interval $[a,b]$:

$$\Delta : a = x_0 < x_1 < ... < x_{n-1} < x_n = b,$$

$h_i = x_{i+1}-x_i$, $\forall i=\overline{0,n-1}$ it obtains the following perturbed trapezoidal quadrature rule:

$$\int_a^b f(x)dx = \frac{1}{2}\sum_{i=0}^{n-1} h_i[f(x_i)+f(x_{i+1})] - \frac{1}{12}\sum_{i=0}^{n-1} h_i^2[f'(x_{i+1})-f'(x_i)] + R_n(f,\Delta) \tag{1.14}$$

with the error estimate,

$$|R_n(f,\Delta)| \le \begin{cases} \frac{\|f'''\|_\infty}{160} \sum\limits_{i=0}^{n-1} h_i^4, & \text{if } f \in C^3[a,b] \\ \frac{\|f^{IV}\|_\infty}{720} \sum\limits_{i=0}^{n-1} h_i^5, & \text{if } f \in C^4[a,b] \\ \frac{L'''}{720} \sum\limits_{i=0}^{n-1} h_i^5, & \text{if } f''' \text{ is Lipschitzian.} \end{cases} \tag{1.15}$$

If the partition Δ is uniform then $h_i = h = \frac{b-a}{n}$, $\forall i = \overline{0, n-1}$, and formula (1.14) becomes

$$\int_a^b f(x)dx = \frac{b-a}{2n}[f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)] - \frac{(b-a)^2}{12n^2}[f'(b) - f'(a)] + R_n(f). \tag{1.16}$$

The error estimate is in this case:

$$|R_n(f)| \le \begin{cases} \frac{(b-a)^4 \cdot \|f'''\|_\infty}{160n^3}, & \text{if } f \in C^3[a,b] \\ \frac{(b-a)^5 \cdot \|f^{IV}\|_\infty}{720n^4}, & \text{if } f \in C^4[a,b] \\ \frac{(b-a)^5 \cdot L'''}{720n^4}, & \text{if } f''' \text{ is Lipschitzian.} \end{cases} \tag{1.17}$$

The quadrature rule (1.16)-(1.17) was used in [47] for ordinary Cauchy problems of first order, in [61] for initial value problems of first order with constant delay, and in [45] and [49] for Volterra integral equations with constant delay.

1.2.2 Trapezoidal quadrature rule for Lipschitzian vector-valued functions

The trapezoidal quadrature rule and its error estimate (1.6) was extended for functions $f : [a,b] \longrightarrow X$ taking values in Banach spaces in the framework of the Bochner integral. The trapezoidal inequality for the Bochner integral $\int_a^b f(t)\,dt$ when $f \in C^1([a,b],X)$, with X Banach space, was obtained in [96]:

$$\left\| \int_a^b f(t)\,dt - \frac{b-a}{2} \cdot [f(a) + f(b)] \right\|_X \le \frac{(b-a)^2}{4} \cdot \|f'\|_C$$

where

$$\|f'\|_C = \max\{\|f'(t)\|_X : t \in [a,b]\}.$$

The trapezoidal quadrature rule was extended for Lipschitzian fuzzy-number-valued functions in [30]. In the case of Lipschitzian functions $f : [a,b] \longrightarrow X$ we have generalized the quadrature rule in [54], as follows:

Let X be a Banach space and $f : [a,b] \to X$, $g : [a,b] \to \mathbb{R}$.

Definition 7 *(see [182]) Consider an arbitrary sequence* $(\delta_p)_{p\in\mathbb{N}} \subset Div[a,b]$

$$\delta_p : a = t_0 < t_1 < ... < t_{n_p-1} < t_{n_p} = b$$

such that $\lim\limits_{p\to\infty} \nu(\delta_p) = 0$, *where*

$$\nu(\delta_p) = \sup_{1\le i\le n_p} |t_i - t_{i-1}|.$$

If there exists $I \in X$ such that

$$\lim_{p\to\infty}\sum_{i=1}^{n_p} S_{n_p}(f,g)=\lim_{p\to\infty}\sum_{i=1}^{n_p}[g(t_i)-g(t_{i-1})]\cdot f(\tau_i)=I$$

for any points τ_i with $t_{i-1}\le\tau_i\le t_i$, then f is Bochner-Stieltjes integrable with respect by g and the element $I\in X$ is named the Bochner-Stieltjes integral of f with respect by g, being denoted by

$$I=\int_a^b f(t)\,dg(t)\,.$$

It is proved in [182] that the element I from the above definition is uniquely determined by this property and if f is continuous and g has bounded variation, then f is Bochner-Stieltjes integrable with respect by g.

Now, we consider the dual type of Bochner-Stieltjes integral. Let $\Delta\in Div[a,b]$

$$\Delta : a=t_0<t_1<...<t_{n-1}<t_n=b.$$

The function $f:[a,b]\to X$ has bounded variation if

$$\sup_{\Delta\in Div[a,b]}\sum_{i=1}^{n}\|f(t_i)-f(t_{i-1})\|_X<\infty.$$

Definition 8 *(see [189]) Suppose that f has bounded variation. Consider an arbitrary sequence $(\delta_p)_{p\in\mathbb{N}}\subset Div[a,b]$*

$$\delta_p : a=t_0<t_1<...<t_{n_p-1}<t_{n_p}=b$$

such that $\lim_{p\to\infty}\nu(\delta_p)=0$, where

$$\nu(\delta_p)=\sup_{1\le i\le n_p}|t_i-t_{i-1}|\,.$$

If there exists $I\in X$ such that

$$\lim_{p\to\infty}\sum_{i=1}^{n_p} S_{n_p}(g,f)=\lim_{p\to\infty}\sum_{i=1}^{n_p} g(\tau_i)[f(t_i)-f(t_{i-1})]=I$$

for any points τ_i with $t_{i-1}\le\tau_i\le t_i$, then $g:[a,b]\to\mathbb{R}$ is Bochner-Stieltjes integrable with respect by f and the element $I\in X$ is named the Bochner-Stieltjes integral of g with respect by f, being denoted by

$$I=\int_a^b g(t)\,df(t)\,.$$

Theorem 9 *(the formula of integration by parts for the Bochner-Stieltjes integrals, see [189]) Let $g:[a,b]\to X$, $f:[a,b]\to\mathbb{R}$, both having bounded variation. If f is Bochner-Stieltjes integrable with respect by g, then g is Bochner-Stieltjes integrable with respect by f (and conversely) and*

$$\int_a^b f(x)\,dg(x)=f(b)\cdot g(b)-f(a)\cdot g(a)-\int_a^b g(x)\,df(x)\,. \tag{1.18}$$

Now, let us consider a Banach space X and a Lipschitzian function $f : [a,b] \to X$ with the Lipschitz constant $L > 0$. Since f is Lipschitzian then it is continuous and consequently f is measurable with $\|f\| : [a,b] \to \mathbb{R}$ Lebesgue integrable. Then f is Bochner integrable. So, the Bochner-Stieltjes integral

$$\int_a^b f(t)\,dt = -\int_a^b f(t)\,d(x-t)$$

exists, and using the integration by parts we infer that the integral $\int_a^b (x-t)df(t)$ also exists for any $x \in [a,b]$, having

$$\int_a^b (x-t)df(t) = (x-t)\,f(t)\mid_a^b + \int_a^b f(t)\,dt.$$

Denoting $u(t) = x - t$ and using the Bochner-Stieltjes integral sums we have:

$$\left\| \int_a^b (x-t)df(t) \right\|_X = \left\| \lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} u\left(\xi_i^{(n)}\right) \cdot [f(t_{i+1}^{(n)}) - f(t_i^{(n)})] \right\|_X \leq$$

$$\leq \lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} \left| u\left(\xi_i^{(n)}\right) \right| \cdot \left\| f(t_{i+1}^{(n)}) - f(t_i^{(n)}) \right\|_X \leq$$

$$\leq L \lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} \left| u\left(\xi_i^{(n)}\right) \right| \cdot (t_{i+1}^{(n)} - t_i^{(n)}) =$$

$$= L \cdot \int_a^b |u(t)|\,dt = L \cdot \int_a^b |x-t|\,dt = L[\frac{(b-a)^2}{4} + \left(x - \frac{a+b}{2}\right)^2].$$

Then, for any $x \in [a,b]$, it follows :

$$\left\| \int_a^b f(t)\,dt - [(x-a)\cdot f(a) + (b-x)\cdot f(b)] \right\|_X = \left\| \int_a^b (x-t)df(t) \right\|_X \leq$$

$$\leq L[\frac{(b-a)^2}{4} + \left(x - \frac{a+b}{2}\right)^2].$$

So, we have obtained:

Theorem 10 *(see [54]) Let X be a Banach space and $f : [a,b] \to X$ a Lipschitzian function with the constant $L > 0$. Then, for any $x \in [a,b]$ we have:*

$$\left\| \int_a^b f(t)\,dt - [(x-a)\cdot f(a) + (b-x)\cdot f(b)] \right\|_X \leq L[\frac{(b-a)^2}{4} + \left(x - \frac{a+b}{2}\right)^2].$$

Taking $x = \frac{a+b}{2}$ it obtains the trapezoidal quadrature formula in Banach spaces for Lipschitzian functions:

Corollary 11 *(see [54]) Let X be a Banach space and $f : [a,b] \to X$ a Lipschitzian function with the constant $L > 0$. Then,*

$$\left\| \int_a^b f(t)\,dt - \frac{b-a}{2} \cdot [f(a) + f(b)] \right\|_X \le L \cdot \frac{(b-a)^2}{4}. \tag{1.19}$$

Consider the partition of the interval $[a,b]$,

$$\Delta : a = t_0 < t_1 < ... < t_{n-1} < t_n = b.$$

Applying on each interval $[t_i, t_{i+1}]$ the trapezoidal inequality from the previous Corollary, we have

$$\int_{t_i}^{t_{i+1}} f(t)\,dt = \frac{(t_{i+1} - t_i)}{2} \cdot [f(t_i) + f(t_{i+1})] + R_i(f)$$

with

$$\|R_i(f)\|_X \le \frac{L}{4} \cdot (t_{i+1} - t_i)^2, \qquad i = \overline{0, n-1}.$$

Then,

$$\int_a^b f(t)\,dt = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f(t)\,dt = \sum_{i=0}^{n-1} \frac{(t_{i+1} - t_i)}{2} \cdot [f(t_i) + f(t_{i+1})] + \sum_{i=0}^{n-1} R_i(f)$$

and

$$\|R_n(f)\|_X \le \sum_{i=0}^{n-1} \|R_i(f)\|_X \le \frac{L}{4} \cdot \sum_{i=0}^{n-1} (t_{i+1} - t_i)^2,$$

obtaining the trapezoidal quadrature rule in Banach spaces for Lipschitzian functions:

Corollary 12 *Let X be a Banach space and $f : [a,b] \to X$ a Lipschitzian function with the constant $L > 0$. Then,*

$$\left\| \int_a^b f(t)\,dt - \sum_{i=0}^{n-1} \frac{(t_{i+1} - t_i)}{2} \cdot [f(t_i) + f(t_{i+1})] \right\|_X \le \frac{L}{4} \cdot \sum_{i=0}^{n-1} (t_{i+1} - t_i)^2.$$

If the partition Δ is uniform, then

$$t_i = a + \frac{i \cdot (b-a)}{n}, \qquad i = \overline{0, n}.$$

In this case, $t_{i+1} - t_i = \frac{b-a}{n}$ for any $i = \overline{0, n-1}$ and we obtain:

$$\left\| \int_a^b f(t)\,dt - \frac{(b-a)}{2n} \cdot \left[f(a) + 2 \sum_{i=1}^{n-1} f\left(a + \frac{i \cdot (b-a)}{n}\right) + f(b)\right] \right\|_X \le L \cdot \frac{(b-a)^2}{4n}. \tag{1.20}$$

The quadrature rule (1.20) is applied in [54] to the following boundary value problem associated to second order functional-differential equation of mixed type with constant delay and advance:

$$\begin{cases} -x''(t) = f(t, x(t), x(t-h), x(t+h)), & t \in [a,b] \\ x(t) = \varphi(t), & t \in [a-h, a] \\ x(t) = \psi(t), & t \in [b, b+h] \end{cases} \tag{1.21}$$

where $h > 0$, $f \in C([a,b] \times X \times X \times X, X)$, $\varphi \in C[a,b]$, $\psi \in C[a,b]$ and X is Banach space.

The boundary value problem (1.21) is equivalent with the following functional integral equation of mixed type

$$x(t) = \begin{cases} \varphi(t), & t \in [a-h,a] \\ w(t) + \int_a^b G(t,s) \cdot f(s, x(s), x(s-h), x(s+h))ds, & t \in [a,b] \\ \psi(t), & t \in [b, b+h] \end{cases} \tag{1.22}$$

where

$$w(t) = \frac{t-a}{b-a} \cdot \psi(b) + \frac{b-t}{b-a} \cdot \varphi(a), \qquad t \in [a,b]$$

and $G(t,s)$ is the Green function,

$$G(t,s) = \begin{cases} \frac{(s-a)(b-t)}{b-a}, & \text{if } s \le t \\ \frac{(t-a)(b-s)}{b-a}, & \text{if } s \ge t. \end{cases} \tag{1.23}$$

Applying the fixed point technique it obtains:

Theorem 13 *(see [54]) Suppose that :*
(i) $f \in C([a,b] \times X \times X \times X, X)$, $\varphi \in C[a,b]$, $\psi \in C[a,b]$
(ii) there exists $L > 0$ *such that*

$$\|f(t,u_1,v_1,z_1) - f(t,u_2,v_2,z_2)\|_X \le L \cdot (\|u_1 - u_2\|_X + \|v_1 - v_2\|_X + \|z_1 - z_2\|_X)$$

for any $t \in [a,b]$, $u_1, v_1, z_1, u_2, v_2, z_2 \in X$
(iii) $\frac{3}{4} \cdot L(b-a)^2 < 1$.
Then, the equation (1.22) has in $Y = C([a,b], X)$ *a unique solution* x^* *such that* $x^* \in C^2([a,b], X) \cap C([a-h, b+h], X)$ *is also the unique solution of the problem (1.21). Moreover, this solution can be approximated by the terms of the sequence of successive approximations* $(x_m)_{m \in \mathbb{N}}$, $x_m = A(x_{m-1})$ *starting from any element* $x_0 \in Y$ *and the apriori error estimate is:*

$$\|x_m - x^*\|_C \le \frac{\left[\frac{3}{4} \cdot L(b-a)^2\right]^m}{1 - \frac{3}{4} \cdot L(b-a)^2} \cdot \|x_1 - x_0\|_C \ , \qquad \forall m \in \mathbb{N}^*.$$

Choosing

$$x_0(t) = \begin{cases} \varphi(t), & t \in [a-h,a] \\ \frac{t-a}{b-a} \cdot \psi(b) + \frac{b-t}{b-a} \cdot \varphi(a), & t \in [a,b] \\ \psi(t), & t \in [b, b+h]. \end{cases}$$

the terms of the sequence of successive approximations are

$$x_m(t) = \frac{t-a}{b-a} \cdot \psi(b) + \frac{b-t}{b-a} \cdot \varphi(a) +$$

$$+ \int_a^b G(t,s) \cdot f(s, x_{m-1}(s), x_{m-1}(s-h), x_{m-1}(s+h))ds, \qquad t \in [a,b], \quad m \in \mathbb{N}^*$$

$$x_m(t) = \varphi(t), \qquad t \in [a-h, a], \quad m \in \mathbb{N}^*$$

$$x_m(t) = \psi(t), \qquad t \in [b, b+h], \quad m \in \mathbb{N}^*.$$

Suppose that there exists $l \in \mathbb{N}^*$ such that $b - a = l \cdot h$. Let $n \in \mathbb{N}^*$ and the uniform partition of $[a-h, b+h]$,

$$\Delta : a - h = t_0 < t_1 < ... < t_n = a < t_{n+1} <$$

$$< ... < t_{q-1} < t_q = b < t_{q+1} < ... < t_{q+n} = b + h \tag{1.24}$$

such that $t_i - t_{i-1} = \frac{h}{n}$, $\forall i = \overline{1, q+n}$ and $q = (l+1) \cdot n$. On the knots of the partition Δ we have:

$$x_m(t_i) = \begin{cases} \varphi(t_i), & i = \overline{0, n}, \quad m \in \mathbb{N} \\ \psi(t_i), & i = \overline{q, q+n}, \quad m \in \mathbb{N} \end{cases}$$

$$x_0(t_i) = \frac{t_i - a}{b - a} \cdot \psi(b) + \frac{b - t_i}{b - a} \cdot \varphi(a), \qquad i = \overline{n+1, q-1}$$

and

$$x_m(t_i) = \frac{t_i - a}{b - a} \cdot \psi(b) + \frac{b - t_i}{b - a} \cdot \varphi(a) + \int_a^b G(t_i, s) \cdot$$

$$\cdot f(s, x_{m-1}(s), x_{m-1}(s-h), x_{m-1}(s+h))ds, \quad i = \overline{n+1, q-1}, \quad m \in \mathbb{N}^*. \tag{1.25}$$

In order to realize the effective computation of the integrals from (1.25) we use the quadrature rule obtained in (1.20) and we get:

$$x_m(t_i) = \frac{t_i - a}{b - a} \cdot \psi(b) + \frac{b - t_i}{b - a} \cdot \varphi(a) + \frac{b - a}{2nl} \cdot$$

$$\cdot [G(t_i, a) \cdot f(t_n, x_{m-1}(t_n), x_{m-1}(t_0), x_{m-1}(t_{2n})) +$$

$$+ 2 \sum_{j=n+1}^{q-1} G(t_i, t_j) \cdot f(t_j, x_{m-1}(t_j), x_{m-1}(t_{j-n}), x_{m-1}(t_{j+n})) + G(t_i, b) \cdot$$

$$\cdot f(t_q, x_{m-1}(t_q), x_{m-1}(t_{q-n}), x_{m-1}(t_{q+n}))] + R_{m,i}, \quad i = \overline{n+1, q-1}, \quad m \in \mathbb{N}^*. \tag{1.26}$$

The relations (1.26) lead to the following algorithm :

$$x_1(t_i) = \frac{t_i - a}{b - a} \cdot \psi(b) + \frac{b - t_i}{b - a} \cdot \varphi(a) + \frac{b - a}{2nl} \cdot [G(t_i, a) \cdot$$

$$\cdot f(t_n, x_0(t_n), x_0(t_0), x_0(t_{2n})) + 2 \sum_{j=n+1}^{q-1} G(t_i, t_j) \cdot f(t_j, x_0(t_j), x_0(t_{j-n}), x_0(t_{j+n})) +$$

$$+ G(t_i, b) \cdot f(t_q, x_0(t_q), x_0(t_{q-n}), x_0(t_{q+n}))] + R_{1,i} =$$

$$= \overline{x_1(t_i)} + R_{1,i}, \quad i = \overline{n+1, q-1}.$$

$$x_2(t_i) = \frac{t_i - a}{b - a} \cdot \psi(b) + \frac{b - t_i}{b - a} \cdot \varphi(a) + \frac{b - a}{2nl} \cdot [G(t_i, a) \cdot$$

$$\cdot f(t_n, x_1(t_n) + R_{1,n}, x_1(t_0) + R_{1,0}, x_1(t_{2n}) + R_{1,2n}) + 2 \sum_{j=n+1}^{q-1} G(t_i, t_j) \cdot$$

$$\cdot f(t_j, x_1(t_j) + R_{1,j}, x_1(t_{j-n}) + R_{1,j-n}, x_1(t_{j+n}) + R_{1,j+n}) +$$

$$+ G(t_i, b) \cdot f(t_q, x_1(t_q) + R_{1,q}, x_1(t_{q-n}) + R_{1,q-n}, x_1(t_{q+n}) + R_{1,q+n})] +$$

$$+R_{2,i}=\frac{t_i-a}{b-a}\cdot\psi(b)+\frac{b-t_i}{b-a}\cdot\varphi(a)+\frac{b-a}{2nl}\cdot[G(t_i,a)\cdot$$

$$\cdot f(t_n,x_1(t_n),x_1(t_0),x_1(t_{2n}))+2\sum_{j=n+1}^{q-1}G(t_i,t_j)\cdot f(t_j,x_1(t_j),x_1(t_{j-n}),x_1(t_{j+n}))+$$

$$+G(t_i,b)\cdot f(t_q,x_1(t_q),x_1(t_{q-n}),x_1(t_{q+n}))]+\overline{R_{2,i}}=\overline{x_2(t_i)}+\overline{R_{2,i}},\quad i=\overline{n+1,q-1}.$$

By induction, for $m\in\mathbb{N}^*$, $m\geq 3$, we obtain :

$$x_m(t_i)=\frac{t_i-a}{b-a}\cdot\psi(b)+\frac{b-t_i}{b-a}\cdot\varphi(a)+\frac{b-a}{2nl}\cdot[G(t_i,a)\cdot$$

$$\cdot f(t_n,\overline{x_{m-1}(t_n)}+\overline{R_{m-1,n}},\overline{x_{m-1}(t_0)}+\overline{R_{m-1,0}},\overline{x_{m-1}(t_{2n})}+\overline{R_{m-1,2n}})+$$

$$+2\sum_{j=n+1}^{q-1}G(t_i,t_j)\cdot f(t_j,\overline{x_{m-1}(t_j)}+\overline{R_{m-1,j}},\overline{x_{m-1}(t_{j-n})}+\overline{R_{m-1,j-n}},$$

$$,\overline{x_{m-1}(t_{j+n})}+\overline{R_{m-1,j+n}})+G(t_i,b)\cdot$$

$$\cdot f(t_q,\overline{x_{m-1}(t_q)}+\overline{R_{m-1,q}},\overline{x_{m-1}(t_{q-n})}+\overline{R_{m-1,q-n}},\overline{x_{m-1}(t_{q+n})}+\overline{R_{m-1,q+n}})]+$$

$$+R_{m,i}=\frac{t_i-a}{b-a}\cdot\psi(b)+\frac{b-t_i}{b-a}\cdot\varphi(a)+\frac{b-a}{2nl}\cdot[G(t_i,a)\cdot$$

$$\cdot f(t_n,\overline{x_{m-1}(t_n)},\overline{x_{m-1}(t_0)},\overline{x_{m-1}(t_{2n})})+$$

$$+2\sum_{j=n+1}^{q-1}G(t_i,t_j)\cdot f(t_j,\overline{x_{m-1}(t_j)},\overline{x_{m-1}(t_{j-n})},\overline{x_{m-1}(t_{j+n})})+$$

$$+G(t_i,b)\cdot f(t_q,\overline{x_{m-1}(t_q)},\overline{x_{m-1}(t_{q-n})},\overline{x_{m-1}(t_{q+n})})]+\overline{R_{m,i}}=$$

$$=\overline{x_m(t_i)}+\overline{R_{m,i}},\quad\forall i=\overline{n+1,q-1}.$$

We have introduced in [54] the notion of uniformly Lipschitz family of functions, as follows.

Definition 14 *(see [54]) Let X be a Banach space and $I\subset\mathbb{R}$, an interval. Consider $F(I,X)$ be the set of all functions $f:I\to X$. A subset $Y\subset F(I,X)$ is uniformly Lipschitz on I if there exists $\dot{L}>0$ such that for any $f\in Y$ we have:*

$$\|f(u)-f(v)\|_X\leq L\,|u-v|\,,\qquad\forall u,v\in I.$$

For any $m\in\mathbb{N}$ and $i=\overline{n+1,q-1}$, let $H_{m,.i}:[a,b]\to X$, given by

$$H_{m,.i}(s)=G(t_i,s)\cdot F_m(s)=G(t_i,s)\cdot f(s,x_m(s),x_m(s-h),x_m(s+h)).$$

From (1.25) we see that

$$x_m(t_i)=\frac{t_i-a}{b-a}\cdot\psi(b)+\frac{b-t_i}{b-a}\cdot\varphi(a)+\int_a^b H_{m-1,.i}(s)\,ds,\quad\forall i=\overline{n+1,q-1},\quad\forall m\in\mathbb{N}^*.$$

In order to obtain a uniformly Lipschitz property for the family of functions $(H_{m,.i})_{m\in\mathbb{N}}$, $i=\overline{n+1,q-1}$, we have considered the following supplementary Lipschitz conditions:

(j) there exists $M>0$ such that

$$\|f(t,u,v,z)\|_X\leq M,\qquad\forall t\in[a,b],\forall u,v,z\in X$$

(jj) there exist $\alpha, \beta, \gamma > 0$ such that

$$\|f(s_1, u, v, z) - f(s_2, u, v, z)\|_X \leq \alpha \cdot |s_1 - s_2|, \qquad \forall s_1, s_2 \in [a, b], \forall u, v, z \in X$$

$$\|\varphi(s_1) - \varphi(s_2)\|_X \leq \beta \cdot |s_1 - s_2|, \qquad \forall s_1, s_2 \in [a - h, a]$$

$$\|\psi(s_1) - \psi(s_2)\|_X \leq \gamma \cdot |s_1 - s_2|, \qquad \forall s_1, s_2 \in [b, b + h].$$

Since

$$\|x_m(s_1) - x_m(s_2)\|_X \leq \frac{1}{b-a} \cdot \|\psi(b) - \varphi(a)\|_X \cdot |s_1 - s_2| +$$

$$+ \int_a^b \|f(u, x_{m-1}(u), x_{m-1}(u-h), x_{m-1}(u+h))\|_X \cdot |G(s_1, u) - G(s_2, u)| \, du \leq$$

$$\leq [\frac{1}{b-a} \cdot \|\psi(b) - \varphi(a)\|_X + M(b-a)] \cdot |s_1 - s_2|, \quad \forall s_1, s_2 \in [a, b], \quad \forall m \in \mathbb{N}^*$$

and

$$\|x_0(s_1) - x_0(s_2)\|_X \leq [\max(\beta, \gamma) + \frac{1}{b-a} \cdot \|\psi(b) - \varphi(a)\|_X] \cdot |s_1 - s_2|, \quad \forall s_1, s_2 \in [a, b]$$

we infer that

$$\|F_0(s_1) - F_0(s_2)\|_X =$$

$$= \|f(s_1, x_0(s_1), x_0(s_1 - h), x_0(s_1 + h)) - f(s_2, x_0(s_2), x_0(s_2 - h), x_0(s_2 + h))\|_X \leq$$

$$\leq \alpha \cdot |s_1 - s_2| + L(\|x_0(s_1) - x_0(s_2)\|_X +$$

$$+ \|x_0(s_1 - h) - x_0(s_2 - h)\|_X + \|x_0(s_1 + h) - x_0(s_2 + h)\|_X) \leq$$

$$\leq [\alpha + 3L\left(\max(\beta, \gamma) + \frac{1}{b-a} \cdot \|\psi(b) - \varphi(a)\|_X\right)] \cdot |s_1 - s_2| =$$

$$= L_0 \cdot |s_1 - s_2|, \quad \forall s_1, s_2 \in [a, b]$$

and

$$\|H_{0,.i}(s_1) - H_{0,.i}(s_2)\|_X = \|G(t_i, s_1) \cdot F_0(s_1) - G(t_i, s_2) \cdot F_0(s_2)\|_X \leq$$

$$\leq M \cdot |G(t_i, s_1) - G(t_i, s_2)| + \|G\|_C \cdot \|F_0(s_1) - F_0(s_2)\|_X \leq$$

$$\leq [M + \frac{b-a}{4} \cdot L_0] \cdot |s_1 - s_2|, \quad \forall s_1, s_2 \in [a, b], \quad \forall i = \overline{n+1, q-1}.$$

So, for any $m \in \mathbb{N}^*$ it follows that

$$\|F_m(s_1) - F_m(s_2)\|_X \leq \alpha \cdot |s_1 - s_2| + L(\|x_m(s_1) - x_m(s_2)\|_X +$$

$$+ \|x_m(s_1 - h) - x_m(s_2 - h)\|_X + \|x_m(s_1 + h) - x_m(s_2 + h)\|_X) \leq$$

$$\leq \alpha \cdot |s_1 - s_2| + L([\frac{1}{b-a} \cdot \|\psi(b) - \varphi(a)\|_X + M(b-a)] \cdot |s_1 - s_2| +$$

$$+[\beta + \frac{1}{b-a} \cdot \|\psi(b) - \varphi(a)\|_X + M(b-a)] \cdot |s_1 - s_2| +$$

$$+[\gamma + \frac{1}{b-a} \cdot \|\psi(b) - \varphi(a)\|_X + M(b-a)] \cdot |s_1 - s_2|) \leq$$

$$\leq [\alpha + 3L(\max(\beta, \gamma) + \frac{1}{b-a} \cdot \|\psi(b) - \varphi(a)\|_X + M(b-a))] \cdot |s_1 - s_2| =$$

$$= L_m \cdot |s_1 - s_2|, \quad \forall s_1, s_2 \in [a, b], \quad \forall m \in \mathbb{N}^*.$$

Finally, we get :

$$\|H_{m,.i}(s_1) - H_{m,.i}(s_2)\|_X = \|G(t_i, s_1) \cdot F_m(s_1) - G(t_i, s_2) \cdot F_m(s_2)\|_X \leq$$

$$\leq [M + \frac{b-a}{4} \cdot L_m] \cdot |s_1 - s_2| =$$

$$= \overline{L} \cdot |s_1 - s_2|, \quad \forall s_1, s_2 \in [a,b], \quad \forall i = \overline{n+1, q-1}, \quad \forall m \in \mathbb{N}^*.$$

In this way, we can present the following results:

Theorem 15 *(see [54]) Under the conditions (i), (ii), (iii), (j) and (jj), the family of functions* $\{H_{m,.i} : m \in \mathbb{N},\ i = \overline{n+1, q-1}\}$ *is uniformly Lipschitz.*

Theorem 16 *(see [54]) Under the conditions (i), (ii), (iii), (j) and (jj), the solution* x^**, of the boundary value problem (1.21), is approximated on the knots* t_i*,* $i = \overline{n+1, q-1}$ *of the partition (1.24) by the terms of the sequence* $\left(\overline{x_m(t_i)}\right)_{m \in \mathbb{N}^*}$ *and the apriori error estimate is:*

$$\left\|x^*(t_i) - \overline{x_m(t_i)}\right\|_X \leq \frac{\left[\frac{3}{4} \cdot L(b-a)^2\right]^m}{1 - \frac{3}{4} \cdot L(b-a)^2} \cdot \frac{M(b-a)^2}{4} + \frac{\overline{L} \cdot (b-a)^2}{4nl[1 - \frac{3}{4} \cdot L(b-a)^2]}$$

for any $m \in \mathbb{N}^*$ *and* $i = \overline{n+1, q-1}$*.*

Sketch of proof: Observing that

$$\|R_{m,i}\|_X \leq \frac{\overline{L} \cdot (b-a)^2}{4nl}, \quad \forall i = \overline{n+1, q-1}, \quad \forall m \in \mathbb{N}^*$$

by induction, in an iterative way it obtains

$$\|R_{1,i}\|_X \leq \frac{\overline{L} \cdot (b-a)^2}{4nl}, \quad \forall i = \overline{n+1, q-1},$$

$$\left\|\overline{R_{2,i}}\right\|_X \leq \|R_{2,i}\|_X + \frac{b-a}{2nl} \cdot [\|G\|_C \cdot L(\|R_{1,n}\|_X + \|R_{1,0}\|_X + \|R_{1,2n}\|_X) +$$

$$+2\sum_{j=n+1}^{q-1} \|G\|_C \cdot L(\|R_{1,j}\|_X + \|R_{1,j-n}\|_X + \|R_{1,j+n}\|_X) + \|G\|_C \cdot L(\|R_{1,q}\|_X +$$

$$+\|R_{1,q-n}\|_X + \|R_{1,q+n}\|_X)] \leq [1 + \frac{3}{4} \cdot L(b-a)^2] \cdot \frac{\overline{L}(b-a)^2}{4nl}, \quad \forall i = \overline{n+1, q-1},$$

$$\left\|\overline{R_{m,i}}\right\|_X \leq \|R_{m,i}\|_X + \frac{b-a}{2nl} \cdot [\|G\|_C \cdot L(\left\|\overline{R_{m-1,n}}\right\|_X + \left\|\overline{R_{m-1,0}}\right\|_X + \left\|\overline{R_{m-1,2n}}\right\|_X) +$$

$$+2\sum_{j=n+1}^{q-1} \|G\|_C \cdot L(\left\|\overline{R_{m-1,j}}\right\|_X + \left\|\overline{R_{m-1,j-n}}\right\|_X + \left\|\overline{R_{m-1,j+n}}\right\|_X) + \|G\|_C \cdot L(\left\|\overline{R_{m-1,q}}\right\|_X +$$

$$+\left\|\overline{R_{m-1,q-n}}\right\|_X + \left\|\overline{R_{m-1,q+n}}\right\|_X)] \leq [1 + \frac{3}{4} \cdot L(b-a)^2 + ... + \left(\frac{3}{4} \cdot L(b-a)^2\right)^{m-1}] \cdot$$

$$\cdot \frac{\overline{L}(b-a)^2}{4nl} \leq \frac{\overline{L} \cdot (b-a)^2}{4nl[1 - \frac{3}{4} \cdot L(b-a)^2]}, \quad \forall i = \overline{n+1, q-1}, \quad \forall m \in \mathbb{N}^*.$$

Finally,

$$\left\|x^*(t_i)-\overline{x_m(t_i)}\right\|_X \le \|x^*(t_i)-x_m(t_i)\|_X+\left\|x_m(t_i)-\overline{x_m(t_i)}\right\|_X \le$$

$$\le \frac{\left[\frac{3}{4}\cdot L(b-a)^2\right]^m}{1-\frac{3}{4}\cdot L(b-a)^2}\cdot\|x_1-x_0\|_C+\left\|\overline{R_{m,i}}\right\|_X \le$$

$$\le \frac{\left[\frac{3}{4}\cdot L(b-a)^2\right]^m}{1-\frac{3}{4}\cdot L(b-a)^2}\cdot\frac{M(b-a)^2}{4}+\frac{\overline{L}\cdot(b-a)^2}{4nl[1-\frac{3}{4}\cdot L(b-a)^2]}.$$

1.3 Integro-differential equations with constant delay

Here we present the results obtained in [60]. As an application of the Perov's fixed point theorem, the following integro-differential equation with constant delay arising in epidemiology

$$x(t)=\int_{t-\tau}^{t} f\left(s,x(s),x'(s)\right)ds, \quad t\in I\subset\mathbb{R} \tag{1.27}$$

is studied in [60]. This equation is a model for the spread of certain infectious disease with a contact rate that varies seasonally, where,

(i) $x(t)$ is the proportion of infectious in population at time t;

(ii) $\tau>0$ is the length of time in which an individual remains infectious;

(iii) $x'(t)$ is the speed of infectivity at the moment t;

(iv) $f(t,x(t),x'(t))$ is the proportion of new infections on unit time.

When the kernel function is time-periodic we have $I\subset\mathbb{R}$ and the existence of periodic solutions is investigated. For $I=[0,T]$, with $T>0$, an initial condition is attached on the delay interval $[-\tau,0]$, $x(t)=\varphi(t)$, with given differentiable function φ and searching for continuous solutions we need to impose a compatibility condition

$$\varphi(0)=\int_{-\tau}^{0} f\left(s,\varphi(s),\varphi'(s)\right)ds.$$

The solution of the obtained initial value problem and the periodic solution, in the case of periodic kernel, are investigated in the following.

1.3.1 The initial value problem

Existence, uniqueness and Lipschitz properties

Consider the initial value problem

$$\begin{cases} x(t)=\displaystyle\int_{t-\tau}^{t} f(s,x(s),x'(s))\,ds, & t\in[0,T] \\ x(t)=\varphi(t), & t\in[-\tau,0] \end{cases} \tag{1.28}$$

We denote by X the product space $C[-\tau,T]\times C[-\tau,T]$ which is generalized metric space with the Bielecki's metric type given by

$$d_B: X\times X\to\mathbb{R}^2,\ d_B\left((x_1,y_1),(x_2,y_2)\right)=\left(\|x_1-x_2\|_B,\|y_1-y_2\|_B\right),$$

where

$$\|u\|_B = \max_{t\in[-\tau,T]} \mid u(t) \mid e^{-\theta(t+\tau)},$$

for a chosen $\theta > 0$ and for any $u \in C[-\tau, T]$. Also, we consider the subset of X,

$$X_+ = \{(x,y) \in X\colon\ x(t) \geq 0, \text{ for any } t \in [-\tau, T]\}.$$

Since X_+ is closed in X and X is a complete metric space, we infer that X_+ is also a complete metric space.

We suppose that the bellow conditions are satisfied:

(C_1) (boundedness conditions): $\exists m, M \in \mathbb{R}$ such that

$$0 \leq m \leq f(t,u,v) \leq M,\ \forall t \in [-\tau, T], \forall u \in [0,\infty), \forall v \in \mathbb{R}$$

and $\exists m_1, M_1 \in \mathbb{R}$ such that

$$0 \leq m_1 \leq \varphi(t) \leq M_1,\ \forall t \in [-\tau, 0]$$

(C_2) (Lipschitz condition): $f \in C([-\tau, T] \times [0,\infty) \times \mathbb{R})$ and $\exists \alpha, \beta > 0$ such that

$$\left|f(t,u,v) - f\left(t,u',v'\right)\right| \leq \alpha \mid u - u' \mid + \beta \mid v - v' \mid$$

$\forall t \in [-\tau, T], \forall u, u' \in [0,\infty), \forall v, v' \in \mathbb{R}$

(C_3) (compatibility conditions)

$$\varphi(0) = \int_{-\tau}^{0} f\left(s, \varphi(s), \varphi'(s)\right) ds$$

and

$$\varphi'(0) = f\left(0, \varphi(0), \varphi'(0)\right) - f\left(-\tau, \varphi(-\tau), \varphi'(-\tau)\right).$$

If we derive (1.28) with respect by t and denoting $y(t) = x'(t)$, we obtain for $t \in [0,T]$

$$y(t) = f(t, x(t), y(t)) - f(t-\tau, x(t-\tau), y(t-\tau))$$

and

$$y(t) = \varphi'(t),\ \text{ for } t \in [-\tau, 0].$$

Let $T : X_+ \to X$ be the map given by

$$T(x,y)(t) = \begin{cases} \begin{pmatrix} \int_{t-\tau}^{t} f(s, x(s), y(s))\, ds, \\ f(t, x(t), y(t)) - f(t-\tau, x(t-\tau), y(t-\tau)) \end{pmatrix} \\ \text{for } t \in [0,T] \\ (\varphi(t), \varphi'(t)),\ t \in [-\tau, 0]. \end{cases} \tag{1.29}$$

From condition (C_1) we have that $T(X_+) \subseteq X_+$. In order to apply the fixed point technique based on the Perov's fixed point theorem, for any $(x_1,y_1), (x_2,y_2) \in X_+$ we estimate $d_B((x_1,y_1),(x_2,y_2))$.

In the case $t \in [0,T]$ we have:

$$d_B(T(x_1,y_1), T(x_2,y_2)) = (\max_{t\in[0,T]} \mid \int_{t-\tau}^{t} f(s, x_1(s), y_1(s))\, ds -$$

$$-\int_{t-\tau}^{t} f(s, x_2(s), y_2(s))\, ds \mid e^{-\theta(t+\tau)},$$

$$\max_{t\in[0,T]} \mid f(t, x_1(t), y_1(t)) - f(t-\tau, x_1(t-\tau), y_1(t-\tau)) -$$
$$-f(t, x_2(t), y_2(t)) + f(t-\tau, x_2(t-\tau), y_2(t-\tau)) \mid e^{-\theta(t+\tau)}).$$

and

$$\left|\int_{t-\tau}^{t} f(s, x_1(s), y_1(s))\, ds - \int_{t-\tau}^{t} f(s, x_2(s), y_2(s))\, ds\right| \leq$$
$$\leq \left(\frac{\alpha}{\theta}\|x_1 - x_2\|_B + \frac{\beta}{\theta}\|y_1 - y_2\|_B\right)(e^{\theta(t+\tau)} - e^{\theta t}).$$

Consequently,

$$\left|\int_{t-\tau}^{t} f(s, x_1(s), y_1(s))\, ds - \int_{t-\tau}^{t} f(s, x_2(s), y_2(s))\, ds\right| e^{-\theta(t+\tau)} \leq$$
$$\leq \left(\frac{\alpha}{\theta}\|x_1 - x_2\|_B + \frac{\beta}{\theta}\|y_1 - y_2\|_B\right)(1 - e^{-\theta\tau}) \leq$$
$$\leq \frac{\alpha}{\theta}\|x_1 - x_2\|_B + \frac{\beta}{\theta}\|y_1 - y_2\|_B\,, \forall t \in [0, T]$$

which means

$$\max_{t\in[0,T]}\left|\int_{t-\tau}^{t} f(s, x_1(s), y_1(s))\, ds - \int_{t-\tau}^{t} f(s, x_2(s), y_2(s))\, ds\right| e^{-\theta(t+\tau)} \leq$$
$$\leq \frac{\alpha}{\theta}\|x_1 - x_2\|_B + \frac{\beta}{\theta}\|y_1 - y_2\|_B\,.$$

Moreover,

$$\mid (f(t, x_1(t), y_1(t)) - f(t-\tau, x_1(t-\tau), y_1(t-\tau))) -$$
$$-(f(t, x_2(t), y_2(t)) + f(t-\tau, x_2(t-\tau), y_2(t-\tau))) \mid\leq$$
$$\leq (\alpha\|x_1 - x_2\|_B + \beta\|y_1 - y_2\|_B)(e^{\theta(t+\tau)} + e^{\theta t}).$$

We have

$$\max_{t\in[0,T]} \mid f(t, x_1(t), y_1(t)) - f(t-\tau, x_1(t-\tau), y_1(t-\tau)) -$$
$$-f(t, x_2(t), y_2(t)) + f(t-\tau, x_2(t-\tau), y_2(t-\tau)) \mid e^{-\theta(t+\tau)}$$
$$\leq (\alpha\|x_1 - x_2\|_B + \beta\|y_1 - y_2\|_B)(1 + e^{-\theta\tau}).$$

In the case $t \in [-\tau, 0]$,

$$d_B(T(x_1, y_1), T(x_2, y_2))(t) = (0, 0)$$

Thus we have

$$d_B(T(x_1, y_1), T(x_2, y_2)) \leq A \cdot d_B((x_1, y_1), (x_2, y_2)), \tag{1.30}$$

where

$$A = \begin{pmatrix} \frac{\alpha}{\theta} & \frac{\beta}{\theta} \\ \alpha\left(1 + e^{-\theta\tau}\right) & \beta\left(1 + e^{-\theta\tau}\right) \end{pmatrix}$$

The eigenvalues of A are $\lambda_1 = 0$ and $\lambda_2 = \frac{\alpha}{\theta} + \beta\left(1 + e^{-\theta\tau}\right)$.

Lemma 17 *If* $0 < \beta < \frac{1}{2}$ *then* $\lambda_2 < 1$ *for some* $\theta^* > 0$ *and for any* $\theta \geq \theta^*$.

Proof. We have

$$\lambda_2 < 1 \Longleftrightarrow \frac{\alpha}{\theta} + \beta\left(1 + e^{-\theta\tau}\right) < 1 \Longleftrightarrow \theta > \frac{\alpha}{1 - \beta\left(1 + e^{-\theta\tau}\right)}.$$

Let $G : (0, \infty) \to \mathbb{R}$ be the function given by

$$G(\theta) = \frac{\alpha}{1 - \beta\left(1 + e^{-\theta\tau}\right)}.$$

For any $\theta > \frac{1}{\tau}\ln\left(\frac{\beta}{1-\beta}\right)$ we have $e^{-\theta\tau} < \frac{1-\beta}{\beta}$ which implies $1 - \beta\left(1 + e^{-\theta\tau}\right) > 0$ and thus $G(\theta) > 0$. Moreover, since $0 < \beta < \frac{1}{2}$, we have that $\ln\left(\frac{\beta}{1-\beta}\right) < 0$ and then $\theta > \frac{1}{\tau}\ln\left(\frac{\beta}{1-\beta}\right), \forall\theta > 0$. Also, G is a strictly decreasing function since $G'(\theta) < 0$, for any $\theta > 0$. We define the sequence

$$\theta_{n+1} = G(\theta_n), \text{ for } \theta_n > 0, \forall n \in \mathbb{N}$$

with $\theta_0 > 0$. If $\theta_0 < G(\theta_0)$ then $\theta_1 > \theta_0$ and $G(\theta_1) < \theta_1$, because G is strictly decreasing. Let $\theta_0 = 1$ and we put,

$$\theta^* = \begin{cases} G(\theta_0), \text{ if } \theta_0 < G(\theta_0), \\ \theta_0, \text{ if } \theta_0 > G(\theta_0). \end{cases}$$

We see that $\theta^* > G(\theta^*)$ and for any $\theta \geq \theta^*$ we have $\theta > G(\theta)$. Then $\lambda_2 < 1$ for any $\theta \geq \theta^*$. ■

Now, applying the Perov's fixed point theorem it obtains:

Theorem 18 *(see [60]) Under the conditions* C_1*,* C_2*,* C_3*, if* $0 < \beta < \frac{1}{2}$ *then the initial value problem (1.28) has an unique bounded solution* x_* *such that* $x^* = (x_*, x_*')$ *lies in* X_+*.*

Sketch of proof: The existence and uniqueness follows from (1.30) and from the above Lemma applying the Perov's fixed point theorem to the operator (1.29). We denote by $x^*(t) = (x_*(t), y_*(t))$ this solution. From C_1 we have

$$0 < m\tau \leq x_*(t) = \int_{t-\tau}^{t} f(s, x_*(s), y_*(s))\, ds \leq M\tau, \; \forall t \in [0, T]$$

and

$$0 < m_1 \leq x_*(t) \leq M_1, \; \forall t \in [-\tau, 0].$$

Hence x_* is bounded. We prove now that $(x_*)'(t) = y_*(t), \forall t \in [-\tau, T]$. For this purpose, in the case $t \in [-\tau, 0]$, we have,

$$T(x_*(t), y_*(t)) = (\varphi(t), \varphi'(t)) = (x_*(t), y_*(t)).$$

So, $x_*(t) = \varphi(t)$ and $y_*(t) = \varphi'(t)$. In the case $t \in [0, T]$ it obtains

$$(x_*(t), y_*(t)) = T(x_*(t), y_*(t)), \forall t \in [0, T]$$

and then

$$\begin{cases} x_*(t) = \int_{t-\tau}^{t} f(s, x_*(s), y_*(s))\, ds \\ y_*(t) = f(t, x_*(t), y_*(t)) - f(t - \tau, x_*(t - \tau), y_*(t - \tau)) \end{cases},$$

We derive with respect by t the first relation and obtain

$$(x_*)'(t) = f(t, x_*(t), y_*(t)) - f(t-\tau, x_*(t-\tau), y_*(t-\tau))$$

which means $(x_*)'(t) = y_*(t)$, $\forall t \in [0, T]$.

Since for the matrix A we have $A^m = \lambda_2^{m-1} \cdot A$, $\forall m \in \mathbb{N}^*$ and

$$(I-A)^{-1} = \frac{1}{1-\lambda_2} \begin{pmatrix} 1-\beta\left(1+e^{-\theta\tau}\right) & \frac{\beta}{\theta} \\ \alpha\left(1+e^{-\theta\tau}\right) & 1-\frac{\alpha}{\theta} \end{pmatrix},$$

by the Perov's theorem we get

$$d_B(x^m, x^*) \leq A^m (I-A)^{-1} d_B\left(x^1, x^0\right), \forall m \in \mathbb{N}^*$$

obtaining the following estimate.

Corollary 19 *(see [60]) Under the conditions of Theorem 18 the solution x_* of (1.28), is obtained by the successive approximations method starting from any $x^0 = (x_0, y_0) \in X_+$, for $\theta > \theta^*$. For $x^* = (x_*, y_*)$, the following estimate holds*

$$d_B(x^m, x^*) \leq \frac{\lambda_2^{m-1}}{1-\lambda_2} \begin{pmatrix} \frac{\alpha}{\theta} & \frac{\beta}{\theta} \\ \alpha\left(1+e^{-\theta\tau}\right) & \beta\left(1+e^{-\theta\tau}\right) \end{pmatrix} d_B\left(x^1, x^0\right),$$

where $x^m = T\left(x^{m-1}\right)$, $\forall m \in \mathbb{N}^*$.

In order to obtain the numerical method for the approximation of the solution of (1.28) by combining the method of successive approximations with the trapezoidal quadrature rule we consider for each $m \in \mathbb{N}$ the term of the sequence of successive approximations be $x^m(t) = (x_m(t), y_m(t))$ and define the sequence of kernel functions $F_m : [-\tau, T] \to \mathbb{R}$,

$$F_m(t) = f(t, x_m(t), y_m(t)), \forall t \in [-\tau, T]$$

where

$$F_0(t) = \begin{cases} f(t, \varphi(t), \varphi'(t)), & t \in [-\tau, 0] \\ f(t, \varphi(0), \varphi'(0)), & t \in [0, T] \end{cases},$$

$$x_0(t) = \begin{cases} \varphi(t), & t \in [-\tau, 0] \\ \varphi(0), & t \in [0, T] \end{cases}, \quad y_0(t) = \begin{cases} \varphi'(t), & t \in [-\tau, 0] \\ \varphi'(0), & t \in [0, T] \end{cases},$$

investigating the Lipschitz properties of the terms of this sequence. Therefore we suppose that $\varphi \in C^1[-\tau, 0]$ and impose the following supplementary Lipschitz properties: $\exists \gamma, L' \geq 0$ such that for each $u \in [0, \infty), v \in \mathbb{R}$ we have

$$|f(t_1, u.v) - f(t_2, u, v)| \leq \gamma |t_1 - t_2|, \ \forall t_1, t_2 \in [-\tau, T] \tag{1.31}$$

and

$$\left|\varphi'(t_1) - \varphi'(t_2)\right| \leq L' |t_1 - t_2|, \ \forall t_1, t_2 \in [-\tau, 0]. \tag{1.32}$$

By induction, in the following way it is proved that the functions F_m are Lipschitzian. Firstly,

$$|F_0(t_1) - F_0(t_2)| \leq \gamma |t_1 - t_2| + \alpha |\varphi(t_1) - \varphi(t_2)| +$$
$$+\beta \left|\varphi'(t_1) - \varphi'(t_2)\right| \leq (\gamma + \alpha \cdot \left\|\varphi'\right\|_C + \beta \cdot L') \cdot |t_1 - t_2|$$

for any $t_1, t_2 \in [-\tau, 0]$ and

$$|F_0(t_1) - F_0(t_2)| \leq \gamma |t_1 - t_2|, \quad \forall t_1, t_2 \in [0, T].$$

For $t_1 \in [-\tau, 0]$ and $t_2 \in [0, T]$ it follows that

$$|F_0(t_1) - F_0(t_2)| \leq |F_0(t_1) - F_0(0)| + |F_0(0) - F_0(t_2)|$$

$$\leq (\gamma + \alpha \cdot \|\varphi'\|_C + \beta \cdot L') \cdot |t_1 - t_2| = L_0 \cdot |t_1 - t_2|$$

and thus F_0 is Lipschitzian, with the Lipschitz constant L_0,

$$L_0 = \gamma + \alpha \cdot \|\varphi'\|_C + \beta \cdot L'$$

where $\|\cdot\|_C$ is the Chebyshev's norm in $C[-\tau, 0]$,

$$\|u\|_C = \{\max |u(t)| : t \in [-\tau, 0]\}.$$

Analogously, we have :

$$|F_1(t_1) - F_1(t_2)| \leq \left| f(t_1, \varphi(t_1), \varphi'(t_1) - f(t_2, \varphi(t_1), \varphi'(t_1) \right|$$

$$+ \mid f(t_2, \varphi(t_1), \varphi'(t_1) - f(t_2, \varphi(t_2), \varphi'(t_2) \mid \leq \gamma \, |t_1 - t_2| +$$

$$+ \alpha \, |\varphi(t_1) - \varphi(t_2)| + \beta \left| \varphi'(t_1) - \varphi'(t_2) \right| \leq (\gamma + \alpha \cdot \|\varphi'\|_C +$$

$$+ \beta \cdot L') \cdot |t_1 - t_2| \, , \forall t_1, t_2 \in [-\tau, 0]$$

and for $t_1, t_2 \in [0, \tau]$,

$$|F_1(t_1) - F_1(t_2)| \leq \mid f(t_1, \int_{t_1-\tau}^{t_1} f\left(s, \varphi\left(s\right), \varphi'\left(s\right)\right) ds, \; f(t_1, \varphi(0), \varphi'(0)) -$$

$$- f(t_1 - \tau, \varphi(t_1 - \tau), \varphi'(t_1 - \tau))) - f(t_2, \int_{t_1-\tau}^{t_1} f\left(s, \varphi\left(s\right), \varphi'\left(s\right)\right) ds,$$

$$, \; f(t_1, \varphi(0), \varphi'(0)) - f(t_1 - \tau, \varphi(t_1 - \tau), \varphi'(t_1 - \tau))) \mid +$$

$$+ \mid f(t_2, \int_{t_1-\tau}^{t_1} f\left(s, \varphi\left(s\right), \varphi'\left(s\right)\right) ds, \; f(t_1, \varphi(0), \varphi'(0)) -$$

$$- f(t_1 - \tau, \varphi(t_1 - \tau), \varphi'(t_1 - \tau))) - f(t_2, \int_{t_2-\tau}^{t_2} f\left(s, \varphi\left(s\right), \varphi'\left(s\right)\right) ds,$$

$$, \; f(t_2, \varphi(0), \varphi'(0)) - f(t_2 - \tau, \varphi(t_2 - \tau), \varphi'(t_2 - \tau))) \mid \leq$$

$$\leq [\beta\,(\gamma + 1) + \gamma + \alpha\beta \, \|\varphi'\|_C + \beta^2 L' + 2\alpha M] \cdot |t_1 - t_2| \, .$$

For $t_1, t_2 \in [\tau, T]$, it follows that

$$|F_1(t_1) - F_1(t_2)| \leq (\gamma + 2\alpha M + 2\beta\gamma) \cdot |t_1 - t_2| \, .$$

Denoting $L_1 = \max(L_0, \gamma + \beta + \beta L_0 + 2\alpha M, \gamma + 2\alpha M + 2\beta\gamma)$ we get

$$|F_1(t_1) - F_1(t_2)| \leq L_1 \, |t_1 - t_2| \, , \quad \forall t_1, t_2 \in [-\tau, T],$$

and so, F_1 is Lipschitzian.

By induction for $m \in \mathbb{N}^*$, supposing that F_{m-1} is Lipschitzian with a Lipschitz constant L_{m-1}, it obtains successively,

$$|x_m(t_1) - x_m(t_2)| \leq 2M \cdot |t_1 - t_2|, \forall t_1, t_2 \in [-\tau, T]$$

$$|y_m(t_1) - y_m(t_2)| \leq |F_{m-1}(t_1) - F_{m-1}(t_2)| + | F_{m-1}(t_1 - \tau) -$$
$$- F_{m-1}(t_2 - \tau) | \leq 2L_{m-1} |t_1 - t_2|, \ \forall t_1, t_2 \in [0, T].$$

Since $\varphi \in C^1[-\tau, 0]$ with φ' Lipschitzian and $L' \leq L_{m-1}$ we have

$$|y_m(t_1) - y_m(t_2)| \leq 2L_{m-1} |t_1 - t_2|, \forall t_1, t_2 \in [-\tau, T].$$

Then for $\forall t_1, t_2 \in [-\tau, T]$ we have:

$$|F_m(t_1) - F_m(t_2)| \leq \gamma |t_1 - t_2| + \alpha |x_m(t_1) - x_m(t_2)| +$$

$$+ \beta |y_m(t_1) - y_m(t_2)| \leq (\gamma + 2\alpha M + 2\beta L_{m-1}) \cdot |t_1 - t_2|.$$

and F_m is Lipschitzian $\forall m \in \mathbb{N}^*$. For the Lipschitz constants we have the recurrence

$$L_m = \gamma + 2\alpha M + 2\beta L_{m-1}.$$

We can see that

$$L_{m+1} - L_m = 2\beta(L_m - L_{m-1}) = ... = (2\beta)^m (L_1 - L_0) \geq 0, \quad \forall m \in \mathbb{N}^*$$

because $L_1 \geq L_0$, and then the sequence $(L_m)_{m \in \mathbb{N}^*}$ is increasing. On the other hand we have,

$$|L_{m+p} - L_m| \leq |L_{m+p} - L_{m+p-1}| + ... + |L_{m+1} - L_m| \leq$$
$$\leq [(2\beta)^m + (2\beta)^{m+1} + ... + (2\beta)^{m+p-1}] \cdot |L_1 - L_0| = (2\beta)^m [1 +$$
$$+ 2\beta + ... + (2\beta)^{p-1}] \cdot |L_1 - L_0| = (2\beta)^m \cdot \frac{1 - (2\beta)^p}{1 - 2\beta} \cdot |L_1 - L_0|.$$

Since $\beta < \frac{1}{2}$, the sequence $(L_m)_{m \in \mathbb{N}^*}$ converges. So, $\exists L \geq 0$ such that $L = \lim_{m \to \infty} L_m$ and $L_m \leq L$, $\forall m \in \mathbb{N}$. Moreover, by induction we obtain

$$L_m = (\gamma + 2\alpha M)[1 + 2\beta + ... + (2\beta)^{m-2}] + (2\beta)^{m-1} L_1$$

$\forall m \in \mathbb{N}^*$, and $\gamma + 2\alpha M < L_1$, and infer that $L \leq \frac{L_1}{1 - 2\beta}$.

In this way we have proved the following result:

Theorem 20 *(see [60]) Under the conditions C_1, C_2, C_3, (1.31), (1.32), if $0 < \beta < \frac{1}{2}$ then the functions F_m, $m \in \mathbb{N}$, are Lipschitzian with the Lipschitz's constant L_m such that*

$$L_m \leq \frac{L_1}{1 - 2\beta}, \forall m \in \mathbb{N}.$$

An interesting Lipschitz property can be obtained even for the derivative, x'_* of the solution. Since f is continuous and x_m, y_m are continuous $\forall m \in \mathbb{N}$ then F_m is continuous for each $m \in \mathbb{N}$. We see that

$$\lim_{m \to \infty} \|x_m - x_*\| = \lim_{m \to \infty} \|y_m - y_*\| = 0$$

and consequently,

$$\lim_{m\to\infty} F_m(t) = F^*(t) = f(t, x_*(t), y_*(t)),\ \forall t \in [-\tau, T].$$

Since F_m is Lipschitzian $\forall m \in \mathbb{N}$, for $\forall t_1, t_2 \in [-\tau, T]$ we have

$$-L_m |t_1 - t_2| \le F_m(t_1) - F_m(t_2) \le L_m |t_1 - t_2|,$$

and passing to limit for $m \to \infty$, in this inequality it follows that

$$-L |t_1 - t_2| \le F^*(t_1) - F^*(t_2) \le L |t_1 - t_2|,\ \forall t_1, t_2 \in [-\tau, T].$$

Then F^* is Lipschitzian with the Lipschitz constant L. For $t_1, t_2 \in [0, T]$ we have

$$|y_*(t_1) - y_*(t_2)| \le |f(t_1, x_*(t_1), y_*(t_1)) - f(t_2, x_*(t_2), y_*(t_2))| +$$

$$+ |f(t_1 - \tau, x_*(t_1 - \tau), y_*(t_1 - \tau)) - f(t_2 - \tau, x_*(t_2 - \tau), y_*(t_2 - \tau))|$$
$$\le |F^*(t_1) - F^*(t_2)| + |F^*(t_1 - \tau) - F^*(t_2 - \tau)| \le 2L \cdot |t_1 - t_2|$$

For $t_1, t_2 \in [-\tau, 0]$, $y_* = \varphi'$ and φ' is Lipschitzian with the Lipschitz constant L'. Finally, we infer that $y_* = (x_*)'$ is Lipschitzian with the Lipschitz constant $\max\{L', 2L\}$ on $[-\tau, T]$.

The approximation of the solution

We present now the numerical method based on the successive approximations method and on the trapezoidal quadrature rule, computing the terms of the sequence of successive approximations

$$x^m(t) = (x_m(t), y_m(t)), \forall m \in \mathbb{N} \ \text{ for } t \in [0, T].$$

For $t \in [-\tau, 0]$ we have:

$$x_m(t) = \varphi(t) \ \text{ and } y_m(t) = \varphi'(t), \text{ for } \forall m \in \mathbb{N}$$

and

$$x_0(t) = \varphi(0) \ \text{ and } y_0(t) = \varphi'(0), \quad \forall t \in [0, T].$$

For $m \in \mathbb{N}^*$ and $t \in [0, T]$ we have

$$x_m(t) = \int_{t-\tau}^{t} f(s, x_{m-1}(s), y_{m-1}(s))\, ds,\ \forall m \in \mathbb{N}^*, \tag{1.33}$$

$$y_m(t) = f(t, x_{m-1}(t), y_{m-1}(t)) - f(t - \tau, x_{m-1}(t - \tau), y_{m-1}(t - \tau)).$$

Let Δ_n be a uniform partition of the interval $[-\tau, 0]$

$$\Delta_n : -\tau = t_0 < t_1 < ... < t_n = 0,$$

with $h = t_i - t_{i-1} = \frac{\tau}{n}$, $\forall i = \overline{1, n}$.

We suppose that the values $\varphi_i = \varphi(t_i)$ and $\varphi_i' = \varphi'(t_i)$, $i = \overline{0, n}$, are known.

In order to compute the integrals from (1.33) was used the trapezoidal type quadrature rule deduced by Cerone and Dragomir for Lipschitz functions in [103]:

$$\int_a^b F(t)\, dt = \frac{b-a}{2n}[F(a) + 2\sum_{i=1}^{n-1} F(t_i) + F(b)] + R_n(F), \tag{1.34}$$

where $t_i = a + \frac{(b-a)}{n} i$, $i = \overline{0,n}$ and $|R_n(F)| \le \frac{(b-a)^2}{4n} L$, for $L \ge 0$, be the Lipschitz constant of F.

On the interval $[0,T]$ where $T = l\tau$, $l \in \mathbb{N}$, we consider the uniform partition Δ'_n

$$\Delta'_n : 0 = t_n < t_{n+1} < ... < t_q = T$$

with $q = (l+1) \cdot n$ and $t_{j+1} - t_j = h = \frac{\tau}{n}$, $\forall j = \overline{n, q-1}$. We can see that $t_k - \tau = t_{k-n}$, $\forall k = \overline{n,q}$, and also $\Delta_n \cup \Delta'_n$ is a uniform partition of the interval $[-\tau, T]$. For any $k = \overline{n,q}$ in the interval $[t_k - \tau, t_k]$ we find n knots of the partition $\Delta_n \cup \Delta'_n$. Applying (1.34) to compute the integrals (1.33) we obtain

$$x_m(t_k) = \int_{t_k-\tau}^{t_k} f(s, x_{m-1}(s), y_{m-1}(s))\, ds = \frac{\tau}{2n}[F_{m-1}(t_k) +$$

$$+2\sum_{j=1}^{n-1} F_{m-1}(t_{k+j} - \tau) + F_{m-1}(t_k - \tau)] + r_{m,k}(f),$$

$$y_m(t_k) = F_{m-1}(t_k) - F_{m-1}(t_k - \tau), \ \forall k = \overline{n+1, q}, \ \forall m \in \mathbb{N}^*.$$

with the remainder estimate

$$|r_{m,k}(f)| \le \frac{\tau^2}{4n} L_{m-1} \le \frac{\tau^2}{4n} L.$$

We see that this estimate is not dependent by m or k.

Since at the first step, on the knots we have the values

$$x_0(t_k) = \begin{cases} \varphi(t_k), \ k = \overline{0,n} \\ \varphi(0), \ k = \overline{n+1, q} \end{cases},$$

$$y_0(t_k) = \begin{cases} \varphi'(t_k), \ k = \overline{0,n} \\ \varphi'(0), \ k = \overline{n+1, q} \end{cases},$$

the following algorithm is constructed:

$$x_1(t_k) = \frac{\tau}{2n}[F_0(t_k) + 2\sum_{j=1}^{n-1} F_0(t_{k+j-n}) + F_0(t_{k-n})] +$$

$$+ r_{1,k}(f) \equiv \overline{x_1}(t_k) + r_{1,k}(f), \ \forall k = \overline{n+1, q}.$$

and

$$y_1(t_k) = F_0(t_k) - F_0(t_{k-n}) \equiv \overline{y_1}(t_k), \ \forall k = \overline{n+1, q}.$$

For $m = 2$ and $k = \overline{n+1, q}$ we have

$$x_2(t_k) = \frac{\tau}{2n}[F_1(t_k) + 2\sum_{j=1}^{n-1} F_1(t_{k+j-n}) + F_1(t_{k-n})] + r_{2,k}(f) =$$

$$= \frac{\tau}{2n}[f(t_k, \overline{x_1(t_k)} + r_{1,k}(f), y_1(t_k)) + f(t_{k-n}, \overline{x_1(t_{k-n})} + r_{1,k-n}(f),$$

$$, y_1(t_{k-n})) + 2\sum_{j=1}^{n-1} f(t_{k+j-n}, \overline{x_1(t_{k+j-n})} + r_{1,k+j-n}(f), y_1(t_{k+j-n}))] +$$

$$+r_{2,k}(f)=\frac{\tau}{2n}[f(t_k,\overline{x_1(t_k)},y_1(t_k))+f(t_{k-n},\overline{x_1(t_{k-n})},y_1(t_{k-n}))+$$

$$+2\sum_{j=1}^{n-1}f(t_{k+j-n},\overline{x_1(t_{k+j-n})},y_1(t_{k+j-n}))]+\overline{r_{2,k}(f)}=\overline{x_2\left(t_k\right)}+\overline{r_{2,k}(f)}$$

and

$$y_2\left(t_k\right)=F_1\left(t_k\right)-F_1\left(t_k-\tau\right)=f(t_k,\overline{x_1(t_k)}+r_{1,k}(f),y_1(t_k))-$$

$$-f(t_k-\tau,\overline{x_1(t_{k-n})}+r_{1,k-n}(f),y_1(t_{k-n}))=f(t_k,\overline{x_1(t_k)},y_1(t_k))-$$

$$-f(t_k-\tau,\overline{x_1(t_{k-n})},y_1(t_{k-n}))+\varpi_{2,k}(f)=\overline{y_2\left(t_k\right)}+\varpi_{2,k}(f).$$

The estimates of the remainders are:

$$\left|\overline{r_{2,k}(f)}\right|\le\frac{\tau}{2n}[\alpha\left|r_{1,k}\left(f\right)\right|+\alpha\left|r_{1,k-n}\left(f\right)\right|+2\sum_{j=1}^{n-1}\alpha\left|r_{1,k+j-n}\left(f\right)\right|]+$$

$$+\left|r_{2,k}\left(f\right)\right|\le\frac{\tau}{2n}[\alpha\frac{\tau^2L}{4n}+2\alpha\left(n-1\right)\frac{\tau^2L}{4n}+\alpha\frac{\tau^2L}{4n}]+\frac{\tau^2L}{4n}$$

that is

$$\left|\overline{r_{2,k}(f)}\right|\le\left(\alpha\tau+1\right)\frac{\tau^2L}{4n},\forall k=\overline{n+1,q}$$

and

$$\left|\varpi_{2,k}(f)\right|\le\alpha\left|r_{1,k}\left(f\right)\right|+\alpha\left|r_{1,k-n}\left(f\right)\right|\le 2\alpha\cdot\frac{\tau^2L}{4n},\forall k=\overline{n+1,q}.$$

By induction, for $m\ge 3$ and $k=\overline{n+1,q}$, it obtains

$$x_m\left(t_k\right)=\frac{\tau}{2n}[F_{m-1}\left(t_k\right)+2\sum_{j=1}^{n-1}F_{m-1}\left(t_{k+j-n}\right)+F_{m-1}\left(t_{k-n}\right)]+r_{m,k}\left(f\right)=$$

$$\frac{\tau}{2n}[f(t_k,\overline{x_{m-1}(t_k)}+\overline{r_{m-1,k}(f)},\overline{y_{m-1}(t_k)}+\varpi_{m-1,k}(f))+f(t_{k-n},\overline{x_{m-1}(t_{k-n})}$$

$$+\overline{r_{m-1,k-n}(f)},\overline{y_{m-1}(t_{k-n})}+\varpi_{m-1,k-n}(f))+2\sum_{j=1}^{n-1}f(t_{k+j-n},\overline{x_{m-1}(t_{k+j-n})}+$$

$$+\overline{r_{m-1,k+j-n}(f)},\overline{y_{m-1}(t_{k+j-n})}+\varpi_{m-1,k+j-n}(f))]+r_{m,k}(f)=\frac{\tau}{2n}[f(t_k,$$

$$,\overline{x_{m-1}(t_k)},\overline{y_{m-1}(t_k)})+f(t_{k-n},\overline{x_{m-1}(t_{k-n})},\overline{y_{m-1}(t_{k-n})})+2\sum_{j=1}^{n-1}f(t_{k+j-n},$$

$$,\overline{x_{m-1}(t_{k+j-n})},\overline{y_{m-1}(t_{k+j-n})})]+\overline{r_{m,k}(f)}=\overline{x_m\left(t_k\right)}+\overline{r_{m,k}(f)}$$

and

$$y_m(t_k)=f(t_k,\overline{x_{m-1}(t_k)}+\overline{r_{m-1,k}(f)},\overline{y_{m-1}(t_k)}+\varpi_{m-1,k}(f))-$$

$$-f(t_{k-n},\overline{x_{m-1}(t_{k-n})}+\overline{r_{m-1,k-n}(f)},\overline{y_{m-1}(t_{k-n})}+\varpi_{m-1,k-n}(f))=$$

$$=f(t_k,\overline{x_{m-1}(t_k)},\overline{y_{m-1}(t_k)})-f(t_{k-n},\overline{x_{m-1}(t_{k-n})},\overline{y_{m-1}(t_{k-n})})+$$

$$+\varpi_{m,k}(f)=\overline{y_m(t_k)}+\varpi_{m,k}(f),\ \forall k=\overline{n+1,q}.$$

In order to estimate the remainders, was denoted

$$\left|\overline{r_{m-1}(f)}\right|=\max_k(\max_j(\left|\overline{r_{m-1,k+j-n}(f)}\right|))$$

$$|\varpi_{m-1}(f)| = \max_k(\max_j(|\varpi_{m-1,k+j-n}(f)|))$$

getting the recurrent relations for $m \in \mathbb{N}$, $m \geq 3$:

$$\left|\overline{r_{m,k}(f)}\right| \leq \tau\alpha \left|\overline{r_{m-1}(f)}\right| + \tau\beta \left|\varpi_{m-1}(f)\right| + \left|r_{m,k}(f)\right|,$$

$$|\varpi_{m,k}(f)| \leq 2\alpha \left|\overline{r_{m-1}(f)}\right| + 2\beta \left|\varpi_{m-1}(f)\right|, \forall k = \overline{n+1,q}.$$

In this way, it is obtained the sequence $((\overline{x_m(t_k)}, \overline{y_m(t_k)}))_{m\in\mathbb{N}}$ which approximates the solution $x^*(t) = (x_*(t), x'_*(t))$ on the knots $t_k, k = \overline{n+1,q}$.

Theorem 21 *(see [60]) Under the conditions C_1, C_2, C_3, (1.31), (1.32), if $\beta < \frac{1}{2}$, and $\mu(\alpha+\beta) < 1$, where $\mu = \max(\tau, 2)$, then the sequence $((\overline{x_m(t_k)}, \overline{y_m(t_k)}))_{m\in\mathbb{N}}$ approximates the solution $x^*(t) = (x_*(t), x'_*(t))$ on the knots $t_k, k = \overline{n+1,q}$, and the following error estimation holds*

$$\begin{pmatrix} \left|x_*(t_k) - \overline{x_m(t_k)}\right| \\ \left|x'_*(t_k) - \overline{y_m(t_k)}\right| \end{pmatrix} \leq \begin{pmatrix} \frac{\tau^2 L}{4n(1-\mu\alpha-\mu\beta)} \\ \frac{\tau^2 L}{4n(1-\mu\alpha-\mu\beta)} \end{pmatrix} +$$

$$+\frac{\lambda_2^{m-1}}{1-\lambda_2} \cdot \begin{pmatrix} \frac{\alpha}{\theta} & \frac{\beta}{\theta} \\ \alpha\left(1+e^{-\theta\tau}\right) & \beta\left(1+e^{-\theta\tau}\right) \end{pmatrix} d_B\left(x^1, x^0\right). \tag{1.35}$$

for any $m \geq 2$ and $k = \overline{n+1,q}$.

Sketch of proof: On the knots $t_k, k = \overline{n+1,q}$, we see that

$$\left|x_*(t_k) - \overline{x_m(t_k)}\right| \leq |x_*(t_k) - x_m(t_k)| + \left|x_m(t_k) - \overline{x_m(t_k)}\right| \tag{1.36}$$

$$\left|x'_*(t_k) - \overline{y_m(t_k)}\right| \leq \left|x'_*(t_k) - y_m(t_k)\right| + \left|y_m(t_k) - \overline{y_m(t_k)}\right| \tag{1.37}$$

and for $m \in \mathbb{N}, m \geq 2$,

$$\left|x_m(t_k) - \overline{x_m(t_k)}\right| = \left|\overline{r_{m,k}(f)}\right|$$

$$\left|y_m(t_k) - \overline{y_m(t_k)}\right| = |\varpi_{m,k}(f)|, \forall k = \overline{n+1,q}$$

having

$$\begin{pmatrix} |x_*(t_k) - x_m(t_k)| \\ |x'_*(t_k) - y_m(t_k)| \end{pmatrix} \leq \frac{\lambda_2^{m-1}}{1-\lambda_2} \cdot \begin{pmatrix} \frac{\alpha}{\theta} & \frac{\beta}{\theta} \\ \alpha\left(1+e^{-\theta\tau}\right) & \beta\left(1+e^{-\theta\tau}\right) \end{pmatrix} d_B\left(x^1, x^0\right).$$

For $m = 2$, the remainder estimates are,

$$|\varpi_{2,k}(f)| \leq 2\alpha \cdot \frac{\tau^2 L}{4n} \leq (2\alpha + 2\beta)\frac{\tau^2 L}{4n}$$

and

$$\left|\overline{r_{2,k}(f)}\right| \leq (\alpha\tau + 1)\frac{\tau^2 L}{4n} \leq (1 + \alpha\tau + \beta\tau)\frac{\tau^2 L}{4n}, \forall k = \overline{n+1,q}.$$

By induction, it obtains

$$\left|\overline{r_{m,k}(f)}\right| \leq \tau\alpha \left|\overline{r_{m-1}(f)}\right| + \tau\beta \left|\varpi_{m-1}(f)\right| + \left|r_{m,k}(f)\right| \leq$$

$$\leq \frac{\tau^2 L}{4n} + (\tau\alpha + \tau\beta)\left|\rho_{m-1}\right| \leq \frac{\tau^2 L}{4n} + (\mu\alpha + \mu\beta)[1+$$

$$+(\mu\alpha+\mu\beta)+...+(\mu\alpha+\mu\beta)^{m-2}]\frac{\tau^2 L}{4n}=[1+(\mu\alpha+$$

$$+\mu\beta)+...+(\mu\alpha+\mu\beta)^{m-1}]\frac{\tau^2 L}{4n},\quad \forall k=\overline{n+1,q}$$

and

$$|\varpi_{m,k}(f)|\le 2\alpha\left|\overline{r_{m-1}(f)}\right|+2\beta\,|\varpi_{m-1}(f)|\le(\mu\alpha+$$

$$+\mu\beta)\left|\rho_{m-1}\right|\le[(\mu\alpha+\mu\beta)+...+(\mu\alpha+\mu\beta)^{m-1}]\frac{\tau^2 L}{4n}$$

$\forall k=\overline{n+1,q}$, where $\left|\rho_{m-1}\right|=\max(\left|\overline{r_{m-1}(f)}\right|,|\varpi_{m-1}(f)|)$. Then, because $\mu\alpha+\mu\beta<1$, for any $m\ge 3$ and $k=\overline{n+1,q}$, it follows that

$$\left|\overline{r_{m,k}(f)}\right|\le\frac{[1-(\mu\alpha+\mu\beta)^m]\tau^2 L}{4n[1-(\mu\alpha+\mu\beta)]}<\frac{\tau^2 L}{4n(1-\mu\alpha-\mu\beta)}$$

and

$$|\varpi_{m,k}(f)|\le\frac{[1-(\mu\alpha+\mu\beta)^m]\tau^2 L}{4n[1-(\mu\alpha+\mu\beta)]}<\frac{\tau^2 L}{4n(1-\mu\alpha-\mu\beta)}.$$

Now, from (1.36) and (1.37), the estimate (1.35) follows.

The smooth dependence by parameters of the solution

Applying Theorem 5, in [60], it is studied the smooth dependence, by real parameters $\lambda\in[a,b]$, of the solution of (1.28) written in the form,

$$x(t,\lambda)=\int_{t-\tau}^{t}f\left(s,x(s,\lambda),x_s'(s,\lambda),\lambda\right)ds,$$

by considering the initial value problem

$$x(t,\lambda)=\begin{cases}\int_{t-\tau}^{t}f\left(s,x(s,\lambda),y(s,\lambda),\lambda\right)ds, & t\in[0,T]\\ \varphi(t,\lambda), & t\in[-\tau,0],\end{cases}$$
$$y(t,\lambda)=\begin{cases}f(t,x(t,\lambda),y(t,\lambda),\lambda)-\\ -f(t-\tau,x(t-\tau,\lambda),y(t-\tau,\lambda),\lambda), t\in[0,T]\\ \varphi_t'(t,\lambda),\quad t\in[-\tau,0],\end{cases}\tag{1.38}$$

with $y(t,\lambda)=x_t'(t,\lambda)$, under the following conditions:
(i) $f\in C([-\tau,T]\times\mathbb{R}_+\times\mathbb{R}\times[a,b])$, $\varphi\in C^1([-\tau,0]\times[a,b])$,
(ii) (boundedness condition): $\varphi(t,\lambda)\ge 0,\forall(t,\lambda)\in[-\tau,0]\times[a,b]$ and $\exists m,M\ge 0$ such that

$$m\le f(t,x,y,\lambda)\le M,\ \forall(t,x,y,\lambda)\in[-\tau,T]\times\mathbb{R}_+\times\mathbb{R}\times[a,b]$$

(iii) (first compatibility condition):

$$\varphi(0,\lambda)=\int_{-\tau}^{0}f\left(s,\varphi(s,\lambda),\varphi_s'(s,\lambda),\lambda\right)ds,\forall\lambda\in[a,b]$$

$$\varphi_t'(0,\lambda)=f(0,\varphi(0,\lambda),\varphi_t'(0,\lambda),\lambda)-f(-\tau,\varphi(-\tau,\lambda),$$
$$,\varphi_t'(-\tau,\lambda),\lambda),\ \forall\lambda\in[a,b]$$

(iv) (second compatibility condition): $\varphi \in C^2([-\tau,0]\times[a,b])$ and

$$\varphi'_\lambda(0,\lambda) = \int_{-\tau}^{0} [\tfrac{\partial f}{\partial \lambda}(s,\varphi(s,\lambda),\varphi'_s(s,\lambda),\lambda) + \tfrac{\partial f}{\partial x}(s,\varphi(s,\lambda),\varphi'_s(s,\lambda),\lambda)\cdot$$
$$\cdot\varphi'_\lambda(s,\lambda) + \tfrac{\partial f}{\partial y}(s,\varphi(s,\lambda),\varphi'_s(s,\lambda),\lambda)\cdot\varphi''_{s\lambda}(s,\lambda)]ds,\ \forall\lambda\in[a,b]$$

$$\varphi''_{t\lambda}(0,\lambda) = \tfrac{\partial f}{\partial \lambda}(0,\varphi(0,\lambda),\varphi'_t(0,\lambda),\lambda) + \tfrac{\partial f}{\partial x}(0,\varphi(0,\lambda),\varphi'_t(0,\lambda),\lambda)\cdot$$
$$\cdot\varphi'_\lambda(0,\lambda) + \tfrac{\partial f}{\partial y}(0,\varphi(0,\lambda),\varphi'_t(0,\lambda),\lambda)\cdot\varphi''_{t\lambda}(0,\lambda) -$$
$$-\tfrac{\partial f}{\partial \lambda}(-\tau,\varphi(-\tau,\lambda),\varphi'_t(-\tau,\lambda),\lambda) - \tfrac{\partial f}{\partial x}(-\tau,\varphi(-\tau,\lambda),\varphi'_t(-\tau,\lambda),\lambda)\cdot$$
$$\cdot\varphi'_\lambda(-\tau,\lambda) - \tfrac{\partial f}{\partial y}(-\tau,\varphi(-\tau,\lambda),\varphi'_t(-\tau,\lambda),\lambda)\cdot\varphi''_{t\lambda}(-\tau,\lambda),\forall\lambda\in[a,b]$$

(v) (Lipschitz condition) $\exists\alpha,\beta>0$ such that

$$|f(t,x_1,y_1,\lambda) - f(t,x_2,y_2,\lambda)| \le \alpha|x_1-x_2| + \beta|y_1-y_2|$$

$\forall(t,\lambda)\in[-\tau,T]\times[a,b], \forall(x_1,y_1),(x_2,y_2)\in\mathbb{R}_+\times\mathbb{R}$.
(vi) (smoothness condition): $f(s,\cdot,\cdot,\cdot)\in C^1(\mathbb{R}_+\times\mathbb{R}\times[a,b]), \forall s\in[-\tau,T],\ \varphi\in C^2([-\tau,0]\times[a,b])$.

Are considered the product spaces

$$X = C([-\tau,T]\times[a,b],\mathbb{R}_+)\times C([-\tau,T]\times[a,b])$$

$$Y = C([-\tau,T]\times[a,b])\times C([-\tau,T]\times[a,b])$$

which are complete generalized metric spaces with, $\rho: Y\times Y\to\mathbb{R}^2$

$$\rho((x_1,y_1),(x_2,y_2)) = \max_{t\in[-\tau,T],\lambda\in[a,b]} |x_1(t)-x_2(t)|\,e^{-\theta[(t+\tau)+(\lambda-a)]},$$
$$\max_{t\in[-\tau,T],\lambda\in[a,b]} |y_1(t)-y_2(t)|\,e^{-\theta[(t+\tau)+(\lambda-a)]}$$

and

$$d: X\times X\to\mathbb{R}^2,\quad d=\rho\mid_{X\times X}.$$

Denoting $u(t,\lambda) = x'_\lambda(t,\lambda)$ and $v(t,\lambda) = y'_\lambda(t,\lambda)$, the following initial value problem appears

$$u(t,\lambda) = \begin{cases} \int_{t-\tau}^{t} [\frac{\partial f}{\partial \lambda}(s,x,y,\lambda) + \frac{\partial f}{\partial x}(s,x,y,\lambda)\cdot u(s,\lambda) \\ +\frac{\partial f}{\partial y}(s,x,y,\lambda)\cdot v(s,\lambda)]ds, \quad t\in[0,T] \\ \varphi'_\lambda(t,\lambda), \quad t\in[-\tau,0], \end{cases}$$
$$v(t,\lambda) = \begin{cases} \frac{\partial f}{\partial \lambda}(t,x,y,\lambda) + \frac{\partial f}{\partial x}(t,x,y,\lambda)\cdot u(t,\lambda) + \\ \frac{\partial f}{\partial y}(t,x,y,\lambda)\cdot v(t,\lambda) - \frac{\partial f}{\partial \lambda}(t-\tau,x,y,\lambda) \\ -\frac{\partial f}{\partial x}(t-\tau,x,y,\lambda)\cdot u(t-\tau,\lambda) - \\ \frac{\partial f}{\partial y}(t-\tau,x,y,\lambda)\cdot v(t-\tau,\lambda), t\in[0,T] \\ \varphi''_{t\lambda}(t,\lambda), \quad t\in[-\tau,0]. \end{cases} \tag{1.39}$$

Defining the operators

$$A,C: X\times Y\to X\times Y, B: X\to X$$

by $A(x,y) = (B(x),C(x,y))$,

$$B(x,y)(t,\lambda) = \begin{cases} \begin{pmatrix} \int_{t-\tau}^{t} f(s,x(s,\lambda),y(s,\lambda),\lambda)\,ds \\ f(t,x,y,\lambda) - f(t-\tau,x,y,\lambda) \end{pmatrix} \\ (\varphi(t,\lambda),\varphi'_t(t,\lambda)), \quad t\in[-\tau,0] \end{cases}$$

and

$$C((x,y),(u,v))(t,\lambda)=\begin{cases}(\int\limits_{t-\tau}^{t}[\frac{\partial f}{\partial\lambda}(s,x,y,\lambda)+\frac{\partial f}{\partial x}(s,x,y,\lambda)\cdot u(s,\lambda)+\\ \quad +\frac{\partial f}{\partial y}(s,x,y,\lambda)\cdot v(s,\lambda)]ds,\\ ,\frac{\partial f}{\partial\lambda}(t,x,y,\lambda)+\frac{\partial f}{\partial x}(t,x,y,\lambda)\cdot u(t,\lambda)+\\ \frac{\partial f}{\partial y}(t,x,y,\lambda)\cdot v(t,\lambda)-\frac{\partial f}{\partial\lambda}(t-\tau,x,y,\lambda)-\\ -\frac{\partial f}{\partial x}(t-\tau,x,y,\lambda)\cdot u(t-\tau,\lambda)-\\ \frac{\partial f}{\partial y}(t-\tau,x,y,\lambda)\cdot v(t-\tau,\lambda)),t\in[0,T]\\ (\varphi'_\lambda(t,\lambda),\varphi''_{t\lambda}(t,\lambda)),\quad t\in[-\tau,0]\end{cases}$$

and applying Theorem 5 to the operators A, B, C, the following smooth dependence result is obtained:

Theorem 22 *(see [60]) (a) Under the conditions (i)-(iii) and (v) the initial value problem (1.38) has in X a unique solution (x^*,y^*). Moreover, for any $(x_0,y_0)\in X$, the sequence $(x_n,y_n),n\in\mathbb{N}$ defined by*

$$x_{n+1}(t,\lambda)=\int\limits_{t-\tau}^{t}f\left(s,x_n\left(s,\lambda\right),y_n\left(s,\lambda\right),\lambda\right)ds,\ t\in[0,T]$$

$$x_n(t,\lambda)=\varphi(t,\lambda),\ \forall n\in\mathbb{N},\forall t\in[-\tau,0]$$

$$y_{n+1}(t,\lambda)=f\left(t,x_n\left(t,\lambda\right),y_n\left(t,\lambda\right),\lambda\right)-$$
$$-f\left(t-\tau,x_n\left(t-\tau,\lambda\right),y_n\left(t-\tau,\lambda\right),\lambda\right),\ t\in[0,T]$$

$$y_n(t,\lambda)=\varphi'_t(t,\lambda),\ \forall n\in\mathbb{N},\forall t\in[-\tau,0]$$

converges uniformly to (x^*,y^*), and $y^*(t,\lambda)=(x^*)'_t(t,\lambda),\forall t\in[-\tau,T],\forall\lambda\in[a,b]$.
(b) Under the conditions (i)-(vi) the solution (x^*,y^*) has the following property:

$$x^*(t,\cdot)\in C^1[a,b],\quad y^*(t,\cdot)\in C^1[a,b]$$

and the pair $(u^*,v^*)=(\frac{\partial x^*}{\partial\lambda},\frac{\partial y^*}{\partial\lambda})$ is the unique solution in Y of the initial value problem (1.39).

1.3.2 The periodic solution for time periodic kernel

Now, we present the case when the kernel function is time periodic and investigate the existence and uniqueness of the periodic bounded solution.

We suppose that $f\in C(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R})$ and exists $\varpi>0$ such that

$$f(t+\varpi,x,y)=f(t,x,y),\ \forall(t,x,y)\in\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}.$$

Consider the following functional spaces

$$X(\varpi)=\{x\in C(\mathbb{R}):x(t+\varpi)=x(t),\ \forall t\in\mathbb{R}\}$$

$$X_+(\varpi)=\{x\in X(\varpi):x(t)\geq 0,\ \forall t\in\mathbb{R}\}.$$

and denote by X the product space $X=X_+(\varpi)\times X(\varpi)$ which is a generalized metric space with

$$d_C:X\times X\to\mathbb{R}^2,\ d_C\left((x_1,y_1),(x_2,y_2)\right)=\left(\|x_1-x_2\|,\|y_1-y_2\|\right),$$

where

$$\|u\| = \max\{|u(t)| : t \in [0, \varpi]\}$$

for any $u \in X(\varpi)$.

In order to obtain the existence and uniqueness result for the integro-differential equation (1.27) the Perov's fixed point theorem is used.

If we derive (1.27) with respect by t and denoting $y(t) = x'(t)$ we obtain

$$y(t) = f(t, x(t), y(t)) - f(t-\tau, x(t-\tau), y(t-\tau)), \ \forall t \in \mathbb{R}.$$

which lead to

$$\begin{cases} x(t) = \int\limits_{t-\tau}^{t} f(s, x(s), y(s))\,ds \\ y(t) = f(t, x(t), y(t)) - f(t-\tau, x(t-\tau), y(t-\tau)) \end{cases} \tag{1.40}$$

Let $T : X \to C(\mathbb{R}) \times C(\mathbb{R})$ be the map given by

$$T(x,y) = (T_1(x,y), T_2(x,y))$$

$$\begin{pmatrix} T_1(x,y)(t) \\ T_2(x,y)(t) \end{pmatrix} = \begin{pmatrix} \int\limits_{t-\tau}^{t} f(s, x(s), y(s))\,ds \\ f(t, x(t), y(t)) - f(t-\tau, x(t-\tau), y(t-\tau)) \end{pmatrix} \tag{1.41}$$

The following conditions are imposed:

(i) $f \in C(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R})$ and exist $m, M \geq 0$ such that

$$m \leq f(t,x,y) \leq M, \ \forall (t,x,y) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}.$$

(ii) f has the property

$$f(t+\varpi, x, y) = f(t,x,y), \ \forall (t,x,y) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}.$$

(iii) there exist $\alpha, \beta > 0$ such that

$$\left|f(t,u,v) - f(t,u',v')\right| \leq \alpha\left|u-u'\right| + \beta\left|v-v'\right|$$

$\forall t \in \mathbb{R}, \forall u, u' \in \mathbb{R}_+, \forall v, v' \in \mathbb{R}$.

From condition (i) we see that $T_1(X) \subseteq C^1(\mathbb{R})$ and

$$T_1(x,y)(t) = \int\limits_{t-\tau}^{t} f(s, x(s), y(s))\,ds \geq \int\limits_{t-\tau}^{t} m ds = m\tau$$

$\forall t \in \mathbb{R}$, $\forall (x,y) \in X$. It is obvious that $T_1(x,y)(t) \leq M\tau$ $\forall t \in \mathbb{R}$, $\forall (x,y) \in X$.

Theorem 23 *(see [60]) If the conditions (i)-(iii) are satisfied and $\alpha\tau + 2\beta < 1$, then the integro-differential equation (1.27) has in $X_+(\varpi)$ unique solution.*

In the proof of this theorem, by applying the fixed point technique, it obtains

$$\begin{pmatrix} \|T_1(x_1,y_1) - T_1(x_2,y_2)\| \\ \|T_2(x_1,y_1) - T_2(x_2,y_2)\| \end{pmatrix} \leq$$

$$\leq \begin{pmatrix} \alpha\tau & \beta\tau \\ 2\alpha & 2\beta \end{pmatrix} \cdot \begin{pmatrix} \|x_1 - x_2\| \\ \|y_1 - y_2\| \end{pmatrix}$$

with the matrix $A = \begin{pmatrix} \alpha\tau & \beta\tau \\ 2\alpha & 2\beta \end{pmatrix}$ having the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = \alpha\tau + 2\beta$.

Corollary 24 *(see [60]) Under the conditions of the previous theorem, the solution x_* of (1.27) is obtained by the successive approximations method, starting from any $x^0 = (x_0, y_0) \in X$, with the following error estimation*

$$d_C\left(x^m, x_*\right) \leq \frac{\lambda_2^{m-1}}{1-\lambda_2}\left(\begin{array}{cc} \alpha\tau & \beta\tau \\ 2\alpha & 2\beta \end{array}\right) \cdot d_C\left(x^1, x^0\right),$$

where $x^m = T\left(x^{m-1}\right)$, $x^m = (x_m, y_m), \forall m \in \mathbb{N}^*$.

In [52], the smooth dependence of the periodic solution of

$$x(t,\lambda) = \int_{t-\tau}^{t} f\left(s, x\left(s,\right), x_s'\left(s,\lambda\right), \lambda\right) ds, \quad t \in \mathbb{R}, \quad \lambda \in [a,b]$$

by parameters is investigated.

1.3.3 The smooth dependence by lag of the periodic solution

In this section we present the results obtained in [51], considering the integro-differential equation

$$x(t,\tau) = \int_{t-\tau}^{t} f(s, x(s,\tau), x_s'(s,\tau), \tau) ds, \qquad t \in \mathbb{R}, \quad \tau \in [a,b] \tag{1.42}$$

where $a > 0$, under the following conditions:
(i) (continuity): $f \in C(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \times [a,b])$ and $f(\cdot, u, v, \cdot) \in C^1(\mathbb{R} \times [a,b])$, $\forall (u,v) \in \mathbb{R}_+ \times \mathbb{R}, .$
(ii) (boundedness): there exist $m, M \geq 0$ such that

$$m \leq f(t,u,v,\tau) \leq M, \ \forall (t,u,v,\tau) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \times [a,b]$$

(iii) (Lipschitz condition): there exist $\alpha, \beta > 0$ such that

$$|f(t,u_1,v_1,\tau) - f(t,u_2,v_2,\tau)| \leq \alpha |u_1 - u_2| + \beta |v_1 - v_2|,$$

$\forall t \in \mathbb{R}, \forall u_1, u_2 \in \mathbb{R}_+, \forall v_1, v_2 \in \mathbb{R}, \forall \tau \in [a,b]$
(iv) (periodicity): $\exists \varpi > 0$ such that

$$f(t+\varpi,u,v,\tau) = f(t,u,v,\tau), \ \forall (t,u,v,\tau) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \times [a,b]$$

(v) (smoothness) : $f \in C^1(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \times [a,b])$, and

$$\frac{\partial f}{\partial t}(t+\varpi,u,v,\tau) = \frac{\partial f}{\partial t}(t,u,v,\tau)$$

$$\frac{\partial f}{\partial \tau}(t+\varpi,u,v,\tau) = \frac{\partial f}{\partial \tau}(t,u,v,\tau)$$

$$\frac{\partial f}{\partial x}(t+\varpi,u,v,\tau) = \frac{\partial f}{\partial x}(t,u,v,\tau)$$

$$\frac{\partial f}{\partial y}(t+\varpi,u,v,\tau) = \frac{\partial f}{\partial y}(t,u,v,\tau), \ \forall (t,u,v,\tau) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \times [a,b].$$

Differentiating equation (1.42) with respect by t and denoting $y(t,\tau) = x'_t(t,\tau)$, it obtains

$$y(t,\tau) = f(t, x(t,\tau), y(t,\tau), \tau) - f(t-\tau, x(t-\tau,\tau), y(t-\tau,\tau), \tau). \tag{1.43}$$

Consider the spaces,

$$X(\varpi) = \{x \in C(\mathbb{R}\times[a,b]) \mid x(t+\varpi,\tau) = x(t,\tau),\ \forall t\in\mathbb{R},\ \forall\tau\in[a,b]\}$$

$$X_+(\varpi) = \{x \in X(\varpi) \mid\ x(t,\tau)\geq 0,\ \forall t\in\mathbb{R},\ \forall\tau\in[a,b]\}$$

and denote $X = X_+(\varpi)\times X(\varpi)$, $Y = X(\varpi)\times X(\varpi)$. On Y is defined the generalized metric $\rho : Y\times Y \longrightarrow \mathbb{R}^2$

$$\rho((x_1,y_1),(x_2,y_2)) =$$
$$= \Big(\max_{t\in[0,\varpi],\tau\in[a,b]} |x_1(t,\tau)-x_2(t,\tau)|\,,\ \max_{t\in[0,\varpi],\tau\in[a,b]} |y_1(t,\tau)-y_2(t,\tau)|\Big)$$

and $d : X\times X \longrightarrow \mathbb{R}^2$, $d = \rho\,|_{X\times X}$.

It is easy to see that (Y,ρ) and (X,d) are complete generalized metric spaces.

In the study of the smooth dependence by τ of the solution of (1.42) we have considered the systems

$$\begin{pmatrix} x(t,\tau)\\ y(t,\tau)\end{pmatrix} = \begin{pmatrix} \int\limits_{t-\tau}^{t} f\left(s,x\left(s,\tau\right),y\left(s,\tau\right),\tau\right)ds \\ f(t,x(t,\tau),y(t,\tau),\tau) - f(t-\tau,x(t-\tau,\tau),y(t-\tau,\tau),\tau), \\ ,\ t\in\mathbb{R},\ \tau\in[a,b], \end{pmatrix} \tag{1.44}$$

and

$$\begin{pmatrix} u(t,\tau)\\ v(t,\tau)\end{pmatrix} = \tag{1.45}$$

$$\begin{pmatrix} \int\limits_{t-\tau}^{t} [\frac{\partial f}{\partial\tau}(s,x\left(s,\tau\right),y\left(s,\tau\right),\tau) + \frac{\partial f}{\partial x}(s,x\left(s,\tau\right),y\left(s,\tau\right),\tau)\cdot u(s,\tau) + \\ +\frac{\partial f}{\partial y}(s,x\left(s,\tau\right),y\left(s,\tau\right),\tau)\cdot v(s,\tau)]ds, \\ \frac{\partial f}{\partial\tau}(t,x(t,\tau),y(t,\tau),\tau) + \frac{\partial f}{\partial x}(t,x(t,\tau),y(t,\tau),\tau)\cdot u(t,\tau) + \\ +\frac{\partial f}{\partial y}(t,x(t,\tau),y(t,\tau),\tau)\cdot v(t,\tau) + \frac{\partial f}{\partial t}(t-\tau,x(t-\tau,\tau),y(t-\tau,\tau),\tau) - \\ -\frac{\partial f}{\partial\tau}(t-\tau,x(t-\tau,\tau),y(t-\tau,\tau),\tau) - \frac{\partial f}{\partial x}(t-\tau,x(t-\tau,\tau),y(t-\tau,\tau),\tau)\cdot \\ \cdot[-y(t-\tau,\tau)+u(t-\tau,\tau)] - \frac{\partial f}{\partial y}(t-\tau,x(t-\tau,\tau),y(t-\tau,\tau),\tau)\cdot \\ \cdot[-\frac{\partial}{\partial t}y(t-\tau,\tau) + (v(t-\tau,\tau)],\quad t\in\mathbb{R},\ \tau\in[a,b]. \end{pmatrix}$$

where

$$u(t,\tau) = \frac{d}{d\tau}x(t,\tau), \qquad v(t,\tau) = \frac{d}{d\tau}y(t,\tau)$$

and we have defined the operators, $B : X \longrightarrow X$, $C : X\times Y \longrightarrow Y$, $A : X\times Y \longrightarrow X\times Y$ by,

$$B(x,y)(t,\tau) = (B_1(x,y)(t,\tau), B_2(x,y)(t,\tau)) = \tag{1.46}$$

$$= \begin{pmatrix} \int\limits_{t-\tau}^{t} f\left(s,x\left(s,\tau\right),y\left(s,\tau\right),\tau\right)ds \\ f(t,x(t,\tau),y(t,\tau),\tau) - f(t-\tau,x(t-\tau,\tau),y(t-\tau,\tau),\tau), \\ ,\ t\in\mathbb{R},\ \tau\in[a,b], \end{pmatrix}$$

$$C((x,y),(u,v))(t,\tau) = (C_1((x,y),(u,v))(t,\tau), C_2((x,y),(u,v))(t,\tau)) = \tag{1.47}$$

$$\left(\begin{array}{c} \int\limits_{t-\tau}^{t}[\frac{\partial f}{\partial \tau}(s,x\left(s,\tau\right),y\left(s,\tau\right),\tau)+\frac{\partial f}{\partial x}(s,x\left(s,\tau\right),y\left(s,\tau\right),\tau)\cdot u(s,\tau)+ \\ +\frac{\partial f}{\partial y}(s,x\left(s,\tau\right),y\left(s,\tau\right),\tau)\cdot v(s,\tau)]ds, \\ \frac{\partial f}{\partial \tau}(t,x(t,\tau),y(t,\tau),\tau)+\frac{\partial f}{\partial x}(t,x(t,\tau),y(t,\tau),\tau)\cdot u(t,\tau)+ \\ +\frac{\partial f}{\partial y}(t,x(t,\tau),y(t,\tau),\tau)\cdot v(t,\tau)+\frac{\partial f}{\partial t}(t-\tau,x(t-\tau,\tau),y(t-\tau,\tau),\tau)- \\ -\frac{\partial f}{\partial \tau}(t-\tau,x(t-\tau,\tau),y(t-\tau,\tau),\tau)-\frac{\partial f}{\partial x}(t-\tau,x(t-\tau,\tau),y(t-\tau,\tau),\tau)\cdot \\ \cdot[-y(t-\tau,\tau)+u(t-\tau,\tau)]-\frac{\partial f}{\partial y}(t-\tau,x(t-\tau,\tau),y(t-\tau,\tau),\tau)\cdot \\ \cdot[-\frac{\partial}{\partial t}y(t-\tau,\tau)+(v(t-\tau,\tau)], \quad t\in\mathbb{R},\ \tau\in[a,b]. \end{array}\right)$$

$$A((x,y),(u,v))=(B(x,y),C((x,y),(u,v))),\ \forall x,y\in X,\forall u,v\in Y. \tag{1.48}$$

Remark 25 *From condition (v) it follows that* $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ *are bounded and therefore exist* $L_1\geq 0$ *and* $L_2\geq 0$ *such that*

$$\left|\frac{\partial f}{\partial x}(s,u(t,\tau),v(t,\tau),\tau)\right|\leq L_1,\quad \forall t\in\mathbb{R},\forall u\in X_+(\varpi),v\in X(\varpi),\forall\tau\in[a,b]$$

and

$$\left|\frac{\partial f}{\partial y}(s,u(t,\tau),v(t,\tau),\tau)\right|\leq L_2,\quad \forall t\in\mathbb{R},\forall u\in X_+(\varpi),v\in X(\varpi),\forall\tau\in[a,b].$$

Theorem 26 *(see [51]) Under the conditions (i)-(iv), if* $\alpha b+2\beta<1$ *then the equation (1.44) has unique solution* (x^*,y^*) *in* X*. Moreover, for arbitrary* $(x_0,y_0)\in X$*, the sequence of successive approximations* $((x_n,y_n))_{n\in\mathbb{N}}$ *defined by*

$$x_{n+1}(t,\tau)=\int\limits_{t-\tau}^{t}f\left(s,x_n\left(s,\tau\right),y_n\left(s,\tau\right),\tau\right)ds$$

$$y_{n+1}(t,\tau)=f(t,x_n(t,\tau),y_n(t,\tau),\tau)-f(t-\tau,x_n(t-\tau,\tau),y_n(t-\tau,\tau),\tau)$$

uniformly converges on $\mathbb{R}\times[a,b]$ *to* (x^*,y^*)*, and* $x^*\in C^1(\mathbb{R}\times[a,b])$*, with*

$$y^*(t,\tau)=\frac{d}{dt}x^*(t,\tau),\ \forall t\in\mathbb{R},\ \forall\tau\in[a,b].$$

b) Under the conditions (i)-(v), if $bL_1+2L_2<1$ *then* $y^*(t,\cdot)\in C^1[a,b]$ *, and the pair* $(u^*,v^*)=(\frac{\partial x^*}{\partial\lambda},\frac{\partial y^*}{\partial\lambda})\in Y$ *is the unique solution of (1.45).*

Sketch of proof: a) By condition (ii) it follows that

$$B_1(x,y)(t,\tau)\geq 0,\quad \forall(t,\tau)\in\mathbb{R}\times[a,b]$$
$$B_1(x,y)(t,\tau)\leq M\tau\leq Mb,\quad \forall(t,\tau)\in\mathbb{R}\times[a,b].$$

and using the transformation of variable $u=s+\varpi$, we get

$$B_1(x,y)(t+\varpi,\tau)=B_1(x,y)(t,\tau),\quad \forall(t,\tau)\in\mathbb{R}\times[a,b],\forall(x,y)\in X,$$

and

$$B_2(x,y)(t+\varpi,\tau)=B_2(x,y)(t,\tau),\ \forall(t,\tau)\in\mathbb{R}\times[a,b],\forall(x,y)\in X,$$

that is $B\left(X\right)\subset X$. Using the fixed point technique we obtain

$$d(B(x_1,y_1),B(x_2,y_2))\leq\left(\begin{array}{cc}\alpha b & \beta b\\ 2\alpha & 2\beta\end{array}\right)d((x_1,y_1),(x_2,y_2)),$$

$\forall (x_1, y_1), (x_2, y_2) \in X$. The eigenvalues of the matrix

$$Q = \begin{pmatrix} \alpha b & \beta b \\ 2\alpha & 2\beta \end{pmatrix}$$

are $\lambda_1 = 0$ and $\lambda_2 = 2\beta + \alpha b$. By condition $\alpha b + 2\beta < 1$ we infer that $Q^m \to 0$ for $m \to \infty$ and using the Perov's fixed point theorem, the operator B has unique fixed point $(x^*, y^*) \in X$ and the sequence $((x_n, y_n))_{n\in\mathbb{N}}$ uniformly converges to (x^*, y^*) on $\mathbb{R} \times [a, b]$. Moreover, $x^* \in C^1(\mathbb{R} \times [a, b])$ and

$$y^*(t, \tau) = \frac{d}{dt} x^*(t, \tau), \ \forall t \in \mathbb{R}, \ \forall \tau \in [a, b].$$

b) Condition (v) implies that $x_m, y_m \in C^1(\mathbb{R} \times [a, b])$, $\forall m \in \mathbb{N}^*$. For arbitrary $(x, y) \in X$, let $C((x, y), \cdot) : Y \longrightarrow Y$. By (iv) and (v) we obtain

$$C_1((x, y), (u, v))(t + \varpi, \tau) = C_1((x, y), (u, v))(t, \tau), \quad \forall (t, \tau) \in \mathbb{R} \times [a, b]$$

and

$$C_2((x, y), (u, v))(t + \varpi, \tau) = C_2((x, y), (u, v))(t, \tau), \quad \forall (t, \tau) \in \mathbb{R} \times [a, b].$$

So, $C((x, y), Y) \subset Y, \forall (x, y) \in X$. For $(u_1, v_1), (u_2, v_2) \in Y$, applying the fixed point technique we get

$$\rho(C((x, y), (u_1, v_1)), C((x, y), (u_2, v_2))) \leq \begin{pmatrix} L_1 b & L_2 b \\ 2L_1 & 2L_2 \end{pmatrix} \cdot \rho((u_1, v_1), (u_2, v_2)),$$

$\forall (x, y) \in X$, $\forall (u_1, v_1), (u_2, v_2) \in Y$. Since, $bL_1 + 2L_2 < 1$, we infer that the operator $C((x^*, y^*), \cdot)$ has unique fixed point $(u^*, v^*) \in Y$, and therefore $((x^*, y^*), (u^*, v^*)) \in X \times Y$ is the unique fixed point of the operator A. For arbitrary $x_0 \in X_+(\varpi) \cap C^2(\mathbb{R} \times [a, b])$, choosing

$$y_0 = \frac{\partial x_0}{\partial t}, \quad u_0 = \frac{\partial x_0}{\partial \tau}, \quad v_0 = \frac{\partial y_0}{\partial \tau},$$

the sequence given by

$$(A^n((x_0, y_0), (u_0, v_0)))_n = ((x_n, y_n), (u_n, v_n))$$

converges on $X \times Y$ to $((x^*, y^*), (u^*, v^*))$. Then, $y^*(t, \cdot) \in C^1[a, b], \forall t \in \mathbb{R}$ and $(u^*, v^*) = (\frac{\partial x^*}{\partial \tau}, \frac{\partial y^*}{\partial \tau}) \in Y$ is the unique solution of equation (1.45).

1.4 Delay integro-differential equations with nonintegral term

We present now, the results obtained in [65] concerning the existence and uniqueness of the solution of the neutral Volterra delay integro-differential equation of the form

$$\begin{cases} x'(t) = f(t, x(t), x'(t - \tau)) + \int\limits_{t-\tau}^{t} g(t, s, x(s), x'(s)) ds, & t \in [0, b] \\ x(t) = \varphi(t), & t \in [-\tau, 0] \end{cases} \tag{1.49}$$

which is a generalization of the equation (1.28), as can be easily viewed. For this purpose, the Perov's fixed point theorem is applied on the product functional space $X = C[-\tau, b] \times C[-\tau, b]$, where

$$C[-\tau, b] = \{f : [-\tau, b] \longrightarrow \mathbb{R} : f \text{ continuous }\}$$

On this space it is defined the generalized metric

$$d_B : X \times X \longrightarrow \mathbb{R}^2,$$

by

$$d_B((x_1, y_1), (x_2, y_2)) = (\|x_1 - x_2\|_B, \|y_1 - y_2\|_B), \tag{1.50}$$

where the Bielecki's type norm on $C[-\tau, b]$ is,

$$\|u\|_B = \max\{|u(t)| \cdot e^{-\theta(t+\tau)} : t \in [-\tau, b]\}, \quad \forall u \in C[-\tau, b], \tag{1.51}$$

with $\theta > 0$ convenient chosen.

We observe that (X, d_B) is a complete generalized metric space and denoting $x' = y$, it obtains the following system equivalent with (1.49):

$$\begin{cases} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \varphi(0) + \int_0^t f(s, x(s), y(s-\tau))ds + \\ + \int_0^t \left(\int_{\eta-\tau}^{\eta} g(\eta, s, x(s), y(s))ds \right) d\eta \ , \\ f(t, x(t), y(t-\tau)) + \int_{t-\tau}^{t} g(t, s, x(s), y(s))ds \end{pmatrix}, & t \in [0, b] \\ (x(t), y(t)) = (\varphi(t), \varphi'(t)), & t \in [-\tau, 0]. \end{cases} \tag{1.52}$$

It is defined the map $A : X \longrightarrow X$ by, $A = (A_1, A_2)$ with

$$(A_1(x(t), y(t)), A_2(x(t), y(t))) = (\varphi(t), \varphi'(t)), \qquad \forall t \in [-\tau, 0] \tag{1.53}$$

and

$$A_1(x(t), y(t)) = \varphi(0) + \int_0^t f(s, x(s), y(s-\tau))ds + \int_0^t \left(\int_{\eta-\tau}^{\eta} g(\eta, s, x(s), y(s))ds \right) d\eta \tag{1.54}$$

$$A_2(x(t), y(t)) = f(t, x(t), y(t-\tau)) + \int_{t-\tau}^{t} g(t, s, x(s), y(s))ds, \qquad \forall t \in [0, b]. \tag{1.55}$$

and the following conditions are imposed:
(CC) (continuity conditions) :

$$f \in C([0, b] \times \mathbb{R} \times \mathbb{R}), \quad g \in C([0, b] \times [-\tau, b] \times \mathbb{R} \times \mathbb{R}), \quad \varphi \in C^1[-\tau, 0]$$

(BC) (boundedness condition) : $\exists M, K > 0$ such that

$$|f(t, u, v)| \le M, \quad \forall (t, u, v) \in [0, b] \times \mathbb{R} \times \mathbb{R}$$

and

$$|g(t, s, u, v)| \le K, \quad \forall (t, s, u, v) \in [0, b] \times [-\tau, b] \times \mathbb{R} \times \mathbb{R}.$$

(CPC) (compatibility condition) :

$$\varphi'(0) = f(0, \varphi(0), \varphi'(-\tau)) + \int_{-\tau}^{0} g(0, s, \varphi(s), \varphi'(s))ds \tag{1.56}$$

(LC) (Lipschitz conditions) : $\exists \alpha, \beta > 0$ and $\exists L_1, L_2 > 0$ such that

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \leq \alpha |u_1 - u_2| + \beta |v_1 - v_2|, \\ ,\forall t \in [0,b], \quad \forall u_1,u_2,v_1,v_2 \in \mathbb{R}$$

and

$$|g(t,s,u_1,v_1) - g(t,s,u_2,v_2)| \leq L_1 |u_1 - u_2| + L_2 |v_1 - v_2|, \\ ,\forall (t,s) \in [0,b] \times [-\tau, b], \quad \forall u_1,u_2,v_1,v_2 \in \mathbb{R}.$$

Applying the fixed point technique it obtains,

$$|A_1(x_1,y_1)(t) - A_1(x_2,y_2)(t)| \leq$$

$$\leq \int_0^t |f(s,x_1(s),y_1(s-\tau) - f(s,x_2(s),y_2(s-\tau))| \, ds +$$

$$+ \int_0^t \left(\int_{\eta-\tau}^{\eta} |g(\eta,s,x_1(s),y_1(s)) - g(\eta,s,x_2(s),y_2(s))| \, ds \right) d\eta \leq$$

$$\leq \left(\frac{\alpha}{\theta} \|x_1 - x_2\|_B + \frac{\beta}{\theta} e^{-\theta\tau} \cdot \|y_1 - y_2\|_B \right) \int_0^t \theta e^{\theta(s+\tau)} ds +$$

$$+ \left(\frac{L_1}{\theta} \|x_1 - x_2\|_B + \frac{L_2}{\theta} \|y_1 - y_2\|_B \right) \int_0^t \left(\int_{\eta-\tau}^{\eta} \theta e^{\theta(s+\tau)}] ds \right) d\eta \leq$$

$$\leq \left[\left(\frac{\alpha}{\theta} + \frac{L_1}{\theta^2} \right) \|x_1 - x_2\|_B + \left(\frac{\beta}{\theta} \cdot e^{-\theta\tau} + \frac{L_2}{\theta^2} \right) \|y_1 - y_2\|_B \right] \cdot e^{\theta(t+\tau)}, \quad \forall t \in [0,b],$$

which leads to

$$\|A_1(x_1,y_1) - A_1(x_2,y_2)\|_B \leq$$

$$\leq \left(\frac{\alpha}{\theta} + \frac{L_1}{\theta^2} \right) \|x_1 - x_2\|_B + \left(\frac{\beta}{\theta} \cdot e^{-\theta\tau} + \frac{L_2}{\theta^2} \right) \|y_1 - y_2\|_B . \tag{1.57}$$

On the other hand, for $t \in [0,b]$,

$$|A_2(x_1,y_1)(t) - A_2(x_2,y_2)(t)| \leq$$

$$\leq |f(t,x_1(t),y_1(t-\tau)) - f(t,x_2(t),y_2(t-\tau))| +$$

$$+ \int_{t-\tau}^t |g(t,s,x_1(s),y_1(s)) - g(t,s,x_2(s),y_2(s))| \, ds \leq$$

$$\leq \left(\alpha \|x_1 - x_2\|_B + \beta e^{-\theta\tau} \|y_1 - y_2\|_B \right) e^{\theta(t+\tau)} + \left(\frac{L_1}{\theta} \|x_1 - x_2\|_B + \frac{L_2}{\theta} \|y_1 - y_2\|_B \right) \cdot$$

$$\cdot [e^{\theta(t+\tau)} - e^{\theta t}] < \left[\left(\alpha + \frac{L_1}{\theta} \right) \|x_1 - x_2\|_B + \left(\beta \cdot e^{-\theta\tau} + \frac{L_2}{\theta} \right) \|y_1 - y_2\|_B \right] \cdot e^{\theta(t+\tau)}.$$

Then,

$$\|A_2(x_1,y_1) - A_2(x_2,y_2)\|_B \leq$$

$$\leq \left(\alpha + \frac{L_1}{\theta} \right) \|x_1 - x_2\|_B + \left(\beta \cdot e^{-\theta\tau} + \frac{L_2}{\theta} \right) \|y_1 - y_2\|_B \tag{1.58}$$

and

$$d_B(A(x_1,y_1),A(x_2,y_2)) \leq \begin{pmatrix} \frac{\alpha}{\theta}+\frac{L_1}{\theta^2} & \frac{\beta}{\theta}\cdot e^{-\theta\tau}+\frac{L_2}{\theta^2} \\ \alpha+\frac{L_1}{\theta} & \beta\cdot e^{-\theta\tau}+\frac{L_2}{\theta} \end{pmatrix} \cdot d_B((x_1,y_1),(x_2,y_2)). \quad (1.59)$$

The eigenvalues of the matrix

$$Q = \begin{pmatrix} \frac{\alpha}{\theta}+\frac{L_1}{\theta^2} & \frac{\beta}{\theta}\cdot e^{-\theta\tau}+\frac{L_2}{\theta^2} \\ \alpha+\frac{L_1}{\theta} & \beta\cdot e^{-\theta\tau}+\frac{L_2}{\theta} \end{pmatrix}$$

are $\lambda_1 = 0$ and

$$\lambda_2 = \frac{\alpha}{\theta}+\frac{L_1}{\theta^2}+\beta\cdot e^{-\theta\tau}+\frac{L_2}{\theta} > 0.$$

We have,

$$0 < \lambda_2 < 1 \Longleftrightarrow h(\theta) = \theta^2-(\alpha+L_2)\theta - L_1 > \theta^2\beta\cdot e^{-\theta\tau}.$$

The quadratic equation $\theta^2-(\alpha+L_2)\theta-L_1 = 0$ has the roots $\theta_1 < 0$ and $\theta_2 > 0$, and the peak $V(\frac{\alpha+L_2}{2}, -\frac{\Delta}{4})$, where

$$\Delta = (\alpha+L_2)^2+4L_1.$$

If we represent geometric the graphs of the functions $h(\theta)$ and $u(\theta) = \theta^2\beta\cdot e^{-\theta\tau}$, then we see that there exists an unique point $\theta^* > \theta_2$ such that $h(\theta^*) = u(\theta^*)$ and $h(\theta) > \theta^2\beta\cdot e^{-\theta\tau}$, $\forall\theta > \theta^*$ (on the other hand, this fact follows from the properties:

$$h(\theta) < 0, \forall\theta\in[0,\theta_2), \quad u(\theta) > 0, \forall\theta > 0, \quad \lim_{\theta\to\infty} h(\theta) = \infty, \quad \lim_{\theta\to\infty}\theta^2\beta\cdot e^{-\theta\tau} = 0$$

and because the function $u(\theta) = \theta^2\beta\cdot e^{-\theta\tau}$ has in $\theta = 0$ a global minimum and in $\theta = \frac{2}{\tau}$ a local maximum). If we choose a value $\theta > \theta^*$, then the operator $A = (A_1, A_2)$ given by (1.53), (1.54), (1.55) is Q-contraction, and has an unique fixed point $(x^*, y^*) \in X$. The pair (x^*, y^*) will be the unique solution of the initial value problem (1.52), hence for any $t \in [0,b]$ and any $\eta \in [0,b]$,

$$x^*(t) = \varphi(0)+\int_0^t f(s,x^*(s),y^*(s-\tau))ds+\int_0^t\left(\int_{\eta-\tau}^{\eta} g(\eta,s,x^*(s),y^*(s))ds\right)d\eta \quad (1.60)$$

and

$$y^*(t) = f(t,x^*(t),y^*(t-\tau))+\int_{t-\tau}^t g(t,s,x^*(s),y^*(s))ds, \qquad \forall t\in[0,b]. \quad (1.61)$$

Using the continuity conditions (CC) and the compatibility condition (CPC), since $x^*, y^* \in C[-\tau,b]$ and $x^*(t) = \varphi(t)$, $\forall t\in[-\tau,0]$, we infer that $x^* \in C^1[-\tau,b]$. If we differentiate with respect by t the equality (1.60), it obtains,

$$(x^*)'(t) = f(t,x^*(t),y^*(t-\tau))+\int_{t-\tau}^t g(t,s,x^*(s),y^*(s))ds, \qquad \forall t\in[0,b]$$

and together with the equality (1.61) it follows that $(x^*)' = y^*$.
Now, let see how can be obtained the point θ^*.
We have

$$h(\theta) = \theta^2\beta\cdot e^{-\theta\tau} \Longleftrightarrow \theta = H(\theta) = \alpha+L_2+\theta\beta\cdot e^{-\theta\tau}+\frac{L_1}{\theta}$$

which means that θ^* is a fixed point of H. Moreover,

$$H'(\theta) < 0 \Longleftrightarrow -\frac{L_1}{\theta^2} + \beta \cdot e^{-\theta\tau}(1 - \theta\tau) < 0.$$

If $\theta \geq \frac{1}{\tau}$ then $H'(\theta) < 0$ and $H'(\frac{1}{\tau}) = -\tau^2 L_1 < 0$. So, $H'(\theta) < 0$, $\forall \theta \geq \frac{1}{\tau}$.
If $\frac{1}{\tau} < \theta_2$ then we can take $\overline{\theta} = H(\theta_2) > \theta^*$ and for any $\theta > \overline{\theta}$ we have $0 < \lambda_2 < 1$.
If $\frac{1}{\tau} > \theta_2$ then we have two possibilities :
1) If $h(\frac{1}{\tau}) < \frac{1}{\tau^2} \cdot \beta e^{-1}$ then we take $\overline{\theta} = H(\frac{1}{\tau}) > \theta^*$ and for any $\theta > \overline{\theta}$ we have $0 < \lambda_2 < 1$.
2) If $h(\frac{1}{\tau}) > \frac{1}{\tau^2} \cdot \beta e^{-1}$ then it is clear that $\frac{1}{\tau} > \theta^*$ and for any $\theta > \frac{1}{\tau}$ we will have $0 < \lambda_2 < 1$.
Consequently, we can choose $\overline{\theta}$ (which can be $H(\theta_2)$, or $H(\frac{1}{\tau})$, or $\frac{1}{\tau}$) such that $0 < \lambda_2 < 1$, $\forall \theta > \overline{\theta}$. In this way, the following result is obtained:

Theorem 27 *(see [65]) Under the conditions (CC), (CPC) and (LC), the initial value problem (1.52) has in $C[-\tau, b] \times C[-\tau, b]$ an unique solution (x^*, y^*) such that $x^* \in C^1[-\tau, b]$ and $(x^*)' = y^*$.*

Remark 28 *(i) Under the conditions (CC)-(LC), the initial value problem (1.49) has an unique bounded solution in $C[-\tau, b]$. Indeed,*

$$|x^*(t)| \leq |\varphi(0)| + \int_0^t \left| f(s, x^*(s), (x^*)'(s-\tau)) \right| ds + \int_{t-\tau}^t \left| g(t, s, x^*(s), (x^*)'(s)) \right| ds \leq$$

$$\leq |\varphi(0)| + Mb + K\tau, \quad \forall t \in [0, b]$$

and $x^*(t) = \varphi(t)$, $\forall t \in [-\tau, 0]$.
(ii) Also, we obtain the positiveness of the solution if we consider the above conditions and the conditions $\varphi(t) > 0$, $\forall t \in [-\tau, 0]$,

$$f(t, u, v) > 0, \quad \forall (t, u, v) \in [0, b] \times \mathbb{R} \times \mathbb{R}$$

and

$$g(t, s, u, v) > 0, \quad \forall (t, s, u, v) \in [0, b] \times [-\tau, b] \times \mathbb{R} \times \mathbb{R}.$$

In this case it follows that $0 < x^*(t) \leq \varphi(0) + Mb + K\tau$, $\forall t \in [0, b]$.

In the same context, it is obtained in [65] the uniformly Lipschitz property for the family of functions $F_m : [0, b] \to \mathbb{R}$, given by

$$F_m(t) = f(t, x_m(t), y_m(t-\tau)), \qquad t \in [0, b],\ m \in \mathbb{N}.$$

1.5 Boundary value problems of neutral type

1.5.1 Neutral type Fredholm integro-differential equations in Banach spaces

We present here, the results obtained in [48] for neutral type Fredholm integro-differential equations in Banach spaces

$$x(t) = \int_a^b f\left(t, s, x(s), x'(s)\right) ds + g(t), \quad t \in [a, b] \tag{1.62}$$

where $f : [a, b] \times [a, b] \times X \times X \to X$ is continuous, X is a Banach space and $g \in C^1([a, b], X)$. If $f(\cdot, s, u, v) \in C^1([a, b], X)$, $\forall s \in [a, b]$, $\forall u, v \in X$, and if we differentiate with respect by t the equation (1.62), then denoting $x' = y$, the study of this equation it reduces to the following system of Fredholm integral equations:

$$\begin{cases} x(t) = \int_a^b f(t,s,x(s),y(s))\,ds + g(t) \\ y(t) = \int_a^b \frac{\partial f}{\partial t}(t,s,x(s),y(s))\,ds + g'(t) \end{cases}, \quad t \in [a,b]. \tag{1.63}$$

The following conditions are imposed:

(i) (continuity): $f \in C([a,b] \times [a,b] \times X \times X, X)$, $g \in C^1([a,b], X)$ and $f(\cdot, s, u, v) \in C^1([a,b], X)$,for any $s \in [a,b]$, $u, v \in X$

(ii) (Lipschitz conditions): there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \eta \in \mathbb{R}_+^*$ such that for any $t, t', s, s_1, s_2 \in [a,b]$ and $u, v, u_1, u_2, v_1, v_2 \in X$, we have:

$$\|f(t,s,u_1,v_1) - f(t,s,u_2,v_2)\|_X \le \alpha_1 \|u_1 - u_2\|_X + \beta_1 \|v_1 - v_2\|_X \tag{1.64}$$

$$\left\|\frac{\partial f}{\partial t}(t,s,u_1,v_1) - \frac{\partial f}{\partial t}(t,s,u_2,v_2)\right\|_X \le \alpha_2 \|u_1 - u_2\|_X + \beta_2 \|v_1 - v_2\|_X \tag{1.65}$$

$$\|f(t,s_1,u,v) - f(t,s_2,u,v)\|_X \le \gamma_1 |s_1 - s_2| \tag{1.66}$$

$$\left\|\frac{\partial f}{\partial t}(t,s_1,u,v) - \frac{\partial f}{\partial t}(t,s_2,u,v)\right\|_X \le \gamma_2 |s_1 - s_2| \tag{1.67}$$

$$\left\|f(t,s,u,v) - f(t',s,u,v)\right\|_X \le \delta_1 |t - t'| \tag{1.68}$$

$$\left\|\frac{\partial f}{\partial t}(t,s,u,v) - \frac{\partial f}{\partial t}(t',s,u,v)\right\|_X \le \delta_2 |t - t'| \tag{1.69}$$

$$\left\|g'(t) - g'(t')\right\|_X \le \eta \cdot |t - t'|. \tag{1.70}$$

Denoting

$$\|u\|_C = \max\{\|u(t)\|_X : t \in [a,b]\},$$

for $u \in C([a,b], X)$, on the product space $Y = C([a,b], X) \times C([a,b], X)$ are defined the generalized metric $d : Y \times Y \to \mathbb{R}^2$ by,

$$d((u_1,v_1),(u_2,v_2)) = (\|u_1 - u_2\|_C, \|v_1 - v_2\|_C),$$

for $(u_1,v_1),(u_2,v_2) \in Y$ and the operator $A : Y \to Y$, $A = (A_1 A_2)$:

$$A_1(x,y)(t) = \int_a^b f(t,s,x(s),y(s))\,ds + g(t)$$

and

$$A_2(x,y)(t) = \int_a^b \frac{\partial f}{\partial t}(t,s,x(s),y(s))\,ds + gt(t), t \in [a,b]. \tag{1.71}$$

Of course, (Y, d) is a complete generalized metric space and applying the Perov's fixed point theorem, the following result is obtained:

Theorem 29 *(see [48]) Under the conditions (i), (1.64), (1.65), if* $(\alpha_1 + \beta_2)(b - a) < 2$ *and*

$$1 + (b-a)^4 (\alpha_1\beta_2 - \alpha_2\beta_1)^2 + 2\alpha_2\beta_1 (b-a)^2 > (b-a)^2 \left(\alpha_1^2 + \beta_2^2\right) \tag{1.72}$$

then the operator A has a unique fixed point (x^*, y^*) such that

$$x^* \in C^1([a,b], X), \qquad y^* = (x^*)'$$

and x^* is the unique solution of the equation (1.62). Moreover, the sequence of the successive approximations given by,

$$(x_0(t), y_0(t)) = (g(t), g'(t)), \quad t \in [a,b]$$

$$x_{m+1}(t) = \int_a^b f(t, s, x_m(s), y_m(s))\, ds + g(t), \quad t \in [a,b]$$

and

$$y_{m+1}(t) = \int_a^b \frac{\partial f}{\partial t}(t, s, x_m(s), y_m(s))\, ds + g'(t), \quad t \in [a,b],$$

converges in Y to (x^*, y^*) and the following error estimation holds:

$$d((x_m, y_m), (x^*, y^*)) \le Q^m (I_n - Q)^{-1} \cdot d((x_0, y_0), (x_1, y_1)), \quad \text{for any } m \in \mathbb{N}^*,$$

where

$$Q = (b-a)\begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}.$$

In the proof of this theorem (in [48]) it is showed that the inequalities (1.72) and $(\alpha_1 + \beta_2)(b-a) < 2$ lead to $\lambda_1, \lambda_2 \in (-1,1)$, where λ_1 and λ_2 are the eigenvalues of the matrix Q. In addition with this existence result, it is developed in [48] a method to approximate the solution of (1.63) by combining the method of successive approximations with the quadrature rule (1.20) and using the conditions (1.64)-(1.70) in the proof of the convergence of this method. A similar method is presented in the next section for neutral type two-point boundary value problems as a particular case of the equation (1.62).

1.5.2 Two-point boundary value problems for second order differential equations of neutral type

Existence, uniqueness, boundedness and Lipschitz properties

Now, we present the results obtained in [58], which represent an improvement of those obtained in [39], pages 243-248, and in [201], considering the two-point boundary value problem

$$\begin{cases} y''(x) = f(x, y(x), y'(x)), & x \in [a,b] \\ y(a) = c, \quad y(b) = d. \end{cases} \tag{1.73}$$

The existence and uniqueness of the solution of (1.73) on the real axis is studied in [39] and [201] using the Perov's fixed point theorem. In this section we approach (1.73) in Banach spaces for the existence, uniqueness and approximation of the solution.

It is known that the two-point boundary value problem (1.73) is equivalent with the following second kind Fredholm integro-differential equation:

$$y(x) = \frac{x-a}{b-a} \cdot d + \frac{b-x}{b-a} \cdot c - \int_a^b G(x,s) f(s, y(s), y'(s))\, ds, \quad x \in [a,b] \tag{1.74}$$

where

$$G(x,s) = \begin{cases} \frac{(s-a)(b-x)}{b-a}, & \text{if } s \le x \\ \frac{(x-a)(b-s)}{b-a}, & \text{if } s \ge x \end{cases}$$

is the well-known Green's function and differentiating equation (1.74) we get

$$y'(x) = \frac{d-c}{b-a} - \int_a^b \frac{\partial G}{\partial x} \cdot f(s, y(s), y'(s))\, ds, \quad x \in [a,b]$$

with

$$\frac{\partial G}{\partial x}(x,s)=\begin{cases} -\frac{(s-a)}{b-a}, & \text{if } s<x \\ \frac{(b-s)}{b-a}, & \text{if } s>x. \end{cases}$$

In that follows we denote $\frac{1}{b-a}\cdot(d-c)=\frac{d-c}{b-a}$. Denoting $z=y'$ we obtain the following system of Fredholm integral equations equivalent with (1.74):

$$\begin{cases} y(x)=\frac{x-a}{b-a}\cdot d+\frac{b-x}{b-a}\cdot c-\int\limits_a^b G(x,s)\cdot f(s,y(s),z(s))\,ds \\ z(x)=\frac{d-c}{b-a}-\int\limits_a^b \frac{\partial G}{\partial x}(x,s)\cdot f(s,y(s),z(s))\,ds, \end{cases} \quad x\in[a,b]. \tag{1.75}$$

Let X be a real Banach space. The following notations will be used:

$$C([a,b],X)=\{f:[a,b]\to X \mid f \text{ is continuous on } [a,b]\}$$

$$C^k([a,b],X)=\{f:[a,b]\to X \mid f \text{ is } k \text{ times differentiable on } [a,b] \text{ with } f^{(k)} \text{ continuous on } [a,b]\},\quad k\in\mathbb{N}^*$$

$$Lip([a,b],X)=\{f:[a,b]\to X \mid f \text{ is Lipschitzian on } [a,b]\}.$$

Remember that $f:[a,b]\to X$ is Lipschitzian on $[a,b]$ iff there exists $L\geq 0$ such that

$$\left\|f(x)-f(x')\right\|_X\leq L\cdot\left|x-x'\right|,\ \text{for any } x,x'\in[a,b]$$

and $f:[a,b]\to X$ is bounded on $[a,b]$ iff there exists $M\geq 0$ such that $\|f(x)\|_X\leq M$ for all $x\in[a,b]$. Concerning to the boundary value problem (1.73) we consider the following conditions:

(i) $f\in C([a,b]\times X\times X,X)$, and $c,d\in X$

(ii) there exist $\alpha,\beta\geq 0$ such that

$$\left\|f(x,u,v)-f(x,u',v')\right\|_X\leq\alpha\cdot\left\|u-u'\right\|_X+\beta\cdot\left\|v-v'\right\|_X$$

for any $u,v,u',v'\in X$.

(iii) $\frac{\alpha}{8}(b-a)^2+\frac{\beta}{2}(b-a)<1$.

In order to prove the convergence of the approximation method we will impose the following supplementary Lipschitz condition:

(iv) there exist $\gamma\geq 0$ such that

$$\left\|f(x,u,v)-f(x',u,v)\right\|_X\leq\gamma\cdot\left|x-x'\right|,\ \text{for any } x,x'\in[a,b].$$

Consider the generalized metric $d_C:Y\times Y\to\mathbb{R}^2$, defined by

$$d_C((y_1,z_1),(y_2,z_2))=(\|y_1-y_2\|_C\ ,\ \|z_1-z_2\|_C),\quad \text{for } (y_1,z_1),(y_2,z_2)\in Y$$

where $Y=C([a,b],X)\times C([a,b],X)$ and

$$\|y\|_C=\max\{\|y(x)\|_X : x\in[a,b]\} \text{ for } y\in C([a,b],X).$$

Let $g,h:[a,b]\to X$, $g(x)=\frac{x-a}{b-a}\cdot d+\frac{b-x}{b-a}\cdot c$, $h(x)=\frac{d-c}{b-a}$, $x\in[a,b]$. We see that

$$\|g\|_C\leq\max(\|c\|_X,\|d\|_X)\overset{notation}{=}r,\quad \|h\|_C\leq\frac{\|c\|_X+\|d\|_X}{b-a}\overset{notation}{=}q$$

and since $f \in C([a,b] \times X \times X, X)$ we infer that the function $f_0 : [a,b] \to X$

$$f_0(s) = f(s, g(s), h(s)), \quad s \in [a,b]$$

is bounded on the compact $[a,b]$ having $\|f_0(s)\|_X \leq M_0$ for all $s \in [a,b]$. In order to obtain the existence and uniqueness of the solution, the following operator $A = (A_1, A_2) : Y \to Y$ is defined,

$$A_1(u,v)(x) = \frac{x-a}{b-a} \cdot d + \frac{b-x}{b-a} \cdot c - \int_a^b G(x,s) \cdot f(s, u(s), v(s))\, ds$$

$$A_2(u,v)(x) = \frac{d-c}{b-a} - \int_a^b \frac{\partial G}{\partial x}(x,s) \cdot f(s, u(s), v(s))\, ds$$

and using the fixed point technique, after elementary calculus, we get

$$\|A_1(u_1,v_1) - A_1(u_2,v_2)\|_C \leq \frac{\alpha}{8}(b-a)^2 \cdot \|u_1 - u_2\|_C + \frac{\beta}{8}(b-a)^2 \cdot \|v_1 - v_2\|_C$$

and

$$\|A_2(u_1,v_1) - A_2(u_2,v_2)\|_C \leq \frac{\alpha}{2}(b-a) \cdot \|u_1 - u_2\|_C + \frac{\beta}{2}(b-a) \cdot \|v_1 - v_2\|_C$$

for any $(u_1,v_1), (u_2,v_2) \in Y$. So,

$$d_C(A(u_1,v_1), A(u_2,v_2)) \leq \begin{pmatrix} \frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2 \\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a) \end{pmatrix} \cdot d_C((u_1,v_1),(u_2,v_2))$$

and the eigenvalues of this matrix are $\lambda_1 = 0$, $\lambda_2 = \frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a) < 1$. Then, A is contraction having a unique fixed point $(y^*, z^*) \in Y$, that is

$$\begin{cases} y^*(x) = \frac{x-a}{b-a} \cdot d + \frac{b-x}{b-a} \cdot c - \int_a^b G(x,s) \cdot f(s, y^*(s), z^*(s))\, ds \\ z^*(x) = \frac{d-c}{b-a} - \int_a^b \frac{\partial G}{\partial x}(x,s) \cdot f(s, y^*(s), z^*(s))\, ds, \end{cases} \quad \forall x \in [a,b]. \tag{1.76}$$

Differentiating in (1.76) we obtain $y^* \in C^2([a,b], X)$ and $z^* = (y^*)'$. Applying the Perov's fixed point theorem we obtain

$$d_C((y_m,z_m),(y^*,(y^*)')) \leq \frac{\left[\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)\right]^{m-1}}{1 - \left[\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)\right]} \cdot Q \cdot d_C((y_1,z_1),(y_0,z_0)) \leq$$

$$\leq \frac{\left[\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)\right]^{m-1}}{1 - \left[\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)\right]} \cdot Q \cdot \begin{pmatrix} (b-a)^2 \cdot \frac{M_0}{8} \\ (b-a) \cdot \frac{M_0}{2} \end{pmatrix}, \quad \forall m \in \mathbb{N}^*$$

since

$$d_C((y_1,z_1),(y_0,z_0)) \leq \begin{pmatrix} \int_a^b |G(x,s)| \cdot \|f_0(s)\|_X\, ds \\ \int_a^b \left|\frac{\partial G}{\partial x}(x,s)\right| \cdot \|f_0(s)\|_X\, ds \end{pmatrix} \leq \begin{pmatrix} (b-a)^2 \cdot \frac{M_0}{8} \\ (b-a) \cdot \frac{M_0}{2} \end{pmatrix},$$

and thus, y^* is the unique solution of the boundary value problem (1.73) being the limit of the sequence of successive approximations. Now, we intend to prove the boundedness of the terms of the sequence of successive approximations and observe that

$$\left\|y_m(x) - y_{m-1}(x)\right\|_X \leq \int_a^b |G(x,s)| \cdot \left(\alpha \left\|y_{m-1} - y_{m-2}\right\|_C + \beta \left\|z_{m-1} - z_{m-2}\right\|_C\right) ds \leq$$

$$\leq \frac{\alpha}{8}(b-a)^2 \cdot \left\|y_{m-1} - y_{m-2}\right\|_C + \frac{\beta}{8}(b-a)^2 \cdot \left\|z_{m-1} - z_{m-2}\right\|_C$$

and

$$\left\|z_m(x) - z_{m-1}(x)\right\|_X \leq \int_a^b \left|\frac{\partial G}{\partial x}(x,s)\right| \cdot \left(\alpha \left\|y_{m-1} - y_{m-2}\right\|_C + \beta \left\|z_{m-1} - z_{m-2}\right\|_C\right) ds \leq$$

$$\leq \frac{\alpha}{2}(b-a) \cdot \left\|y_{m-1} - y_{m-2}\right\|_C + \frac{\beta}{2}(b-a) \cdot \left\|z_{m-1} - z_{m-2}\right\|_C$$

for any $x \in [a,b]$ and consequently, by induction it obtains:

$$\begin{pmatrix} \|y_m - y_{m-1}\|_C \\ \|z_m - z_{m-1}\|_C \end{pmatrix} \leq \begin{pmatrix} \frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2 \\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a) \end{pmatrix} \cdot \begin{pmatrix} \|y_{m-1} - y_{m-2}\|_C \\ \|z_{m-1} - z_{m-2}\|_C \end{pmatrix} \leq$$

$$\leq ... \leq \begin{pmatrix} \frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2 \\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a) \end{pmatrix}^{m-1} \cdot \begin{pmatrix} \|y_1 - y_0\|_C \\ \|z_1 - z_0\|_C \end{pmatrix} \leq$$

$$\leq \left[\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)\right]^{m-2} \cdot \begin{pmatrix} \frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2 \\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a) \end{pmatrix} \cdot \begin{pmatrix} (b-a)^2 \cdot \frac{M_0}{8} \\ (b-a) \cdot \frac{M_0}{2} \end{pmatrix}.$$

Then,

$$\begin{pmatrix} \|y_m - y_1\|_C \\ \|z_m - z_1\|_C \end{pmatrix} \leq \begin{pmatrix} \|y_m - y_{m-1}\|_C \\ \|z_m - z_{m-1}\|_C \end{pmatrix} + \begin{pmatrix} \|y_{m-1} - y_{m-2}\|_C \\ \|z_{m-1} - z_{m-2}\|_C \end{pmatrix} + ... + \begin{pmatrix} \|y_2 - y_1\|_C \\ \|z_2 - z_1\|_C \end{pmatrix} \leq$$

$$\leq \begin{pmatrix} \frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2 \\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a) \end{pmatrix}^{m-1} \cdot \begin{pmatrix} \|y_1 - y_0\|_C \\ \|z_1 - z_0\|_C \end{pmatrix} + \begin{pmatrix} \frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2 \\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a) \end{pmatrix}^{m-2}$$

$$\cdot \begin{pmatrix} \|y_1 - y_0\|_C \\ \|z_1 - z_0\|_C \end{pmatrix} + ... + \begin{pmatrix} \frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2 \\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a) \end{pmatrix} \cdot \begin{pmatrix} \|y_1 - y_0\|_C \\ \|z_1 - z_0\|_C \end{pmatrix} \leq$$

$$\leq \left(Q^{m-1} + Q^{m-2} + ... + Q\right) \cdot \begin{pmatrix} \|y_1 - y_0\|_C \\ \|z_1 - z_0\|_C \end{pmatrix} \leq$$

$$\leq \left(\left[\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)\right]^{m-2} + \left[\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)\right]^{m-3} + ... + 1\right) \cdot Q \cdot \begin{pmatrix} \|y_1 - y_0\|_C \\ \|z_1 - z_0\|_C \end{pmatrix} \leq$$

$$\leq \frac{1}{1 - \left[\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)\right]} \cdot \begin{pmatrix} \frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2 \\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a) \end{pmatrix} \cdot \begin{pmatrix} (b-a)^2 \cdot \frac{M_0}{8} \\ (b-a) \cdot \frac{M_0}{2} \end{pmatrix} \overset{notation}{=} \begin{pmatrix} M_1 \\ M_1' \end{pmatrix}.$$

So,

$$\begin{pmatrix} \|y_m - y_0\|_C \\ \|z_m - z_0\|_C \end{pmatrix} \leq \begin{pmatrix} M_1 \\ M_1' \end{pmatrix} + \begin{pmatrix} (b-a)^2 \cdot \frac{M_0}{8} \\ (b-a) \cdot \frac{M_0}{2} \end{pmatrix}$$

and

$$\begin{pmatrix} \|y_m\|_C \\ \|z_m\|_C \end{pmatrix} \le \begin{pmatrix} \|y_m - y_0\|_C \\ \|z_m - z_0\|_C \end{pmatrix} + \begin{pmatrix} \|y_0\|_C \\ \|z_0\|_C \end{pmatrix} \le$$

$$\le \begin{pmatrix} M_1 \\ M_1' \end{pmatrix} + \begin{pmatrix} (b-a)^2 \cdot \frac{M_0}{8} \\ (b-a) \cdot \frac{M_0}{2} \end{pmatrix} + \begin{pmatrix} r \\ q \end{pmatrix} \overset{notation}{=} \begin{pmatrix} R \\ R' \end{pmatrix} \tag{1.77}$$

for any $m \in \mathbb{N}^*$. In this way, the following result is obtained:

Theorem 30 *(see [58]) Under the conditions (i)-(iii), the boundary value problem (1.73) has unique solution in* $y^* \in C^2([a,b],X)$ *and the sequence of successive approximations* $(y_m, z_m)_{m\in\mathbb{N}} \subset C([a,b],X) \times C([a,b],X)$ *given by*

$$y_0(x) = \frac{x-a}{b-a} \cdot d + \frac{b-x}{b-a} \cdot c, \quad z_0(x) = \frac{d-c}{b-a}, \quad x \in [a,b]$$

$$y_m(x) = \frac{x-a}{b-a} \cdot d + \frac{b-x}{b-a} \cdot c - \int_a^b G(x,s) \cdot f(s, y_{m-1}(s), z_{m-1}(s))\, ds, \quad m \in \mathbb{N}^* \tag{1.78}$$

$$z_m(x) = (y_m)'(x) = \frac{d-c}{b-a} - \int_a^b \frac{\partial G}{\partial x}(x,s) \cdot f(s, y_{m-1}(s), z_{m-1}(s))\, ds, \quad m \in \mathbb{N}^* \tag{1.79}$$

has the following properties:
(i) $\lim_{m\to\infty} y_m(x) = y^*(x)$ *and* $\lim_{m\to\infty} z_m(x) = (y^*)'(x)$ *uniformly in* $C([a,b],X)$
(ii) the following estimates hold $(\forall m \in \mathbb{N}^*)$:

$$d_C((y_m, z_m), (y^*, (y^*)')) \le \frac{\left[\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)\right]^{m-1}}{1 - \left[\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)\right]} \cdot Q \cdot \begin{pmatrix} (b-a)^2 \cdot \frac{M_0}{8} \\ (b-a) \cdot \frac{M_0}{2} \end{pmatrix} \tag{1.80}$$

$$d_C((y_m, z_m), (y^*, (y^*)')) \le \frac{\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)}{1 - \left[\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)\right]} \cdot Q \cdot d_C((y_m, z_m), (y_{m-1}, z_{m-1})) \tag{1.81}$$

where

$$Q = \begin{pmatrix} \frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2 \\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a) \end{pmatrix}.$$

(iii) the terms of the sequence of successive approximations are uniformly bounded.

Remark 31 *The existence and uniqueness of the solution of (1.73) was obtained on the real axis in [39], pages 243-248, and in [201]. Here, in the previous theorem this result is extended for solutions taking values in Banach spaces, and moreover the uniformly boundedness of the terms of the sequence of successive approximations is obtained under the same conditions. In addition, we see that the contraction condition* $\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a) < 1$ *is better than the condition* $\frac{\alpha}{4}(b-a)^2 + \beta(b-a) < 1$ *imposed in [39], page 243. In that follows, we provide a method to approximate the solution.*

We define the functions $F_m : [a,b] \to X$,

$$F_m(s) = f(s, y_m(s), z_m(s)), \quad \forall s \in [a,b], \quad \forall m \in \mathbb{N}.$$

Corollary 32 *(see [58]) The solution of the boundary value problem (1.73) and its first and second derivative are bounded. The functions F_m, $m \in \mathbb{N}$, are uniformly bounded.*

In the proof of this Corollary, the following inequalities are obtained:

$$\begin{pmatrix} \|y^*\|_C \\ \left\|(y^*)'\right\|_C \end{pmatrix} \leq \begin{pmatrix} \|y^* - y_m\|_C \\ \left\|(y^*)' - z_m\right\|_C \end{pmatrix} + \begin{pmatrix} \|y_m\|_C \\ \|z_m\|_C \end{pmatrix} \leq$$

$$\leq \frac{1}{1 - \left[\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)\right]} \cdot Q \cdot \begin{pmatrix} (b-a)^2 \cdot \frac{M_0}{8} \\ (b-a) \cdot \frac{M_0}{2} \end{pmatrix} + \begin{pmatrix} R \\ R' \end{pmatrix}, \quad \forall m \in \mathbb{N}^*$$

Since the functions F_m, $m \in \mathbb{N}$, are continuous it follows that $\|F_m\|_C : [a,b] \to \mathbb{R}$, $\|F_m\|_C = \|\cdot\|_C \circ F_m$, is continuous on $[a,b]$ and therefore exists $M \geq 0$ such that

$$\|F_m\|_C \leq M = \max\{\|f(s, y_m(s), z_m(s))\|_X : s \in [a,b]\}, \quad \forall m \in \mathbb{N},$$

By

$$(y^*)''(x) = f\left(x, y^*(x), (y^*)'(x)\right)$$

and

$$y_m''(x) = f\left(x, y_{m-1}(x), y_{m-1}'(x)\right), \quad \forall m \in \mathbb{N}^*,$$

we get $\|y_m''\|_C \leq M$ for all $m \in \mathbb{N}$ and $\left\|(y^*)''\right\|_C \leq M$. Moreover,

$$\left\|(y^*)'' - y_m''\right\|_C \leq \left(\frac{\left[\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)\right]^{m-1}}{1 - \left[\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)\right]} \right) \cdot [\alpha \left(\frac{\alpha}{64}(b-a)^4 + \frac{\beta}{16}(b-a)^3 \right) +$$

$$+\beta \left(\frac{\alpha}{16}(b-a)^3 + \frac{\beta}{4}(b-a)^2 \right)] M_0, \quad \forall m \in \mathbb{N}^*. \tag{1.82}$$

From (1.80) and (1.82), since $z_m = (y_m)'$, it follows that $\lim_{m\to\infty} y_m(x) = y^*(x)$, $\lim_{m\to\infty} z_m(x) = (y^*)'(x)$, $\lim_{m\to\infty} y_m''(x) = (y^*)''(x)$, $\forall x \in [a,b]$ uniformly in $C([a,b], X)$.

In order to compute the integrals from (1.78) and (1.79) we apply a quadrature rule considering an uniform partition of the interval $[a,b]$:

$$\Delta : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$$

with

$$x_i = a + \frac{i \cdot (b-a)}{n}, \quad i = \overline{0,n}.$$

On these knots the relations (1.78) and (1.79) can be written as follows:

$$y_m(x_i) = \frac{x_i - a}{b-a} \cdot d + \frac{b - x_i}{b-a} \cdot c - \int_a^b G(x_i, s) \cdot f(s, y_{m-1}(s), z_{m-1}(s))\, ds, \quad i = \overline{0,n} \tag{1.83}$$

$$z_m(x_i) = \frac{d-c}{b-a} - \int_a^b \frac{\partial G}{\partial x}(x_i, s) \cdot f(s, y_{m-1}(s), z_{m-1}(s))\, ds, \quad i = \overline{0,n}. \tag{1.84}$$

Define the functions $H_{m,i}$, $K_{m,i} : [a,b] \to X$,

$$H_{m,i}(s) = G(x_i, s) \cdot F_m(s) = G(x_i, s) \cdot f(s, y_m(s), z_m(s)), \quad \forall s \in [a,b], \quad \forall m \in \mathbb{N},\ i = \overline{0,n}$$

$$K_{m,i}(s) = \frac{\partial G}{\partial x}(x_i, s) \cdot F_m(s) = \frac{\partial G}{\partial x}(x_i, s) \cdot f(s, y_m(s), z_m(s)), \quad \forall s \in [a,b], \quad \forall m \in \mathbb{N}, \ i = \overline{0,n}.$$

Now, we investigate the Lipschitz properties of these families of functions.

Firstly, we can see that

$$\left\| y_m(x) - y_m(x') \right\|_X \le \left\| y_m' \right\|_C \cdot |x - x'| \le \left(\frac{\|c\|_X + \|d\|_X}{b-a} + \frac{M(b-a)^2}{2} \right) \cdot |x - x'| = \delta \cdot |x - x'|$$

$$\left\| z_m(x) - z_m(x') \right\|_X = \left\| y_m'(x) - y_m'(x') \right\|_X \le \left\| y_m'' \right\|_C \cdot |x - x'| \le M \cdot |x - x'|,$$

$$\left\| F_0(x) - F_0(x') \right\|_X \le \left(\gamma + \alpha \left(\frac{\|c\|_X + \|d\|_X}{b-a} \right) \right) \cdot |x - x'|$$

and

$$\left\| F_m(x) - F_m(x') \right\|_X \le \gamma \cdot |x - x'| + \alpha \cdot \left\| y_m(x) - y_m(x') \right\|_X +$$
$$+ \beta \cdot \left\| z_m(x) - z_m(x') \right\|_X \le (\gamma + \alpha\delta + \beta M) \cdot |x - x'| = L_0 \cdot |x - x'|$$

for any $x, x' \in [a,b]$ and $m \in \mathbb{N}$ with

$$L_0 = \gamma + \alpha \left(\frac{\|c\|_X + \|d\|_X}{b-a} + \frac{M(b-a)^2}{2} \right) + \beta M.$$

On the other hand,

$$\left\| H_{m,i}(s) - H_{m,i}(s') \right\|_X =$$
$$= \left\| G(x_i, s) \cdot f(s, y_m(s), z_m(s)) - G(x_i, s') \cdot f(s', y_m(s'), z_m(s')) \right\|_X \le$$
$$\le M \cdot \left| G(x_i, s) - G(x_i, s') \right| + \left| G(x_i, s') \right| \cdot \left\| f(s, y_m(s), z_m(s)) - f(s', y_m(s'), z_m(s')) \right\|_X \le$$
$$\le [M + \frac{(b-a)}{4} \cdot L_0] \cdot |s - s'| = L_1 \cdot |s - s'|$$

and

$$\left\| K_{m,i}(s) - K_{m,i}(s') \right\|_X =$$
$$= \left\| \frac{\partial G}{\partial x}(x_i, s) \cdot f(s, y_m(s), z_m(s)) - \frac{\partial G}{\partial x}(x_i, s') \cdot f(s', y_m(s'), z_m(s')) \right\|_X \le$$
$$\le M \cdot \left| \frac{\partial G}{\partial x}(x_i, s) - \frac{\partial G}{\partial x}(x_i, s') \right| + \left| \frac{\partial G}{\partial x}(x_i, s') \right| \cdot \left\| f(s, y_m(s), z_m(s)) - f(s', y_m(s'), z_m(s')) \right\|_X$$
$$\le \left(\frac{M}{b-a} + L_0 \right) \cdot |s - s'| = L_2 \cdot |s - s'|$$

for any $s, s' \in [a,b]$ and $m \in \mathbb{N}$ with

$$L_1 = M + \frac{(b-a)}{4} \cdot \left[\gamma + \alpha \left(\frac{\|c\|_X + \|d\|_X}{b-a} + \frac{M(b-a)^2}{2} \right) + \beta M \right]$$

$$L_2 = \frac{M}{b-a} + \gamma + \alpha \left(\frac{\|c\|_X + \|d\|_X}{b-a} + \frac{M(b-a)^2}{2} \right) + \beta M.$$

In this way, was obtained:

Theorem 33 *(see [58]) Under the conditions (i)-(iv), the functions F_m, $m \in \mathbb{N}$, are uniformly Lipschitz with the Lipschitz constant*

$$L_0 = \gamma + \alpha \left(\frac{\|c\|_X + \|d\|_X}{b-a} + \frac{M(b-a)^2}{2} \right) + \beta M$$

and the functions $H_{m,i}$, $K_{m,i}$, $m \in \mathbb{N}$, $i = \overline{0,n}$ are Lipschitzian with the same Lipschitz constant (uniformly Lipschitz)

$$L_1 = M + \frac{(b-a)}{4} \cdot \left[\gamma + \alpha \left(\frac{\|c\|_X + \|d\|_X}{b-a} + \frac{M(b-a)^2}{2} \right) + \beta M \right]$$

and

$$L_2 = \frac{M}{b-a} + \gamma + \alpha \left(\frac{\|c\|_X + \|d\|_X}{b-a} + \frac{M(b-a)^2}{2} \right) + \beta M,$$

respectively.

The approximation algorithm

In order to compute the Bochner integrals from (1.83) and (1.84) we apply the trapezoidal quadrature rule on Banach spaces (see [54] and [96]):

$$\int_a^b F(x)\,dx = \frac{(b-a)}{2n} \cdot \sum_{i=1}^{n} \left[F\left(a + \frac{i \cdot (b-a)}{n}\right) + F\left(a + \frac{(i-1)\cdot(b-a)}{n}\right) \right] + R_n(F) \tag{1.85}$$

with the remainder estimation

$$\|R_n(F)\|_X \le \begin{cases} \frac{L(b-a)^2}{4n}, & \text{if } F \in Lip([a,b],X), \quad \text{(see [54])} \\ \frac{(b-a)^2}{4n} \cdot \|F'\|_C, & \text{if } F \in C^1([a,b],X), \quad \text{(see [96])} \end{cases} \tag{1.86}$$

Applying the quadrature rule (1.85)-(1.86) to the integrals from (1.83) and (1.84) on the uniform partition we obtain the following numerical method:

$$y_0(x_i) = \frac{x_i - a}{b-a} \cdot d + \frac{b - x_i}{b-a} \cdot c, \qquad z_0(x_i) = \frac{d-c}{b-a}, \quad i = \overline{0,n}$$

$$y_m(x_0) = c, \qquad y_m(x_n) = d, \quad m \in \mathbb{N}^*$$

$$y_m(x_i) = \frac{x_i - a}{b-a} \cdot d + \frac{b - x_i}{b-a} \cdot c - \int_a^b H_{m-1,i}(s)\,ds = \frac{x_i - a}{b-a} \cdot d + \frac{b - x_i}{b-a} \cdot c -$$

$$-\frac{(b-a)}{2n} \cdot \sum_{j=1}^{n} \left[H_{m-1,i}(x_j) + H_{m-1,i}(x_{j-1}) \right] + R_{m,i}, \quad i = \overline{1,n-1}, \; m \in \mathbb{N}^*,$$

$$z_m(x_i) = \frac{d-c}{b-a} - \int_a^b K_{m-1,i}(s)\,ds = \frac{d-c}{b-a} -$$

$$-\frac{(b-a)}{2n} \cdot \sum_{j=1}^{n} \left[K_{m-1,i}(x_j) + K_{m-1,i}(x_{j-1}) \right] + \omega_{m,i}, \quad i = \overline{0,n}, \; m \in \mathbb{N}^*$$

with

$$\|R_{m,i}\|_X \leq \frac{(b-a)^2 \cdot L_1}{4n}, \qquad \|\omega_{m,i}\|_X \leq \frac{(b-a)^2 \cdot L_2}{4n}, \quad i=\overline{0,n},\ m\in\mathbb{N}^*.$$

These lead to the following algorithm:

$$y_0(x_i) = \frac{x_i-a}{b-a}\cdot d + \frac{b-x_i}{b-a}\cdot c, \qquad z_0(x_i) = \frac{d-c}{b-a}, \quad i=\overline{0,n} \tag{1.87}$$

$$y_m(x_0) = c, \qquad y_m(x_n) = d, \quad m\in\mathbb{N}^* \tag{1.88}$$

$$y_1(x_i) = \frac{x_i-a}{b-a}\cdot d + \frac{b-x_i}{b-a}\cdot c-$$

$$-\frac{(b-a)}{2n}\cdot\sum_{j=1}^{n}\left[G(x_i,x_j)\cdot f(x_j,y_0(x_j),z_0(x_j)) + G(x_i,x_{j-1})\cdot f(x_{j-1},y_0(x_{j-1}),z_0(x_{j-1}))\right]+$$

$$+R_{1,i} = \overline{y_1(x_i)} + R_{1,i}, \quad i=\overline{1,n-1} \tag{1.89}$$

$$z_1(x_i) = \frac{d-c}{b-a} - \frac{(b-a)}{2n}\cdot\sum_{j=1}^{n}[\frac{\partial G}{\partial x}(x_i,x_j)\cdot f(x_j,y_0(x_j),z_0(x_j))+$$

$$+\frac{\partial G}{\partial x}(x_i,x_{j-1})\cdot f(x_{j-1},y_0(x_{j-1}),z_0(x_{j-1}))] + \omega_{1,i} = \overline{z_1(x_i)} + \omega_{1,i}, \quad i=\overline{0,n} \tag{1.90}$$

$$y_2(x_i) = \frac{x_i-a}{b-a}\cdot d + \frac{b-x_i}{b-a}\cdot c-$$

$$-\frac{(b-a)}{2n}\cdot\sum_{j=1}^{n}[G(x_i,x_j)\cdot f\left(x_j,\overline{y_1(x_j)}+R_{1,j},\overline{z_1(x_j)}+\omega_{1,j}\right)+$$

$$+G(x_i,x_{j-1})\cdot f\left(x_{j-1},\overline{y_1(x_{j-1})}+R_{1,j-1},\overline{z_1(x_{j-1})}+\omega_{1,j-1}\right)]+R_{2,i} = \frac{x_i-a}{b-a}\cdot d+\frac{b-x_i}{b-a}\cdot c-$$

$$-\sum_{j=1}^{n}[G(x_i,x_j)\cdot f\left(x_j,\overline{y_1(x_j)},\overline{z_1(x_j)}\right) + G(x_i,x_{j-1})\cdot f\left(x_{j-1},\overline{y_1(x_{j-1})},\overline{z_1(x_{j-1})}\right)+$$

$$+\overline{R_{2,i}} = \overline{y_2(x_i)} + \overline{R_{2,i}}, \quad i=\overline{1,n-1} \tag{1.91}$$

$$z_2(x_i) = \frac{d-c}{b-a} - \frac{(b-a)}{2n}\cdot\sum_{j=1}^{n}[\frac{\partial G}{\partial x}(x_i,x_j)\cdot f\left(x_j,\overline{y_1(x_j)}+R_{1,j},\overline{z_1(x_j)}+\omega_{1,j}\right)+$$

$$+\frac{\partial G}{\partial x}(x_i,x_{j-1})\cdot f\left(x_{j-1},\overline{y_1(x_{j-1})}+R_{1,j-1},\overline{z_1(x_{j-1})}+\omega_{1,j-1}\right)] + \omega_{2,i} = \frac{d-c}{b-a}-$$

$$-\frac{(b-a)}{2n}\cdot\sum_{j=1}^{n}[\frac{\partial G}{\partial x}(x_i,x_j)\cdot f\left(x_j,\overline{y_1(x_j)},\overline{z_1(x_j)}\right)+\frac{\partial G}{\partial x}(x_i,x_{j-1})\cdot f\left(x_{j-1},\overline{y_1(x_{j-1})},\overline{z_1(x_{j-1})}\right)]+$$

$$+\overline{\omega_{2,i}} = \overline{z_2(x_i)} + \overline{\omega_{2,i}}, \quad i=\overline{0,n} \tag{1.92}$$

and by induction for $m\geq 3$,

$$y_m(x_i) =$$

$$= \frac{x_i-a}{b-a}\cdot d+\frac{b-x_i}{b-a}\cdot c-\frac{(b-a)}{2n}\cdot\sum_{j=1}^{n}[G(x_i,x_j)\cdot f\left(x_j,\overline{y_{m-1}(x_j)}+\overline{R_{m-1,j}},\overline{z_{m-1}(x_j)}+\overline{\omega_{m-1,j}}\right)+$$

$$+G(x_i,x_{j-1})\cdot f\left(x_{j-1},\overline{y_{m-1}(x_{j-1})}+\overline{R_{m-1,j-1}},\overline{z_{m-1}(x_{j-1})}+\overline{\omega_{m-1,j-1}}\right)]+R_{m,i} = \frac{x_i-a}{b-a}\cdot d+$$

$$+\frac{b-x_i}{b-a}\cdot c-\frac{(b-a)}{2n}\cdot\sum_{j=1}^{n}[G(x_i,x_j)\cdot f\left(x_j,\overline{y_{m-1}(x_j)},\overline{z_{m-1}(x_j)}\right)+G(x_i,x_{j-1})\cdot$$

$$\cdot f\left(x_{j-1},\overline{y_{m-1}(x_{j-1})},\overline{z_{m-1}(x_{j-1})}\right)]+\overline{R_{m,i}}=\overline{y_m(x_i)}+\overline{R_{m,i}},\quad i=\overline{1,n-1}\tag{1.93}$$

$$z_m(x_i)=\frac{d-c}{b-a}-\frac{(b-a)}{2n}\cdot\sum_{j=1}^{n}[\frac{\partial G}{\partial x}(x_i,x_j)\cdot f\left(x_j,\overline{y_{m-1}(x_j)}+\overline{R_{m-1,j}},\overline{z_{m-1}(x_j)}+\overline{\omega_{m-1,j}}\right)+$$

$$+\frac{\partial G}{\partial x}(x_i,x_{j-1})\cdot f\left(x_{j-1},\overline{y_{m-1}(x_{j-1})}+\overline{R_{m-1,j-1}},\overline{z_{m-1}(x_{j-1})}+\overline{\omega_{m-1,j-1}}\right)]+\omega_{m,i}=$$

$$=\frac{d-c}{b-a}-\frac{(b-a)}{2n}\cdot\sum_{j=1}^{n}[\frac{\partial G}{\partial x}(x_i,x_j)\cdot f\left(x_j,\overline{y_{m-1}(x_j)},\overline{z_{m-1}(x_j)}\right)+$$

$$+\frac{\partial G}{\partial x}(x_i,x_{j-1})\cdot f\left(x_{j-1},\overline{y_{m-1}(x_{j-1})},\overline{z_{m-1}(x_{j-1})}\right)]+\overline{\omega_{m,i}}=\overline{z_m(x_i)}+\overline{\omega_{m,i}},\quad i=\overline{0,n}.\tag{1.94}$$

The effective computed values are $\overline{y_m(x_i)}$, $\overline{z_m(x_i)}$, $i=\overline{0,n}$ approximating on the knots of the uniform partition the solution of the system (1.75).

The convergence analysis

Theorem 34 *(see [58]) Under the conditions (i)-(iv), if* $\frac{\alpha}{4}(b-a)^2+\beta(b-a)<1$, *then the effective computed values* $\overline{y_m(x_i)}$, $\overline{z_m(x_i)}$, $i=\overline{0,n}$, $m\in\mathbb{N}^*$ *approximate the solution of the system (1.75) on the knots of the partition with the apriori error estimate:*

$$\begin{pmatrix}\left\|y^*(t_i)-\overline{y_m(x_i)}\right\|_X\\ \left\|(y^*)'(t_i)-\overline{z_m(x_i)}\right\|_X\end{pmatrix}\le\frac{\left[\frac{\alpha}{8}(b-a)^2+\frac{\beta}{2}(b-a)\right]^{m-1}}{1-\left[\frac{\alpha}{8}(b-a)^2+\frac{\beta}{2}(b-a)\right]}\cdot\begin{pmatrix}\frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2\\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a)\end{pmatrix}\cdot$$

$$\cdot\begin{pmatrix}(b-a)^2\cdot\frac{M_0}{8}\\ (b-a)\cdot\frac{M_0}{2}\end{pmatrix}+\begin{pmatrix}1-\frac{\alpha(b-a)^2}{4} & -\frac{\beta(b-a)^2}{4}\\ -\alpha(b-a) & 1-\beta(b-a)\end{pmatrix}^{-1}\cdot\begin{pmatrix}\frac{(b-a)^2\cdot L_1}{4n}\\ \frac{(b-a)^2\cdot L_2}{4n}\end{pmatrix}\tag{1.95}$$

for any $i=\overline{0,n}$ *and* $m\in\mathbb{N}^*$.

Sketch of proof: Firstly, we see that

$$\begin{pmatrix}\left\|y^*(t_i)-\overline{y_m(x_i)}\right\|_X\\ \left\|(y^*)'(t_i)-\overline{z_m(x_i)}\right\|_X\end{pmatrix}\le\begin{pmatrix}\|y^*(t_i)-y_m(x_i)\|_X\\ \|(y^*)'(t_i)-z_m(x_i)\|_X\end{pmatrix}+\begin{pmatrix}\left\|y_m(x_i)-\overline{y_m(x_i)}\right\|_X\\ \left\|z_m(x_i)-\overline{z_m(x_i)}\right\|_X\end{pmatrix}\le$$

$$\le\frac{\left[\frac{\alpha}{8}(b-a)^2+\frac{\beta}{2}(b-a)\right]^{m-1}}{1-\left[\frac{\alpha}{8}(b-a)^2+\frac{\beta}{2}(b-a)\right]}\cdot Q\cdot\begin{pmatrix}(b-a)^2\cdot\frac{M_0}{8}\\ (b-a)\cdot\frac{M_0}{2}\end{pmatrix}+\begin{pmatrix}\left\|\overline{R_{m,i}}\right\|_X\\ \left\|\overline{\omega_{m,i}}\right\|_X\end{pmatrix},\quad i=\overline{0,n},\ m\in\mathbb{N}^*.$$

In order to estimate the remainders $\overline{R_{m,i}}$, $\overline{\omega_{m,i}}$, $i=\overline{0,n}$, $m\in\mathbb{N}^*$ we obtain from (1.93), (1.94) the recurrences:

$$\left\|\overline{R_{m,i}}\right\|_X\le\|R_{m,i}\|_X+\frac{(b-a)}{2n}\cdot\sum_{j=1}^{n}[G(x_i,x_j)\cdot\left(\alpha\cdot\left\|\overline{R_{m-1,j}}\right\|_X+\beta\cdot\left\|\overline{\omega_{m-1,j}}\right\|_X\right)+$$

$$+G(x_i,x_{j-1})\cdot\left(\alpha\cdot\left\|\overline{R_{m-1,j-1}}\right\|_X+\beta\cdot\left\|\overline{\omega_{m-1,j-1}}\right\|_X\right)],\quad i=\overline{0,n},\ m\in\mathbb{N}^*,$$

$$\|\overline{\omega_{m,i}}\|_X \leq \|\omega_{m,i}\|_X + \frac{(b-a)}{2n} \cdot \sum_{j=1}^{n} [\frac{\partial G}{\partial x}(x_i, x_j) \cdot (\alpha \cdot \|\overline{R_{m-1,j}}\|_X + \beta \cdot \|\overline{\omega_{m-1,j}}\|_X) +$$

$$+\frac{\partial G}{\partial x}(x_i, x_{j-1}) \cdot (\alpha \cdot \|\overline{R_{m-1,j-1}}\|_X + \beta \cdot \|\overline{\omega_{m-1,j-1}}\|_X)], \quad i = \overline{0,n}, \ m \in \mathbb{N}^*.$$

that lead to:

$$\begin{pmatrix} \|\overline{R_{m,i}}\|_X \\ \|\overline{\omega_{m,i}}\|_X \end{pmatrix} \leq \begin{pmatrix} \frac{(b-a)^2 \cdot L_1}{4n} \\ \frac{(b-a)^2 \cdot L_2}{4n} \end{pmatrix} + \begin{pmatrix} \frac{\alpha(b-a)^2}{4} & \frac{\beta(b-a)^2}{4} \\ \alpha(b-a) & \beta(b-a) \end{pmatrix} \cdot \begin{pmatrix} \|\overline{R_{m-1}}\|_X \\ \|\overline{\omega_{m-1}}\|_X \end{pmatrix}, \quad i = \overline{0,n}$$

where

$$\|\overline{R_{m-1}}\|_X = \max\{\|\overline{R_{m-1,j}}\|_X : j = \overline{0,n}\}, \qquad \|\overline{\omega_{m-1}}\|_X = \max\{\|\overline{\omega_{m-1,j}}\|_X : j = \overline{0,n}\}.$$

By induction for $m \geq 3$ we obtain,

$$\begin{pmatrix} \|\overline{R_{m,i}}\|_X \\ \|\overline{\omega_{m,i}}\|_X \end{pmatrix} \leq$$

$$\leq \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \frac{\alpha(b-a)^2}{4} & \frac{\beta(b-a)^2}{4} \\ \alpha(b-a) & \beta(b-a) \end{pmatrix} + \ldots + \begin{pmatrix} \frac{\alpha(b-a)^2}{4} & \frac{\beta(b-a)^2}{4} \\ \alpha(b-a) & \beta(b-a) \end{pmatrix}^{m-1} + \ldots\right]$$

$$\cdot \begin{pmatrix} \frac{(b-a)^2 \cdot L_1}{4n} \\ \frac{(b-a)^2 \cdot L_2}{4n} \end{pmatrix} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{\alpha(b-a)^2}{4} & \frac{\beta(b-a)^2}{4} \\ \alpha(b-a) & \beta(b-a) \end{pmatrix}\right]^{-1} \cdot \begin{pmatrix} \frac{(b-a)^2 \cdot L_1}{4n} \\ \frac{(b-a)^2 \cdot L_2}{4n} \end{pmatrix}, \quad i = \overline{0,n}$$

because the eigenvalues of the matrix $Q_2' = \begin{pmatrix} \frac{\alpha(b-a)^2}{4} & \frac{\beta(b-a)^2}{4} \\ \alpha(b-a) & \beta(b-a) \end{pmatrix}$ are $\lambda_1 = 0$ and $\lambda_2 = \frac{\alpha}{4}(b-a)^2 + \beta(b-a) < 1$.

From the estimates (1.80) and (1.95) we infer that

$$\lim_{m\to\infty, n\to\infty} \begin{pmatrix} \left\|y^*(t_i) - \overline{y_m(x_i)}\right\|_X \\ \left\|(y^*)'(t_i) - \overline{z_m(x_i)}\right\|_X \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

that is the convergence of the algorithm to the solution of the system (1.75).

Remark 35 *We see that the apriori (1.80) and a posteriori (1.81) error estimates can offer a practical stopping criterion of the algorithm. This can be stated as follows: for given $\varepsilon' > 0$ and given $n \in \mathbb{N}^*$ (previously chosen) we determine the first natural number $m \in \mathbb{N}^*$ for which*

$$\left\|\overline{y_m(x_i)} - \overline{y_{m-1}(x_i)}\right\|_X < \varepsilon', \text{ for all } i = \overline{1, n-1}$$

and

$$\left\|\overline{z_m(x_i)} - \overline{z_{m-1}(x_i)}\right\|_X < \varepsilon', \text{ for all } i = \overline{0,n}$$

and we stop to this m retaining the approximations $\overline{y_m(x_i)}$, $\overline{z_m(x_i)}$, $i = \overline{0,n}$ of the solution. The demonstration of this criterion is the following. We denote

$$\Omega = \begin{pmatrix} 1 - \frac{\alpha(b-a)^2}{4} & -\frac{\beta(b-a)^2}{4} \\ -\alpha(b-a) & 1 - \beta(b-a) \end{pmatrix}^{-1} \cdot \begin{pmatrix} \frac{(b-a)^2 \cdot L_1}{4n} \\ \frac{(b-a)^2 \cdot L_2}{4n} \end{pmatrix}$$

and we have

$$\begin{pmatrix} \left\| y^*(t_i) - \overline{y_m(x_i)} \right\|_X \\ \left\| (y^*)'(t_i) - \overline{z_m(x_i)} \right\|_X \end{pmatrix} \le \begin{pmatrix} \| y^*(t_i) - y_m(x_i) \|_X \\ \left\| (y^*)'(t_i) - z_m(x_i) \right\|_X \end{pmatrix} + \begin{pmatrix} \left\| y_m(x_i) - \overline{y_m(x_i)} \right\|_X \\ \left\| z_m(x_i) - \overline{z_m(x_i)} \right\|_X \end{pmatrix} \le$$

$$\le \frac{\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)}{1 - \left[\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)\right]} \begin{pmatrix} \frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2 \\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a) \end{pmatrix} \begin{pmatrix} \| y_m(x_i) - y_{m-1}(x_i) \|_X \\ \| z_m(x_i) - z_{m-1}(x_i) \|_X \end{pmatrix} +$$

$$+ \begin{pmatrix} \| \overline{R_{m,i}} \|_X \\ \| \overline{\omega_{m,i}} \|_X \end{pmatrix}$$

and

$$\begin{pmatrix} \| y_m(x_i) - y_{m-1}(x_i) \|_X \\ \| z_m(x_i) - z_{m-1}(x_i) \|_X \end{pmatrix} \le \begin{pmatrix} \left\| y_m(x_i) - \overline{y_m(x_i)} \right\|_X \\ \left\| z_m(x_i) - \overline{z_m(x_i)} \right\|_X \end{pmatrix} + \begin{pmatrix} \left\| \overline{y_m(x_i)} - \overline{y_{m-1}(x_i)} \right\|_X \\ \left\| \overline{z_m(x_i)} - \overline{z_{m-1}(x_i)} \right\|_X \end{pmatrix} +$$

$$\begin{pmatrix} \left\| \overline{y_{m-1}(x_i)} - y_{m-1}(x_i) \right\|_X \\ \left\| \overline{z_{m-1}(x_i)} - z_{m-1}(x_i) \right\|_X \end{pmatrix} \le \begin{pmatrix} \| \overline{R_{m,i}} \|_X \\ \| \overline{\omega_{m,i}} \|_X \end{pmatrix} + \begin{pmatrix} \| \overline{R_{m-1,i}} \|_X \\ \| \overline{\omega_{m-1,i}} \|_X \end{pmatrix} +$$

$$+ \begin{pmatrix} \left\| \overline{y_m(x_i)} - \overline{y_{m-1}(x_i)} \right\|_X \\ \left\| \overline{z_m(x_i)} - \overline{z_{m-1}(x_i)} \right\|_X \end{pmatrix}.$$

So,

$$\begin{pmatrix} \left\| y^*(t_i) - \overline{y_m(x_i)} \right\|_X \\ \left\| (y^*)'(t_i) - \overline{z_m(x_i)} \right\|_X \end{pmatrix} \le$$

$$\le \begin{pmatrix} \| \overline{R_{m,i}} \|_X \\ \| \overline{\omega_{m,i}} \|_X \end{pmatrix} + \frac{\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)}{1 - \left[\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)\right]} \begin{pmatrix} \frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2 \\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a) \end{pmatrix}$$

$$\cdot \begin{pmatrix} \left\| \overline{y_m(x_i)} - \overline{y_{m-1}(x_i)} \right\|_X \\ \left\| \overline{z_m(x_i)} - \overline{z_{m-1}(x_i)} \right\|_X \end{pmatrix} + \frac{\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)}{1 - \left[\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)\right]} \cdot \begin{pmatrix} \frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2 \\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a) \end{pmatrix} \cdot$$

$$\cdot \left[\begin{pmatrix} \| \overline{R_{m,i}} \|_X \\ \| \overline{\omega_{m,i}} \|_X \end{pmatrix} + \begin{pmatrix} \| \overline{R_{m-1,i}} \|_X \\ \| \overline{\omega_{m-1,i}} \|_X \end{pmatrix} \right].$$

Then

$$\begin{pmatrix} \left\| y^*(t_i) - \overline{y_m(x_i)} \right\|_X \\ \left\| (y^*)'(t_i) - \overline{z_m(x_i)} \right\|_X \end{pmatrix} \le \frac{1 + \frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)}{1 - \left[\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)\right]} \cdot \begin{pmatrix} \frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2 \\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a) \end{pmatrix} \cdot \Omega +$$

$$+ \frac{\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)}{1 - \left[\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)\right]} \cdot \begin{pmatrix} \frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2 \\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a) \end{pmatrix} \cdot \begin{pmatrix} \left\| \overline{y_m(x_i)} - \overline{y_{m-1}(x_i)} \right\|_X \\ \left\| \overline{z_m(x_i)} - \overline{z_{m-1}(x_i)} \right\|_X \end{pmatrix}.$$

For given $\varepsilon > 0$ *we require*

$$\frac{1 + \frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)}{1 - \left[\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)\right]} \cdot \begin{pmatrix} \frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2 \\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a) \end{pmatrix} \cdot \Omega < \begin{pmatrix} \frac{\varepsilon}{2} \\ \frac{\varepsilon}{2} \end{pmatrix} \tag{1.96}$$

and

$$\frac{\frac{\alpha}{8}(b-a)^2+\frac{\beta}{2}(b-a)}{1-\left[\frac{\alpha}{8}(b-a)^2+\frac{\beta}{2}(b-a)\right]}\cdot\begin{pmatrix}\frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2\\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a)\end{pmatrix}\cdot\begin{pmatrix}\left\|\overline{y_m(x_i)}-\overline{y_{m-1}(x_i)}\right\|_X\\ \left\|\overline{z_m(x_i)}-\overline{z_{m-1}(x_i)}\right\|_X\end{pmatrix}<\begin{pmatrix}\frac{\varepsilon}{2}\\ \frac{\varepsilon}{2}\end{pmatrix}. \tag{1.97}$$

From inequality (1.96) we determine the smallest natural number n for which this inequality holds. Afterwards, we find the smallest natural number m for which the inequality (1.97) holds.

Remark 36 *Comparing the hypotheses in Theorems 30 and 34 we see that in Theorem 34 only the supplementary Lipschitz condition (iv) appears and the inequality* $\frac{\alpha}{8}(b-a)^2+\frac{\beta}{2}(b-a)<1$ *becomes* $\frac{\alpha}{4}(b-a)^2+\beta(b-a)<1$ *(the same as in [39], page 243).*

In [58], the notion of stability of the algorithm with respect by the boundary values is introduced as follows.

It is considered the two-point boundary value problem with the same second order differential equation, but with modified boundary values:

$$\begin{cases} y''(x)=f(x,y(x),y'(x)), & x\in[a,b]\\ y(a)=c', \quad y(b)=d'. \end{cases} \tag{1.98}$$

such that $\|c-c'\|_X<\epsilon$ and $\|d-d'\|_X<\varepsilon$.

For the boundary value problem (1.98) the sequence of successive approximations on the same knots is:

$$v_0(x_i)=\frac{x_i-a}{b-a}\cdot d'+\frac{b-x_i}{b-a}\cdot c',\qquad w_0(x_i)=\frac{d'-c'}{b-a},\quad i=\overline{0,n}$$

$$v_m(x_0)=c',\qquad v_m(x_n)=d',\quad m\in\mathbb{N}^*$$

$$v_m(x_i)=\frac{x_i-a}{b-a}\cdot d+\frac{b-x_i}{b-a}\cdot c-\int_a^b G(x_i,s)\cdot f(s,v_{m-1}(s),w_{m-1}(s))\,ds,\quad i=\overline{1,n-1},\ m\in\mathbb{N}^*$$

$$w_m(x_i)=\frac{d-c}{b-a}-\int_a^b\frac{\partial G}{\partial x}(x_i,s)\cdot f(s,v_{m-1}(s),w_{m-1}(s))\,ds,\quad i=\overline{0,n},\ m\in\mathbb{N}^*$$

and the effective computed values are

$$v_0(x_i)=\frac{x_i-a}{b-a}\cdot d+\frac{b-x_i}{b-a}\cdot c,\qquad w_0(x_i)=\frac{d-c}{b-a},\quad i=\overline{0,n}$$

$$v_m(x_0)=c',\qquad v_m(x_n)=d',\quad m\in\mathbb{N}^*,$$

and $\overline{v_m(x_i)}$, $i=\overline{1,n-1}$, $\overline{w_m(x_i)}$, $i=\overline{0,n}$, $m\in\mathbb{N}^*$ with $v_m(x_i)=\overline{v_m(x_i)}+\overline{R'_{m,i}}$ and $w_m(x_i)=\overline{w_m(x_i)}+\overline{\omega'_{m,i}}$. We see that

$$\|y_0(x)-v_0(x)\|_X\le\left\|d-d'\right\|_X+\left\|c-c'\right\|_X<\epsilon+\varepsilon,\ \text{for all } x\in[a,b]$$

and

$$\|z_0(x)-w_0(x)\|_X\le\frac{1}{b-a}\cdot\left(\left\|d-d'\right\|_X+\left\|c-c'\right\|_X\right)<\frac{\epsilon+\varepsilon}{b-a},\ \text{for all } x\in[a,b].$$

Definition 37 *(see [58]) We say that the proposed algorithm is stable with respect to the boundary values if there exist* $p \in \mathbb{N}^*$ *and the matrices* $K_1, K_2, K_3 \in \mathbb{R}^2_+$ *such that*

$$\left(\begin{array}{c} \left\| \overline{y_m(x_i)} - \overline{v_m(x_i)} \right\|_X \\ \left\| \overline{z_m(x_i)} - \overline{w_m(x_i)} \right\|_X \end{array} \right) \leq K_1 \cdot \epsilon + K_2 \cdot \varepsilon + K_3 \cdot h^p$$

for all $i = \overline{0,n}$, $m \in \mathbb{N}^*$, *where* $h = \frac{b-a}{n}$.

Theorem 38 *(see [58]) Under the conditions of Theorem 34 the proposed algorithm of successive approximations for the boundary value problem (1.73) is stable with respect to the boundary values.*

In the proof of this theorem, by induction it obtains

$$\left(\begin{array}{c} \left\| \overline{y_m(x_i)} - \overline{v_m(x_i)} \right\|_X \\ \left\| \overline{z_m(x_i)} - \overline{w_m(x_i)} \right\|_X \end{array} \right) \leq$$

$$\leq \left[\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) - \left(\begin{array}{cc} \frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2 \\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a) \end{array} \right) \right]^{-1} \cdot \left(\begin{array}{c} \epsilon + \varepsilon \\ \frac{\epsilon+\varepsilon}{b-a} \end{array} \right) + \left(\frac{b-a}{2} \right) \cdot$$

$$\cdot \left(\begin{array}{cc} 1 - \frac{\alpha(b-a)^2}{4} & -\frac{\beta(b-a)^2}{4} \\ -\alpha(b-a) & 1 - \beta(b-a) \end{array} \right)^{-1} \cdot \left(\frac{b-a}{n} \right) \cdot \left(\begin{array}{c} L_1 \\ L_2 \end{array} \right) = K_1 \cdot \epsilon + K_2 \cdot \varepsilon + K_3 \cdot h,$$

$\forall i = \overline{0,n}, m \in \mathbb{N}^*$.

Numerical examples

For the two-point boundary value problem (1.73) considered in Banach spaces, the application of the Perov's fixed point theorem permits to obtain the existence, uniqueness and boundedness of the solution and to construct a convergent and stable approximation method for this solution. The convergence of the proposed method can be proved using only Lipschitz conditions, without smoothness or boundedness conditions. These extend the applicability of the method. In order to illustrate the accuracy of the method we choose the case $X = \mathbb{R}$ and consider the following examples.

Example 39 *For the boundary value problem*

$$\left\{ \begin{array}{c} y''(t) = -y(x) \cdot y'(x) + |y(x)|^3 \\ y(0) = 1, \quad y(1) = \frac{1}{2} \end{array} \right. , \quad x \in [0,1]$$

the kernel function $f(s,u,v) = -u \cdot v + |u|^3$ *is nonlinear. The exact solution is* $y^*(x) = \frac{1}{x+1}$ *and applying the above presented algorithm, the error approximation results are in Table 1. For* $n = 10$, $n = 100$, $n = 1000$ *the number of iterations is* $m = 33$, $m = 34$ *and* $m = 35$, *respectively.*

Example 40 *The boundary value problem*

$$\left\{ \begin{array}{c} y''(t) = y(x) + y'(x) \\ y(0) = 1, \quad y(0.5) = \sqrt{e} \end{array} \right. , \quad x \in [0, 0.5]$$

has the exact solution $y^*(x) = e^x$ *and the error approximation results are in Table 1. For* $n = 10, 100, 1000$ *we get the same number of iterations* $m = 8$.

In Table 1, in the second and in the fourth column we present the order of effective errors

$$er = \max\{\left\| y^* (t_i) - \overline{y_m (x_i)} \right\|_X : i = \overline{1, n-1}\}$$

for the above presented examples, corresponding to different stepsize $h = \frac{b-a}{n}$.

h (first ex.)	er, first example	h (second ex.)	er, second example
0.1	2.51×10^{-4}	0.05	8.276×10^{-5}
0.01	3.057×10^{-6}	0.005	8.648×10^{-7}
0.001	3.064×10^{-8}	0.0005	8.681×10^{-9}

Table 1

We see that for stepsize $h = 0.1$ the order of effective error is $O\left(10^{-4}\right)$, for stepsize $h = 0.01$ this order is $O\left(10^{-6}\right)$ and for stepsize $h = 0.001$ this order is $O\left(10^{-8}\right)$. These confirm the convergence of the method.

The results presented in this section extend, enrich and generalize the results obtained in [39], pages 243-248, and in [201], providing in addition an effective method to approximate the solution of the two-point boundary value problem for second order neutral differential equations and generalizing the existence and uniqueness of the solution from the framework of real axis to those of Banach spaces. Moreover, in [50], the smooth dependence of the solution by parameters is studied. The smooth dependence of the same solution by the boundary values a and b is investigated in [53].

1.6 Neutral type initial value problems with constant delay

1.6.1 First order initial value problems of neutral type with constant delay

The results presented in this section are obtained in [44]. For given $\tau > 0$ and $T > 0$, consider the following initial value problem

$$\begin{cases} x'(t) = f(t, x(t), x'(t-\tau)), & t \in [0, T] \\ x(t) = \varphi(t), & t \in [-\tau, 0] \end{cases} \tag{1.99}$$

where, $\varphi : [-\tau, 0] \to \mathbb{R}$, $\varphi \in C^1[-\tau, 0]$. Integrating on $[0, T]$, we get

$$x(t) = \varphi(0) + \int_0^t f(s, x(s), x'(s-\tau))ds, \quad t \in [0, T].$$

In order to obtain the existence and uniqueness of the solution of the initial value problem (1.99), the following system is approached,

$$\begin{cases} x(t) = \varphi(0) + \int_0^t f(s, x(s), y(s-\tau))ds, & t \in [0, T] \\ y(t) = f(t, x(t), y(t-\tau)), & t \in [0, T], \end{cases} \tag{1.100}$$

$$\begin{cases} x(t) = \varphi(t), & t \in [-\tau, 0] \\ y(t) = \varphi'(t), & t \in [-\tau, 0]. \end{cases}$$

On the functional space

$$X = C[-\tau, T] \times C[-\tau, T]$$

it is defined the Bielecki's metric, $d_B : X \times X \to \mathbb{R}^2_+$,

$$d_B((x,y),(u,v)) = (\|x-u\|_B, \|y-v\|_B),$$

where,

$$\|u\|_B = \max\{|u(t)| \cdot e^{-\theta(t+\tau)} : t \in [-\tau, T]\}$$

with $\theta > 0$ suitable chosen. The pair (X, d_B) is a generalized complete metric space. The following operator, $A = (A_1, A_2) : X \to X$,

$$A((x,y))(t) = (\varphi(t), \varphi'(t)), \quad t \in [-\tau, 0],$$

$$A((x,y))(t) = \begin{pmatrix} \varphi(0) + \int_0^t f(s, x(s), y(s-\tau))ds \\ f(t, x(t), y(t-\tau)) \end{pmatrix}, t \in [0, T].$$

is defined and are imposed the following conditions:
(i) $f \in C([-\tau, T] \times \mathbb{R} \times \mathbb{R})$ and there exists $M > 0$ such that

$$|f(t,u,v)| \le M, \quad \forall (t,u,v) \in [-\tau, T] \times \mathbb{R} \times \mathbb{R},$$

(ii) $\exists \alpha > 0$, $\beta > 0$ such that for any x_1, x_2, y_1, $y_2 \in \mathbb{R}$, we have,

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \le \alpha |x_1 - x_2| + \beta |y_1 - y_2|, \ \forall t \in [-\tau, T]$$

(iii) $\varphi \in C^1[-\tau, 0]$ and $\varphi'(0) = f(0, \varphi(0), \varphi'(-\tau))$.

Applying the fixed point technique, for $t \in [0, T]$ and $(x_1, y_1), (x_2, y_2) \in X$ it obtains,

$$(|A_1((x_1,y_1))(t) - A_1((x_2,y_2))(t)|, \quad |A_2((x_1,y_1))(t) - A_2((x_2,y_2))(t)|) \le$$

$$\le \begin{pmatrix} \int_0^t |f(s, x_1(s), y_1(s-\tau)) - f(s, x_2(s), y_2(s-\tau))| ds \\ |f(t, x_1(t), y_1(t-\tau)) - f(t, x_2(t), y_2(t-\tau))| \end{pmatrix} \le$$

$$\le \begin{pmatrix} \frac{\alpha}{\theta} & \frac{\beta}{\theta} \cdot e^{-\theta\tau} \\ \alpha & \beta \cdot e^{-\theta\tau} \end{pmatrix} \cdot \begin{pmatrix} \|x_1 - x_2\|_B \\ \|y_1 - y_2\|_B \end{pmatrix} e^{\theta(t+\tau)}, \quad \forall t \in [0, T],$$

which leads to

$$d_B(A((x_1,y_1)), A((x_2,y_2))) \le \begin{pmatrix} \frac{\alpha}{\theta} & \frac{\beta}{\theta} \cdot e^{-\theta\tau} \\ \alpha & \beta \cdot e^{-\theta\tau} \end{pmatrix} \cdot d_B((x_1,y_1),(x_2,y_2)). \tag{1.101}$$

Let

$$Q = \begin{pmatrix} \frac{\alpha}{\theta} & \frac{\beta}{\theta} \cdot e^{-\theta\tau} \\ \alpha & \beta \cdot e^{-\theta\tau} \end{pmatrix}.$$

The eigenvalues of Q are $\lambda_1 = 0$ and $\lambda_2 = \frac{\alpha}{\theta} + \beta \cdot e^{-\theta\tau} > 0$. We see that

$$\lambda_2 < 1 \Longleftrightarrow \theta > \alpha + \beta\theta \cdot e^{-\theta\tau}, \qquad \text{for } \theta > 0.$$

Let h be the function

$$h : (0, \infty) \to \mathbb{R}, \qquad h(\theta) = \alpha + \beta\theta \cdot e^{-\theta\tau}, \quad \forall \theta > 0.$$

Since

$$\lim_{\theta\to 0} h(\theta) = \alpha \quad \text{and} \quad \lim_{\theta\to\infty} h(\theta) = \alpha$$

we infer that there exists at least one value $\theta > 0$ such that $h(\theta) < \theta$, and for this value of θ, we have $\lambda_2 < 1$. We can conclude that the operator A is $Q-$contraction. After elementary calculus,

$$(I_2 - Q)^{-1} = [\det(I_2 - Q)]^{-1} \begin{pmatrix} 1-\beta\cdot e^{-\theta\tau} & \frac{\beta}{\theta}\cdot e^{-\theta\tau} \\ \alpha & 1-\frac{\alpha}{\theta} \end{pmatrix} \tag{1.102}$$

and

$$\det(I_2 - Q) = 1 - \beta - \frac{\alpha}{\theta}\cdot e^{-\theta\tau} = 1 - \lambda_2. \tag{1.103}$$

By induction, we obtain,

$$Q^n = (\frac{\alpha}{\theta} + \beta\cdot e^{-\theta\tau})^{n-1}\cdot \begin{pmatrix} \frac{\alpha}{\theta} & \frac{\beta}{\theta}\cdot e^{-\theta\tau} \\ \alpha & \beta\cdot e^{-\theta\tau} \end{pmatrix} = \lambda_2^{n-1}\cdot Q, \quad \forall n\in\mathbb{N}^* \tag{1.104}$$

and then $\lim\limits_{n\to\infty} Q^n = 0$. From the Perov's fixed point theorem we infer that the operator A has unique fixed point $(x^*, y^*) \in X$, which is the solution of the system (1.100). For this solution we have

$$\begin{cases} x^*(t) = \varphi(0) + \int\limits_0^t f(s, x^*(s), y^*(s-\tau))ds, & t\in[0,T] \\ y^*(t) = f(t, x^*(t), y^*(t-\tau)), & t\in[0,T] \end{cases}$$

and if we differentiate the first equality, we obtain

$$(x^*)'(t) = f(t, x^*(t), y^*(t-\tau)), \quad \forall t\in[0,T].$$

We conclude that $x^* \in C^1[0,T]$ and $(x^*)'(t) = y^*(t)$, $\forall t\in[0,T]$. On the other hand, we infer that $x^* \in C^1[-\tau, 0]$ and $x^*(t) = \varphi(t)$, $\forall t\in[-\tau,0]$. Finally, using (iii) it follows that $x^*\in C^1[-\tau, T]$.

Now, we present the way to determine the value of $\theta > 0$ such that the operator A is a $Q-$contraction on the corresponding generalized metric space (X, d_B). But, in order to obtain this value of θ we must to impose some conditions for the value of β. In this sense we have $h'(\theta) > 0$, $\forall\theta\in(0,\frac{1}{\tau})$,$\forall\alpha,\beta,\tau > 0$ and $h'(\theta) < 0$, $\forall\theta\in(\frac{1}{\tau},\infty)$, $\forall\alpha,\beta,\tau>0$. The maximum of h is $h(\frac{1}{\tau}) = \alpha + \frac{\beta}{\tau}\cdot e^{-1}$. If

$$\alpha + \frac{\beta}{\tau}\cdot e^{-1} \le \frac{1}{\tau}$$

then we can choose any $\theta > \frac{1}{\tau}$ to obtain $h(\theta) < \theta$, that is $\lambda_2 < 1$. If

$$\alpha + \frac{\beta}{\tau}\cdot e^{-1} > \frac{1}{\tau}$$

then, since $h'(\theta) < 0$, $\forall\theta\in(\frac{1}{\tau},\infty)$, we will apply the method of successive approximations to solve the equation $h(\theta) = \theta$, and afterwards we will choose a value of θ greater than the solution of this equation. We see that $h'(\frac{1}{\tau}) = 0$, $\lim\limits_{\theta\to\infty} h'(\theta) = 0$ and

$$\left|h'(\theta)\right| < \left|h'(\frac{2}{\tau})\right|, \quad \forall\theta\in(\frac{1}{\tau},\infty),\ \theta\ne\frac{2}{\tau}.$$

If $0 < \beta < e^2$ we have $|h'(\theta)| < 1$, $\forall\theta\in(\frac{1}{\tau},\infty)$ and then the method of successive approximations converges to the unique solution θ^* of the equation $h(\theta) = \theta$. Finally, we will choose a value $\theta > \theta^*$.

In this way it obtains:

Theorem 41 *(see [44]) Suppose that the conditions (ii) and (iii) are satisfied. For any* $\alpha > 0$ *and* $0 < \beta < e^2$ *there exists at least one value of* $\theta > 0$ *such that the operator* A *has an unique fixed point in* (X, d_B)*. Consequently, in the generalized metric space* (X, d_B)*, the system (1.100) has unique solution* (x^*, y^*) *for which,* $(x^*)' = y^*$*.*

From the previous theorem it follows that $x^* \in C^1[-\tau, T]$ is the unique solution of the initial value problem (1.99). Moreover, from this theorem we obtain an error estimation in the approximation of this solution by the terms of the sequence of successive approximations.

Similarly as in section 1.3.1, are obtained Lipschitz properties for the family of functions $F_m : [0, T] \to \mathbb{R}$, $m \in \mathbb{N}$, defined by

$$F_m(t) = f(t, x_m(t), y_m(t - \tau)),$$

and a numerical method to approximate the solution of (1.99) is constructed in [44] by combining the method of successive approximations with a suitable quadrature rule.

1.6.2 Second order initial value problems of neutral type with constant delay

In [55] is approached the following initial value problem:

$$\begin{cases} x''(t) = f(t, x(t), x(g(t)), x'(t), x'(g(t))), \ t \in [a, b] \\ x(t) = \varphi(t), \quad t \in [a_1, a] \end{cases} \tag{1.105}$$

with

$$\varphi \in C^1([a_1, a], X), \ g \in C([a, b], [a_1, b])$$

given, where X is a Banach space, $a_1 \le g(t) \le t, \ \forall t \in [a, b]$ and $f \in C([a, b] \times X \times X \times X \times X, X)$. The initial value problem (1.105) is equivalent in $C^1([a_1, b], X) \cap C^2([a, b], X)$ with the neutral Volterra integro-differential equation:

$$\begin{cases} x(t) = \varphi(a) + \varphi'(a)(t - a) + \int_a^t (t - s) f(s, x(s), x(g(s)), x'(s), x'(g(s))) ds, \quad t \in [a, b] \\ x(t) = \varphi(t), \quad t \in [a_1, a]. \end{cases} \tag{1.106}$$

Differentiating (1.106) with respect by t and denoting $y = x'$, we obtain the equivalent system of Volterra integral equations,

$$\begin{cases} x(t) = \varphi(a) + \varphi'(a)(t - a) + \int_a^t (t - s) f(s, x(s), x(g(s)), y(s), y(g(s))) ds \\ y(t) = \varphi'(a) + \int_a^t f(s, x(s), x(g(s)), y(s), y(g(s))) ds, \quad t \in [a, b] \\ x(t) = \varphi(t), \ y(t) = \varphi'(t), \ t \in [a_1, a]. \end{cases} \tag{1.107}$$

In order to obtain the existence and uniqueness of the solution of (1.107), the following conditions are imposed:

(i) $f \in C([a, b] \times X \times X \times X \times X, X)$, $\varphi \in C^1([a, b], X)$;

(ii) $g \in C([a, b], [a_1, b])$ and $a_1 \le g(t) \le t, \ \forall t \in [a, b]$;

(iii) $\exists \alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ such that

$$\|f(s, z_1, u_1, v_1, w_1) - f(s, z_2, u_2, v_2, w_2)\|_X \le \alpha_1 \|z_1 - z_2\|_X + \alpha_2 \|u_1 - u_2\|_X +$$

$+\beta_1 \|v_1 - v_2\|_X + \beta_2 \|w_1 - w_2\|_X, \forall s \in [a,b], \ \forall z_1, z_2, u_1, u_2, v_1, v_2, w_1, w_2 \in X$

(iv) $\exists M > 0$ such that $\|f(s,z,u,v,w)\|_X \leq M, \ \forall s \in [a,b], \ \forall z,u,v,w \in X.$

On the product functional space

$$Y = C([a_1,b], X) \times C([a_1,b], X)$$

it is defined the metric

$$d : Y \times Y \longrightarrow \mathbb{R}^2$$

by

$$d((u_1,v_1),(u_2,v_2)) = (\|u_1 - u_2\|_B, \|v_1 - v_2\|_B),$$

where,

$$\|u\|_B = \max\left\{\|u(t)\|_X e^{-\theta(t-a_1)} : t \in [a_1,b]\right\}, \quad \theta > 0.$$

On the complete metric space (Y,d) the Perov's fixed point theorem is applied to the operator $A : Y \longrightarrow Y, \ A = (A_1, A_2)$ defined by

$$\begin{cases} A_1(x,y)(t) = \varphi(t), & t \in [a_1,a] \\ A_2(x,y)(t) = \varphi'(t), & t \in [a_1,a] \end{cases}$$

$$A_1(x,y)(t) = \varphi(a) + \varphi'(a)(t-a) + \int_a^t (t-s) f(s, x(s), x(g(s)), y(s), y(g(s)))\, ds, \quad t \in [a,b]$$

$$A_2(x,y)(t) = \varphi'(a) + \int_a^t f(s, x(s), x(g(s)), y(s), y(g(s)))\, ds, \quad t \in [a,b].$$

Using the fixed point technique generated by the Perov's theorem, it obtains the following result:

Theorem 42 *(see [55]) Under the conditions* $(i), (ii), (iii)$, *for suitable chosen* $\theta > 0$, *the operator* A *has an unique fixed point* $(x^*, y^*) \in Y$ *such that*

$$x^* \in C^1([a,b], X), \ y^* = (x^*)'$$

and x^* *is the unique solution of the initial value problem (1.105). Moreover, the sequence of successive approximations given by*

$$x_0(t) = \begin{cases} \varphi(t), & t \in [a_1,a] \\ \varphi(a) + \varphi'(a)(t-a), & t \in [a,b] \end{cases}$$

$$y_0(t) = \begin{cases} \varphi'(t), & t \in [a_1,a] \\ \varphi'(a), & t \in [a,b] \end{cases}$$

$$x_m(t) = \varphi(a) + \varphi'(a)(t-a) +$$

$$+ \int_a^t (t-s) f(s, x_{m-1}(s), x_{m-1}(g(s)), y_{m-1}(s), y_{m-1}(g(s)))\, ds, \quad t \in [a,b]$$

$$y_m(t) = \varphi'(a) + \int_a^t f(s, x_{m-1}(s), x_{m-1}(g(s)), y_{m-1}(s), y_{m-1}(g(s)))\, ds, \quad t \in [a,b]$$

$$(x_m(t), y_m(t)) = (\varphi(t), \varphi'(t)), \quad t \in [a_1, a], \quad m \in \mathbb{N}^*$$

converges in Y *to* (x^*, y^*) *and the following error estimate holds:*

$$d((x_m, y_m), (x^*, y^*)) \leq Q^m (I_n - Q)^{-1} d((x_0, y_0), (x_1, y_1))$$

$\forall m \in \mathbb{N}^*$, *where*

$$Q = \begin{pmatrix} \frac{b-a}{\theta}(\alpha_1 + \alpha_2) & \frac{b-a}{\theta}(\beta_1 + \beta_2) \\ \frac{\alpha_1+\alpha_2}{\theta} & \frac{\beta_1+\beta_2}{\theta} \end{pmatrix}.$$

Applying the same technique as in the section 1.3.1, uniformly Lipschitz properties of the families of functions $F_{m,i}, G_m : [a, b] \longrightarrow X$, $m \in \mathbb{N}$, $i = \overline{n, q}$

$$F_{m,i} = (t_i - s) f(s, x_m(s), x_m(s - \tau), y_m(s), y_m(s - \tau))$$

$$G_m(s) = f(s, x_m(s), x_m(s - \tau), y_m(s), y_m(s - \tau)), \quad \forall s \in [a, b]$$

are obtained in [55], and a method to approximate the solution of (1.105) is constructed in the case $g(t) = t - \tau$ and $a_1 = a - \tau$. The corresponding error estimate is derived.

Chapter 2

Optimal properties for cubic splines

The results presented in this chapter are focused on the notion of quadratic oscillation in average (QOA) introduced by the author in [62], which represents a functional intended to be a measure of the deviations of an interpolation function by the data polygon. These results are obtained in [41], [59], [62], [63], [64], [66], [73], [81], and [84] and are related to the minimizing of QOA for splines. The obtained optimal property differs by the existing well-known optimal properties for splines, such as minimal strain energy (see [236] and [245]), minimal curvature (see [8], [95], [163] and [185]) and minimal L^2-norm of a cubic spline and of its derivatives (see [163]).

2.1 The quadratic oscillation in average QOA

Consider arbitrary given points $(x_i, y_i) \in \mathbb{R}^2$, $i = \overline{0,n}$, such that the points situated on the real axis x_i, $i = \overline{0,n}$, realize a partition of an interval $[a,b]$:

$$\Delta : a = x_0 < x_1 < ... < x_{n-1} < x_n = b.$$

Generally, the problem of interpolating the points (x_i, y_i) , $i = \overline{0,n}$, without other information (about tangents or local curvature), than the values y_i, $i = \overline{0,n}$, can be solved by Lagrange polynomial interpolation, B-splines, and polynomial or nonpolynomial splines. In smooth fitting data and computer aided geometric design, this problem is usually solved by polynomial splines, with a special attention for cubic splines with optimal or shape preserving (including monotony or convexity preservation) properties. In this context, the classical Holladay's property (see [8] and [185]) which leads to the minimal curvature for natural, complete, or periodic cubic splines with deficiency one, is generally known. This property is expressed in terms of the second derivative of the cubic spline and it is generalized for odd order natural splines (see [185]).

In [62] we have introduced the notion of quadratic oscillation in average which measures the deviations of a spline by the data polygon. This notion not requires any derivative of the spline, being applicable to splines with arbitrary deficiency (even high deficiency). So, we consider a partition of $[a,b]$ like

$$\Delta : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$$

and the given values $y_0,\, y_1, ..., y_n \in \mathbb{R}$. In the functional space $C[a,b]$ we can form the set $C\left([a,b], \Delta, y\right) = \{f \in C[a,b] : f\left(x_i\right) = y_i, \quad \forall i = \overline{0,n}\}$.

Definition 43 *(see [62] and [73]) The quadratic oscillation in average (QOA) of a given function $f \in C([a,b],\Delta,y)$ is the value of the functional $\rho_2 : C([a,b],\Delta,y) \to \mathbb{R}$ defined by*

$$\rho_2(f) = \sqrt{\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} [f(x) - \frac{x_i - x}{h_i} \cdot y_{i-1} - \frac{x - x_{i-1}}{h_i} \cdot y_i]^2 dx}. \quad (2.1)$$

This notion is considered as a measure of the deviation of an interpolating function by the polygonal line joining the points (x_i, y_i), $i = \overline{0,n}$. These deviations are called oscillations of an interpolation function and can be defined for any function interpolating the points (x_i, y_i), $i = \overline{0,n}$, and for any type of spline functions, not depending by its deficiency. It can be easily seen that the functional ρ_2 is positive, $\rho_2(f) \geq 0$, $\forall f \in C([a,b],\Delta,y)$ and $\rho_2(f) = 0$ if and only if f is the linear first order polynomial spline interpolating the points (x_i, y_i), $i = \overline{0,n}$ (namely, the polygonal line). Moreover, for any $\alpha \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$, we see that $\rho_2(\alpha \cdot f) = \alpha \cdot \rho_2(f)$. So,we can conclude that the QOA is a positive and homogeneous functional.

The notion of QOA differs by the "global deviation" introduced by Floater in [132], page 149, where it is considered that great oscillations of a spline function correspond to the "badness of the interpolating spline" in the interpolation of given points $P_0, P_1, ..., P_{n-1}, P_n \in \mathbb{R}^d$, $d \geq 2$, $P_0 = P_n$, and therefore, as a measure of these oscillations, Floater proposes the notion of '*deviation from the data polygon*'. Firstly, is introduced the local deviation to be the ratio

$$\mu_i(s) = \frac{dist\left(s\mid_{[t_i,t_{i+1}]}, [P_i P_{i+1}]\right)}{|P_{i+1} - P_i|}, \quad i = \overline{0, n-1}$$

and then, the global deviation, is the value

$$\mu(s) = \frac{\max_{0 \leq i \leq n-1} dist\left(s\mid_{[t_i,t_{i+1}]}, [P_i P_{i+1}]\right)}{\max_{0 \leq i \leq n-1} |P_{i+1} - P_i|}$$

where $dist$ is the Hausdorf distance in the euclidean space $\mathbb{R}^d$, and a parametrization is given by $t_0 < t_1 < ... < t_n$ in $\mathbb{R}$. These deviations are studied for periodic C^2 chordal and centripetal cubic splines proving that $\mu_i(s) \leq \frac{3}{4}$, $\forall i = \overline{0, n-1}$, for chordal and centripetal splines, and it is obtained a smaller value of the global deviation $\mu(s) \leq \frac{9}{20}$, for centripetal splines. For other cubic splines it is proved that $\mu(s) \leq \frac{3}{4}$. In [132], Floater considers that "a spline s is 'good' if both its μ_i and μ values are low."

In a given class of splines, we search for the spline with minimal QOA. According to (2.1), the expression

$$\pi \cdot \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} [f(x) - \frac{x_i - x}{h_i} \cdot y_{i-1} - \frac{x - x_{i-1}}{h_i} \cdot y_i]^2 dx$$

represents the volume of the body obtained rotating the plane regions, situated between the graph of f and the polygonal line, round about the axis Ox. Therefore, from geometric point of view, minimizing the QOA of f, the above mentioned volume will be minimized.

We see from (2.1) that the QOA is defined over the all subintervals $I_i = [x_{i-1}, x_i]$, $i = \overline{1,n}$. In the problems where we intend to minimize the quadratic oscillation only on some of these subintervals, more suitable is the notion of partial quadratic oscillation in average (PQOA), which can be defined as follows:

Definition 44 *(see [81]) Let a subset $K \subset \{1, ..., n\}$ be given. The partial quadratic oscillation in average of the function $f \in C([a,b], \Delta, y)$ corresponding to the subset K, is the functional $\rho(K) : C([a,b], \Delta, y) \to \mathbb{R}$ given by:*

$$\rho(K)(f) = \sqrt{\sum_{i \in K} \int_{I_i} [f(x) - \frac{x_i - x}{h_i} \cdot y_{i-1} - \frac{x - x_{i-1}}{h_i} \cdot y_i]^2 dx}. \tag{2.2}$$

When $K = \{1, ..., n\}$ the functional PQOA becomes QOA.

Since in the purpose to minimize the QOA of a spline over the all subintervals $I_i = [x_{i-1}, x_i]$, $i = \overline{1, n}$, we need the degree of freedom at least n+1, we infer that the functional QOA is suitable for splines with deficiency at least 2. For splines with deficiency 1 we can use only the functional PQOA in the problem to determine the free parameters such that the PQOA to be minimized. These considerations will be illustrated in the next sections.

2.2 Hermite type cubic splines with minimal QOA

The Hermite type cubic spline $s : [a,b] \to \mathbb{R}$, $s \in C^1[a,b]$, interpolating the given points $(x_i, y_i) \in \mathbb{R}^2$, $i = \overline{0, n}$, is defined by its restrictions to the intervals $[x_{i-1}, x_i]$, $i = \overline{1, n}$

$$s(x) = A_i(x) \cdot m_{i-1} + B_i(x) \cdot m_i + C_i(x) \cdot y_{i-1} + D_i(x) \cdot y_i, \quad x \in [x_{i-1}, x_i], \ i = \overline{1, n} \tag{2.3}$$

with

$$A_i(x) = \frac{(x_i - x)^2 (x - x_{i-1})}{h_i^2}, \quad B_i(x) = -\frac{(x - x_{i-1})^2 (x_i - x)}{h_i^2},$$

$$C_i(x) = \frac{(x_i - x)^2 [2(x - x_{i-1}) + h_i]}{h_i^3}, \quad D_i(x) = \frac{(x - x_{i-1})^2 [2(x_i - x) + h_i]}{h_i^3},$$

where $h_i = x_i - x_{i-1}$, $i = \overline{1, n}$, and the derivatives $m_i = s'(x_i)$, $i = \overline{0, n}$, remain to be determined. In the case that s interpolates on $[a,b]$ a given function $f \in C^1[a,b]$ with known values $f_i = f'(x_i)$, $i = \overline{0, n}$, then we can specify $m_i = f'(x_i)$, $\forall i = \overline{0, n}$, but when the values $f'(x_i)$, $i = \overline{0, n}$ are unknown, we have to estimate the free n+1 parameters $m_0, ..., m_n$. H. Akima proposes in [9] a method to estimate these derivatives based on geometric local procedures (see the next Section). In [148], the Hermite type cubic splines (2.3) are applied in order to obtain a suboptimal algorithm in least squares fitting-data problems. In [41], the values $m_2, ..., m_{n-2}$ are computed using the Akima's method and m_0, m_1, m_{n-1}, m_n are obtained by partial minimization of the quadratic oscillation in average on the intervals $[x_0, x_2]$ and $[x_{n-2}, x_n]$. In [9], [148], and [41] the cubic spline has the smoothness property $s \in C^1[a,b]$.

Another choice to determine the values $m_0, m_1, ..., m_n$ is to require high order smoothness, $s \in C^2[a,b]$. Denoting by s_i the restriction of s to each subinterval $[x_{i-1}, x_i]$, $i = \overline{1, n}$, the smoothness condition $s \in C^2[a,b]$ leads to $s_i''(x_i) = s_{i+1}''(x_i)$, $i = \overline{1, n-1}$, which are $n - 1$ linear equations for the $n + 1$ unknowns m_i, $i = \overline{0, n}$:

$$\frac{1}{h_i} m_{i-1} + 2\left(\frac{1}{h_i} + \frac{1}{h_{i+1}}\right) m_i + \frac{1}{h_{i+1}} m_{i+1} = \frac{3(y_i - y_{i-1})}{h_i^2} + \frac{3(y_{i+1} - y_i)}{h_{i+1}^2}, \ i = \overline{1, n-1}. \tag{2.4}$$

It follows that two additional linearly independent conditions are needed. These are usually end-point conditions that can be chosen in many forms. The natural cubic spline is obtained with the end conditions $s''(a) = s''(b) = 0$ generating the additional two equations

$$\begin{array}{c} m_0 + \frac{1}{2} m_1 = \frac{3}{2h_1} \cdot (y_1 - y_0) \\ \frac{1}{2} m_{n-1} + m_n = \frac{3}{2h_n} \cdot (y_n - y_{n-1}), \end{array} \tag{2.5}$$

and the well-known De Boor's not-a-knot cubic spline (see [91] and [185]) is generated by the conditions $s_1'''(x_1) = s_2'''(x_1)$ and $s_{n-1}'''(x_{n-1}) = s_n'''(x_{n-1})$. Another kind of end conditions not depending on derivative information were proposed by Behforooz and Papamichael in [31]:

$$\begin{cases} m_0 + \alpha m_1 = \frac{1}{h}\left(\alpha_0 y_0 + \alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3\right) \\ m_n + \alpha m_{n-1} = \frac{1}{h}\left(\alpha_0 y_n + \alpha_1 y_{n-1} + \alpha_2 y_{n-2} + \alpha_3 y_{n-3}\right) \end{cases}$$

where

$$\alpha_0 = \frac{1}{6}\left(-2\alpha - 11\right), \ \alpha_1 = \frac{1}{6}\left(-3\alpha + 18\right), \ \alpha_2 = \frac{1}{6}\left(6\alpha - 9\right), \ \alpha_3 = \frac{1}{6}\left(-\alpha + 2\right),$$

obtaining the so called $E(\alpha)$ cubic splines (see [31], [32], [33], for the results concerning these $E(\alpha)$ cubic splines).

In [73] we have determined the values $m_0, m_1, ..., m_n$ in order to minimize the quadratic oscillation in average of s.

Theorem 45 *(see [73]) For given points (x_i, y_i), $i = \overline{0,n}$, there exists an unique cubic spline of the Hermite type having minimal quadratic oscillation in average. This cubic spline $s \in C^1[a,b]$ can be determined by using an iterative algorithm. If s interpolates a function $f \in C[a,b]$, $f(x_i) = y_i$, $i = \overline{0,n}$, then its error estimation is:*

$$|f(x) - s(x)| \leq \left(1 + \frac{h^3}{4\overline{h}^3}\right) \cdot \varpi(f,h), \quad \forall x \in [a,b] \tag{2.6}$$

where $h = \max\{h_i : i = \overline{1,n}\}$, $\overline{h} = \min\{h_i : i = \overline{1,n}\}$, $\varpi(f,h) \overset{notation}{=} \max\{\varpi(f,h_i) : i = \overline{1,n}\}$, and $\varpi(f,\delta) = \sup\{|f(t) - f(s)| : t,s \in [a,b], |t-s| \leq \delta\}$ is the uniform modulus of continuity. If $f \in C^1[a,b]$ then $|f(x) - s(x)| \leq \left(1 + \frac{h^3}{4\overline{h}^3}\right) \cdot M'h$, where $M' = \max\{|f'(x)| : x \in [a,b]\}$.

Sketch of proof: Consider the residual $R(m_0,...,m_n) = \sum\limits_{i=1}^{n} \int\limits_{x_{i-1}}^{x_i} [A_i(x) \cdot m_{i-1} + B_i(x) \cdot m_i + C_i(x) \cdot y_{i-1} + D_i(x) \cdot y_i - \frac{x_i - x}{h_i} \cdot y_{i-1} - \frac{x - x_{i-1}}{h_i} \cdot y_i]^2 dx$. In order to minimize it we solve the system of normal equations $\frac{\partial R}{\partial m_i} = 0$, $i = \overline{0,n}$. This system is

$$\begin{cases} \left(\int\limits_{x_0}^{x_1} [A_1(x)]^2 dx\right) \cdot m_0 + \left(\int\limits_{x_0}^{x_1} A_1(x) \cdot B_1(x)\, dx\right) \cdot m_1 = d_0 \\ \left(\int\limits_{x_{i-1}}^{x_i} A_i(x) \cdot B_i(x)\, dx\right) \cdot m_{i-1} + \left(\int\limits_{x_{i-1}}^{x_i} [B_i(x)]^2 dx + \int\limits_{x_i}^{x_{i+1}} [A_{i+1}(x)]^2 dx\right) \cdot m_i \\ \quad + \left(\int\limits_{x_i}^{x_{i+1}} A_{i+1}(x) \cdot B_{i+1}(x)\, dx\right) \cdot m_{i+1} = d_i, \qquad i = \overline{1, n-1} \\ \left(\int\limits_{x_{n-1}}^{x_n} A_n(x) \cdot B_n(x)\, dx\right) \cdot m_{n-1} + \left(\int\limits_{x_{n-1}}^{x_n} [B_n(x)]^2 dx\right) \cdot m_n = d_n, \end{cases} \tag{2.7}$$

where

$$d_0 = -\int\limits_{x_0}^{x_1} A_1(x) \cdot [C_1(x) \cdot y_0 + D_1(x) \cdot y_1 - \frac{x_1 - x}{h_1} \cdot y_0 - \frac{x - x_0}{h_1} \cdot y_1] dx$$

$$d_i = -\int_{x_{i-1}}^{x_i} B_i(x)\,[C_i(x)\cdot y_{i-1} + D_i(x)\cdot y_i - \frac{x_i - x}{h_i}\cdot y_{i-1} - \frac{x - x_{i-1}}{h_i}\cdot y_i]dx -$$

$$-\int_{x_i}^{x_{i+1}} A_{i+1}(x)\,[C_{i+1}(x)\cdot y_i + D_{i+1}(x)\cdot y_{i+1} - \frac{x_{i+1} - x}{h_{i+1}}\cdot y_i - \frac{x - x_i}{h_{i+1}}\cdot y_{i+1}]dx,\quad i = \overline{1, n-1},$$

$$d_n = -\int_{x_{n-1}}^{x_n} B_n(x)\cdot[C_n(x)\cdot y_{n-1} + D_n(x)\cdot y_n - \frac{x_n - x}{h_n}\cdot y_{n-1} - \frac{x - x_{n-1}}{h_n}\cdot y_n]dx$$

We see that

$$\int_{x_{i-1}}^{x_i}[A_i(x)]^2dx = \frac{h_i^3}{105},\quad \int_{x_{i-1}}^{x_i}[B_i(x)]^2dx = \frac{h_i^3}{105},\quad \int_{x_{i-1}}^{x_i} A_i(x)\cdot B_i(x)\,dx = -\frac{h_i^3}{140}$$

and $d_0 = \frac{h_1^2}{420}\cdot(y_1 - y_0)\,,\ d_n = \frac{h_n^2}{420}\cdot(y_n - y_{n-1})\,,$

$$d_i = \frac{h_i^2}{420}\cdot(y_i - y_{i-1}) + \frac{h_{i+1}^2}{420}\cdot(y_{i+1} - y_i)\,,\quad i = \overline{1, n-1}.$$

Then, the system (2.7) becomes

$$\begin{cases} m_0 - \frac{3}{4}m_1 = \frac{1}{4h_1}\cdot(y_1 - y_0) = q_0 \\ -\frac{3h_i^3}{4\left(h_i^3 + h_{i+1}^3\right)}\cdot m_{i-1} + m_i - \frac{3h_{i+1}^3}{4\left(h_i^3 + h_{i+1}^3\right)}\cdot m_{i+1} = \\ = \frac{h_i^2}{4\left(h_i^3 + h_{i+1}^3\right)}\cdot(y_i - y_{i-1}) + \frac{h_{i+1}^2}{4\left(h_i^3 + h_{i+1}^3\right)}\cdot(y_{i+1} - y_i) = q_i \\ -\frac{3}{4}m_{n-1} + m_n = \frac{1}{4h_n}\cdot(y_n - y_{n-1}) = q_n. \end{cases} \tag{2.8}$$

This system is diagonally dominant and therefore has unique solution, its matrix $G = I + A$ being nonsingular. Denoting $m = (m_0, ..., m_n)$ and $q = (q_0, ..., q_n)$, the solution of (2.8) is $\overline{m} = (\overline{m_0}, ..., \overline{m_n}) = G^{-1}\cdot q$ and can be obtained by using the iterative algorithm for general diagonally dominant systems presented in [8], pages 14-15. Moreover, $\|A\|_\infty = \frac{3}{4}$ and

$$\left\|G^{-1}\right\|_\infty = \left\|(I + A)^{-1}\right\|_\infty \le \frac{1}{1 - \|A\|_\infty} = 4.$$

The Hessian of the residual R, $H(R) = \left(\frac{\partial^2 R}{\partial m_i \partial m_j}\right)_{i,j=\overline{0,n}}$, is a symmetric tridiagonal matrix, diagonally dominant. Using the Gershgorin's theorem we infer that for all diagonal-minors of this matrix the eigenvalues have strictly positive real parts. Since each diagonal-minor is the product of its eigenvalues, we infer that all diagonal-minors of $H(R)$ are strictly positive. We conclude that the quadratic form associated to the matrix $H(R)$ is positive definite and therefore the point $(\overline{m_0}, ..., \overline{m_n})$ minimizes the residual $R(m_0, ..., m_n)$. Since $\rho_2(s)(m_0, ..., m_n) = \sqrt{R(m_0, ..., m_n)}$, we infer that

$$\rho_2(s)(\overline{m_0}, ..., \overline{m_n}) = \min\{\rho_2(s)(m_0, ..., m_n) : (m_0, ..., m_n) \in \mathbb{R}^{n+1}\}.$$

Now, let $x \in [a, b]$ be an arbitrary point. Then there is $i \in \{1, 2, ..., n\}$ such that $x \in [x_{i-1}, x_i]$ and

$$|s(x) - f(x)| \le |C_i(x)|\cdot|y_{i-1} - f(x)| + |D_i(x)|\cdot|y_i - f(x)| + |A_i(x)|\cdot|\overline{m_{i-1}}| + |B_i(x)|\cdot|\overline{m_i}|\,.$$

Since $|C_i(x)| + |D_i(x)| = C_i(x) + D_i(x) = 1$, $\forall x \in [x_{i-1}, x_i]$ and $\max\{|A_i(x)| + |B_i(x)| : x \in [x_{i-1}, x_i]\} = \frac{h_i}{4}$ we infer that

$$|s(x) - f(x)| \leq \omega(f, h_i) + \frac{h_i}{4} \cdot \max\{|\overline{m_{i-1}}|, |\overline{m_i}|\} \leq \omega(f, h_i) + \frac{h_i}{4} \cdot \|\overline{m}\|_\infty,$$

where $\|\overline{m}\|_\infty = \max\{|\overline{m_i}| : i = \overline{0,n}\} \leq \left\|G^{-1}\right\|_\infty \cdot \|q\|_\infty \leq 4\|q\|_\infty$. We see that $|q_0| \leq \frac{1}{4h_1} \cdot \omega(f, h_1) \leq \frac{1}{4\overline{h}} \cdot \omega(f, h)$, $|q_n| \leq \frac{1}{4h_n} \cdot \omega(f, h_n) \leq \frac{1}{4\overline{h}} \cdot \omega(f, h)$,

$$|q_i| \leq \frac{h_i^2}{4\left(h_i^3 + h_{i+1}^3\right)} \cdot \omega(f, h_i) + \frac{h_{i+1}^2}{4\left(h_i^3 + h_{i+1}^3\right)} \cdot \omega(f, h_{i+1}) \leq \frac{h^2}{4\overline{h}^3} \cdot \omega(f, h), \forall i = \overline{1, n-1}$$

and so, $\|q\|_\infty = \max\{|q_i| : \overline{0,n}\} \leq \max\{\frac{1}{4\overline{h}}, \frac{h^2}{4\overline{h}^3}\} \cdot \omega(f, h) \leq \frac{h^2}{4\overline{h}^3} \cdot \omega(f, h)$. Consequently,

$$|s(x) - f(x)| \leq \omega(f, h) + \frac{h}{4} \cdot \frac{h^2}{\overline{h}^3} \cdot \omega(f, h) = [1 + \frac{h^3}{4\overline{h}^3}] \cdot \omega(f, h), \forall x \in [a, b].$$

For uniform partitions, $h = \overline{h}$ and

$$|s(x) - f(x)| \leq \frac{5}{4} \cdot \omega(f, h), \ \forall x \in [a, b]. \tag{2.9}$$

If $f \in C^1[a, b]$, then $\omega(f, h) \leq M'h$ and $|f(x) - s(x)| \leq \left(1 + \frac{h^3}{4\overline{h}^3}\right) \cdot M'h$.

Remark 46 *An explicit form of the iterative algorithm that solves the system (2.8) is the following: Let* $b_i = 1$, $i = \overline{0,n}$, $a_i = -\frac{3h_i^3}{4\left(h_i^3 + h_{i+1}^3\right)}$, $i = \overline{1, n-1}$, $a_n = -\frac{3}{4}$, $c_0 = -\frac{3}{4}$, $c_i = -\frac{3h_{i+1}^3}{4\left(h_i^3 + h_{i+1}^3\right)}$, $i = \overline{1, n-1}$, $v_{-1} = u_{-1} = 0$, *and for* $i = \overline{0,n}$ *we put* $p_i = a_i \cdot v_{i-1} + b_i$, $v_i = -\frac{c_i}{p_i}$, $u_i = \frac{d_i - a_i \cdot u_{i-1}}{p_i}$. *Finally, the solution of (2.8) is obtained by backward recurrence* $\overline{m_n} = u_n$, $\overline{m_k} = v_k \cdot \overline{m_{k+1}} + u_k$, $k = \overline{n-1, 0}$.

Remark 47 *In addition to the property of minimal QOA, this cubic spline has another remarkable one: In the class of uniformly continuous functions we can observe that the estimates (2.6) and (2.9) are better than some of the classical ones. Indeed, for the case of natural cubic splines, the corresponding system (2.4)-(2.5) can be solved recurrently, and analogous with the proof of the previous theorem (or using the method presented in [184]) we obtain the estimate*

$$|s(x) - f(x)| \leq \left(1 + \frac{3h^2}{4\overline{h}^2}\right) \cdot \omega(f, h), x \in [a, b].$$

For uniform partitions, in the case of natural cubic spline we have $|s(x) - f(x)| \leq \frac{7}{4} \cdot \omega(f, h)$, $\forall x \in [a, b]$. *Analogous, for the not-a-knot cubic spline the estimate has a constant major than* $\frac{7}{4}$, *and for the Akima's cubic spline we obtain the estimates* $|s(x) - f(x)| \leq \frac{5}{4} \cdot \omega(f, h)$, $\forall x \in [x_1, x_{n-1}]$ *and* $|s(x) - f(x)| \leq \frac{7}{4} \cdot \omega(f, h)$, $\forall x \in [x_0, x_1] \cup [x_{n-1}, x_n]$. *In order to complete these special properties, we can mention that the restrictions of this cubic spline s to the intervals* $[x_{i-1}, x_i]$, $i = \overline{1,n}$, *always preserve their third order as polynomial functions, excepting the case of collinear points. More precisely, if and only if three consecutive points* P_{i-1}, P_i, P_{i+1} *are collinear, then on the corresponding intervals* $[x_{i-1}, x_i]$ *and* $[x_i, x_{i+1}]$ *the cubic spline becomes first order polynomial and his graph it reduces to the the line joining the points* P_{i-1}, P_i, P_{i+1}. *So, if* $n = 1$, *then (after elementary calculus) we see that the cubic polynomial (2.3) with minimal QOA on* $[a, b] = [x_0, x_1]$

it reduces to a first order polynomial, and if $n \geq 2$*, then the cubic spline* $s \in C^1[a,b]$ *with minimal QOA has the following supplementary property: the polynomial order of its restrictions* s_i, $i = \overline{1,n}$*, becomes less than* 3*, if and only if the interpolated points* $P_i(x_i, y_i)$, $i = \overline{0,n}$*, are all collinear (all situated on the same line). In this case this spline it reduces to the line joining these points, the polynomial order being* 1 *and never* 2*. If* $n \geq 2$ *and if the points* $P_i(x_i, y_i)$, $i = \overline{0,n}$ *are not all collinear (that is these points are not situated all on the same line), then the restrictions* s_i*, to the intervals* $[x_{i-1}, x_i]$, $i = \overline{1,n}$*, of the cubic spline* $s \in C^1[a,b]$ *with minimal QOA, are third order cubic polynomials (this can be proved by reduction to a contradiction).*

Numerical experiment

In order to test and to illustrate the remarkable properties of the cubic spline of Hermite type with minimal quadratic oscillation in average in the interpolation of experimental data, we present a numerical experiment from diabetology. In July 29-30, 2005, was realized the glycemic profile of a non-diabetic volunteer having 35 years old, by 8 measurements of the blood-glucose levels in specially chosen moments during a day. The moments were 9, 12, 15, 18, 21, and 24 o'clock in July 29, and 3 AM, 6 AM in July 30, and the obtained values, measured in mg/100ml, were 94, 76, 83, 73, 80, 91, 64 and 100, respectively. By a transformation of the moments into an increasing sequence (using a translation), these becomes: 0, 3, 6, 9, 12, 15, 18, 21. These glycemic values were interpolated using various splines of Hermite type: the natural cubic spline, the Akima's cubic spline, the cubic spline with zero derivatives, the Catmull-Rom cubic spline $s \in C^1[a,b]$ (introduced in [100]) with natural end conditions $s''(a+0) = s''(b-0) = 0$, and the cubic spline with minimal quadratic oscillation in average. For all these cubic spline functions, the quadratic oscillation in average was computed, and the obtained values are in Table 1. A comparison of the values in the second line of Table 1 confirm the result of Theorem 45. In Figure 1 are plotted the graphs of the natural cubic spline (in green), the Akima's cubic spline (in blue), the cubic spline with minimal QOA (in red), and the polygonal line (in black) joining the interpolated points, observing which of the three cubic splines is the closest to the polygonal line. According to the third line of Table 1, the Hermite type cubic spline with minimal quadratic oscillation in average has another interesting minimal property in the above presented set of cubic splines: with exception of the cubic spline with zero derivatives, it has minimal length of the graph. The explanation of the fact that the cubic spline with zero derivatives has the smallest graph's length is that this spline is a special type of cardinal spline corresponding to the greatest tension parameter.

spline type:	natural	QOA min.	Akima	zero der.	C-R
$\rho_2(s)$:	12.926	5.7295	12.995	6.2335	9.7596
$L(s)$:	126.31	122.14	122.38	121.61	122.91

Table 1. The QOA and the length of the graphs of cubic splines

As a potential application of the obtained cubic spline with minimal quadratic oscillation in average we can mention the design of aerodynamic profiles, for instance the design of longitudinal and horizontal cross sections in shipping boats (the observed presence of an inflection point in the end intervals could suggest better aerodynamic properties for the angle of the boat). A recent application of this optimal cubic spline in the design of omni-directional humanoid robots can be found in [247].

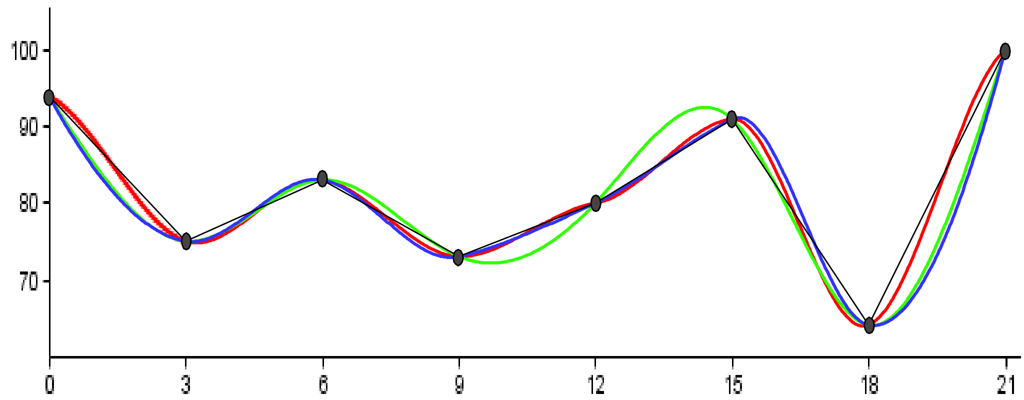

Figure 1. The graphs of the Akima's (blue), natural (green), and minimal QOA (red) cubic splines.

2.3 Optimization at the end points of the Akima's method

The Akima's interpolation method (see [9]) provides a natural and more suitable procedure for the smooth fitting of the data (x_i, y_i), $i = \overline{0, n}$, with $y_i \in \mathbb{R}$, $\forall i = \overline{0, n}$, and x_i, $i = \overline{0, n}$, being the knots of a grid, $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$. In the Akima's method, the values m_i, $i = \overline{0, n}$, (in (2.3)) are determined by using a local procedure based on geometric reasons. More exactly, for five given points $M_i(x_i, y_i)$, $i = \overline{1, 5}$, are computed the slopes $p_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$, $i = \overline{1, 4}$, and then, starting from a proportion that uses some of the obtained segments, it is suggested the following value for the tangent in the point $M_2(x_2, y_2)$:

$$m_2 = \frac{|p_4 - p_3| \cdot p_2 + |p_2 - p_1| \cdot p_3}{|p_4 - p_3| + |p_2 - p_1|}. \tag{2.10}$$

This formula (2.10) is generalized considering the slopes $p_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$, $i = \overline{0, n-1}$ together with the derivatives

$$m_i = \frac{|p_{i+1} - p_i| \cdot p_{i-1} + |p_{i-1} - p_{i-2}| \cdot p_i}{|p_{i+1} - p_i| + |p_{i-1} - p_{i-2}|}, \quad i = \overline{2, n-2}. \tag{2.11}$$

In order to extend formula (2.11) for $i = \overline{0, n}$, the previously computed slopes are not enough and therefore, Akima proposes the construction of four new supplementary slopes $p_{-1}, p_{-2}, p_n, p_{n+1}$, based on a reasoning in the framework of a particular case (equidistant grid and exactness for second order polynomials on the end intervals $[x_0, x_1]$ and $[x_{n-1}, x_n]$):

$$p_{-1} = 2p_0 - p_1, \quad p_{-2} = 3p_0 - 2p_1, \quad p_n = 2p_{n-1} - p_{n-2}, \quad p_{n+1} = 3p_{n-1} - 2p_{n-2}. \tag{2.12}$$

Since the artificial introduction of the four slopes well performs only in the particular case of equidistant grids, this is not a strong point of the Akima's method. For this reason we have proposed in [81] an optimal procedure for the computation of the left unspecified derivatives m_0, m_1, m_{n-1}, m_n (the other derivatives m_i, $i = \overline{2, n-2}$, being computed using (2.11)). This optimal procedure is based on the idea to partially minimize the PQOA of the cubic spline on the end intervals $[x_0, x_2]$ and $[x_{n-2}, x_n]$, and consequently, we consider the subset $K = \{1, 2, n-1, n\}$. Since the unknown derivatives m_0, m_1 and m_{n-1}, m_n appear only in the intervals $[x_0, x_2]$ and $[x_{n-2}, x_n]$, respectively, we define the residual

$$R_K\left(m_0, m_1, m_{n-1}, m_n\right) = \sum_{i \in K} \int_{x_{i-1}}^{x_i} [s(x) - \frac{x_i - x}{h_i} \cdot y_{i-1} - \frac{x - x_{i-1}}{h_i} \cdot y_i]^2 dx \qquad (2.13)$$

for $K = \{1, 2, n-1, n\}$ and we see that in this case,

$$\rho(K)(s) = \sqrt{R_K\left(m_0, m_1, m_{n-1}, m_n\right)}.$$

Theorem 48 *(see [81]) For given data (x_i, y_i), $i = \overline{0, n}$, and with the values m_i, $i = \overline{2, n-2}$, be computed using (2.11), there are uniquely determined the values m_0, m_1, m_{n-1}, m_n such that the partial quadratic oscillation in average, $\rho(K)(s)$ is minimal. Moreover, the error estimate in the interpolation of a given function $f \in C([a,b], \Delta, y)$ by the obtained Hermite-type cubic spline is:*

$$|s(x) - f(x)| \leq \left(1 + \frac{h}{4\underline{h}}\right) \cdot \varpi(f, h), \quad \forall x \in [x_2, x_{n-2}], \qquad (2.14)$$

and

$$|s(x) - f(x)| \leq \left(1 + \frac{h^4}{4\underline{h}^4}\right) \cdot \varpi(f, h), \quad \forall x \in [x_0, x_2] \cup [x_{n-2}, x_n], \qquad (2.15)$$

where $h = \max\{h_i : i = \overline{1, n}\}$, $\underline{h} = \min\{h_i : i = \overline{1, n}\}$, and $\varpi(f, h) = \max\{|f(u) - f(v)| : u, v \in [a, b],\ |u - v| \leq h\}$ is the uniform modulus of continuity. For equidistant grids the error estimate becomes

$$|s(x) - f(x)| \leq \frac{5}{4} \cdot \varpi(f, h), \quad \forall x \in [x_0, x_n]. \qquad (2.16)$$

In the proof of this theorem it is applied the least squares method to the residual $R_K(m_0, m_1, m_{n-1}, m_n)$ analogous as in the previous section. The solution of the system of normal equations, $\frac{\partial R}{\partial m_0} = 0$, $\frac{\partial R}{\partial m_1} = 0$, $\frac{\partial R}{\partial m_{n-1}} = 0$, $\frac{\partial R}{\partial m_n} = 0$, is:

$$m_0 = \frac{1}{\delta} \cdot [\frac{y_1 - y_0}{4h_1} + \frac{9h_2^3}{16\left(h_1^3 + h_2^3\right)} \cdot m_2 + \frac{3h_1^2}{16\left(h_1^3 + h_2^3\right)} \cdot (y_1 - y_0) + \frac{3h_2^2}{16\left(h_1^3 + h_2^3\right)} \cdot (y_2 - y_1)]$$

$$m_1 = \frac{1}{\delta} \cdot [\frac{3h_2^3}{4\left(h_1^3 + h_2^3\right)} \cdot m_2 + \frac{7h_1^2}{16\left(h_1^3 + h_2^3\right)} \cdot (y_1 - y_0) + \frac{h_2^2}{4\left(h_1^3 + h_2^3\right)} \cdot (y_2 - y_1)]$$

where

$$\delta = 1 - \frac{9h_1^3}{16\left(h_1^3 + h_2^3\right)} = \frac{7h_1^3 + 16h_2^3}{16\left(h_1^3 + h_2^3\right)}$$

and

$$m_{n-1} = \frac{1}{\delta'} \cdot \left(\frac{7h_n^2 \cdot (y_n - y_{n-1})}{16\left(h_{n-1}^3 + h_n^3\right)} + \frac{3h_{n-1}^3}{4\left(h_{n-1}^3 + h_n^3\right)} \cdot m_{n-2} + \frac{h_{n-1}^2 \cdot (y_{n-1} - y_{n-2})}{4\left(h_{n-1}^3 + h_n^3\right)}\right)$$

$$m_n = \frac{1}{\delta'} \cdot \left(\frac{y_n - y_{n-1}}{4h_n} + \frac{9h_{n-1}^3}{16\left(h_{n-1}^3 + h_n^3\right)} \cdot m_{n-2} + \frac{3h_{n-1}^2 \cdot (y_{n-1} - y_{n-2})}{16\left(h_{n-1}^3 + h_n^3\right)} + \frac{3h_n^2 \cdot (y_n - y_{n-1})}{16\left(h_{n-1}^3 + h_n^3\right)}\right)$$

where

$$\delta' = 1 - \frac{9h_n^3}{16\left(h_{n-1}^3 + h_n^3\right)} = \frac{7h_n^3 + 16h_{n-1}^3}{16\left(h_{n-1}^3 + h_n^3\right)}.$$

The Hesse matrix of $R_K(m_0, m_1, m_{n-1}, m_n)$ is $H = \left(\frac{\partial^2 R_K}{\partial m_i^2}\right), i = 0, 1, n-1, n$, having the form:

$$H = 2 \cdot \begin{pmatrix} \frac{h_1^3}{105} & -\frac{h_1^3}{140} & 0 & 0 \\ -\frac{h_1^3}{140} & \frac{h_1^3}{105} + \frac{h_2^3}{105} & 0 & 0 \\ 0 & 0 & \frac{h_{n-1}^3}{105} + \frac{h_n^3}{105} & -\frac{h_n^3}{140} \\ 0 & 0 & -\frac{h_n^3}{140} & \frac{h_n^3}{105} \end{pmatrix}$$

and it is proved that all the diagonal minors of H are strictly positive, and therefore the solution (m_0, m_1, m_{n-1}, m_n) is the unique critical point of the residual R_K and minimize it.

We can see that in the case of equidistant grid it obtains:

$$\begin{aligned} m_0 &= \frac{9m_2}{23} + \frac{11(y_1 - y_0)}{23h} + \frac{3(y_2 - y_1)}{23h} \\ m_1 &= \frac{12m_2}{23} + \frac{7(y_1 - y_0)}{23h} + \frac{4(y_2 - y_1)}{23h} \\ m_{n-1} &= \frac{12m_{n-2}}{23} + \frac{7(y_n - y_{n-1})}{23h} + \frac{4(y_{n-1} - y_{n-2})}{23h} \\ m_n &= \frac{9m_{n-2}}{23} + \frac{11(y_n - y_{n-1})}{23h} + \frac{3(y_{n-1} - y_{n-2})}{23h}. \end{aligned}$$

Remark 49 *(see [81]) We can see that in the Akima's method,*

$$|m_i| \leq \frac{1}{\underline{h}} \cdot \varpi(f, h), \quad \forall i = \overline{2, n-2},$$

and $|p_i| \leq \frac{1}{\underline{h}} \cdot \varpi(f,h), \forall i = \overline{0, n-1}$. *Since,* $p_{-1} = 2p_0 - p_1$, $p_{-2} = 2p_{-1} - p_0 = 3p_0 - 2p_1$, $p_n = 2p_{n-1} - p_{n-2}$, $p_{n+1} = 2p_n - p_{n-1} = 3p_{n-1} - 2p_{n-2}$, *we infer that* $|p_i| \leq \frac{3}{\underline{h}}\varpi(f,h)$, *for* $i \in \{-1, n,\}$ *and* $|p_i| \leq \frac{5}{\underline{h}}\varpi(f,h)$, *for* $i \in \{-2, n+1,\}$. *So,*

$$|m_i| \leq \frac{|p_{i+1} - p_i| \cdot |p_{i-1}| + |p_{i-1} - p_{i-2}| \cdot |p_i|}{|p_{i+1} - p_i| + |p_{i-1} - p_{i-2}|} \leq \max\{|p_{i-1}|, |p_i|\}, \forall i = \overline{0, n},$$

that is $|m_0| \leq \frac{3}{\underline{h}}\varpi(f,h)$, $|m_i| \leq \frac{1}{\underline{h}}\varpi(f,h)$, $\forall i = \overline{1, n-1}$ *and* $|m_n| \leq \frac{3}{\underline{h}}\varpi(f,h)$. *Then, the error estimate, in terms of the modulus of continuity, for the Akima's method of interpolation is:*

$$|f(x) - s(x)| \leq \left(1 + \frac{h}{4\underline{h}}\right) \cdot \varpi(f,h), \quad \forall x \in [x_1, x_{n-1}]$$

$$|f(x) - s(x)| \leq \left(1 + \frac{3h}{4\underline{h}}\right) \cdot \varpi(f,h), \quad \forall x \in [x_0, x_1] \cup [x_{n-1}, x_n].$$

For equidistant grids, $h = \underline{h}$, *and we obtain the estimate of the Akima's method:*

$$|f(x) - s(x)| \leq \begin{cases} \frac{5}{4} \cdot \varpi(f,h), & \forall x \in [x_1, x_{n-1}] \\ \frac{7}{4} \cdot \varpi(f,h), & \forall x \in [x_0, x_1] \cup [x_{n-1}, x_n]. \end{cases}$$

Comparing these estimates with the obtained estimates (2.14), (2.15) and (2.16), we observe that, the estimates of our proposed method represent an improvement. Now, considering the Hermite-type natural cubic spline, it is easy to derive the estimate

$$|f(x) - s(x)| \leq \left(1 + \frac{3h^2}{4\underline{h}^2}\right) \cdot \varpi(f,h), \forall x \in [x_0, x_n],$$

and $|f(x) - s(x)| \le \frac{7}{4} \cdot \varpi(f,h)$, *in the equidistant case. Again, the estimates (2.14), (2.15) and (2.16) are better. Since in the estimate (2.16) we recover the same constant* $\frac{5}{4}$ *as in the inequality (2.9) from the previous section (see [73]), we can say that the interpolation procedure proposed here possesses both the properties of the Akima's interpolation method of "natural" derivatives* m_i, $i = \overline{2, n-2}$, *and the property of minimal deviation from the polygonal line (on the first two and last two subintervals) of the method obtained in [73].*

Remark 50 *(see [81]) There is another possibility to optimize at the end-points the Akima's interpolation procedure: to assume the partial smoothness condition*

$$s \in C^2([x_0, x_2] \cup [x_{n-2}, x_n])$$

with natural end conditions $s''(a) = s''(b) = 0$, *that is*

$$s''(x_1 - 0) = s''(x_1 + 0), \quad s''(x_{n-1} - 0) = s''(x_{n-1} + 0), \quad s''(x_0) = s''(x_n) = 0, \tag{2.17}$$

where $s''(x_i - 0) = \lim_{t \to x_i, t < x_i} s''(t)$, $s''(x_i + 0) = \lim_{t \to x_i, t > x_i} s''(t)$. *The obtained cubic spline is* $s \in C^1[x_2, x_{n-2}] \cap C^2([x_0, x_2] \cup [x_{n-2}, x_n])$, *for which the values* m_i, $i = \overline{2, n-2}$ *are computed using the Akima's method (2.11) and the values* m_0, m_1, m_{n-1}, m_n *are obtained solving the systems generated by the conditions (2.17):*

$$\begin{cases} m_0 + \frac{1}{2} m_1 = \frac{3(y_1 - y_0)}{2h_1} = d_0 \\ \frac{h_2}{2(h_1+h_2)} \cdot m_0 + m_1 = -\frac{h_1}{2(h_1+h_2)} \cdot m_2 + \frac{3h_2(y_1-y_0)}{2h_1(h_1+h_2)} + \frac{3h_1(y_2-y_1)}{2h_2(h_1+h_2)} = d_1 \end{cases}$$

and

$$\begin{cases} m_{n-1} + \frac{h_{n-1}}{2(h_{n-1}+h_n)} \cdot m_n = -\frac{h_n}{2(h_{n-1}+h_n)} \cdot m_{n-2} + \frac{3h_n(y_{n-1}-y_{n-2})}{2h_{n-1}(h_{n-1}+h_n)} + \frac{3h_{n-1}(y_n-y_{n-1})}{2h_n(h_{n-1}+h_n)} = d_{n-1} \\ \frac{1}{2} m_{n-1} + m_n = \frac{3(y_n - y_{n-1})}{2h_n} = d_n \end{cases} .$$

It obtains

$$|m_i| \le \frac{12h}{7\underline{h}} \cdot \left(\frac{h}{4\underline{h}} m_j + \frac{3h}{2\underline{h}^2} \cdot \varpi(f,h) \right) \le \frac{3h^2}{\underline{h}^3} \cdot \varpi(f,h)$$

for $i \in \{0,1\}$, $j = 2$, *and* $i \in \{n-1, n\}$, $j = n-2$, *respectively. So,*

$$|f(x) - s(x)| \le \left(1 + \frac{3h^3}{4\underline{h}^3}\right) \cdot \varpi(f,h), \quad \forall x \in [x_0, x_2] \cup [x_{n-2}, x_n].$$

For equidistant grids, it obtains

$$m_0 = \frac{1}{7} m_2 + \frac{9(y_1 - y_0)}{7h} - \frac{3(y_2 - y_1)}{7h}, \quad m_1 = \frac{9(y_1 - y_0)}{7h} - \frac{1}{4} m_2 + \frac{3(y_2 - y_1)}{4h}$$

$$m_{n-1} = \frac{9(y_n - y_{n-1})}{7h} - \frac{1}{4} m_{n-2} + \frac{3(y_{n-1} - y_{n-2})}{4h}$$

$$m_n = \frac{1}{7} m_{n-2} + \frac{9(y_n - y_{n-1})}{7h} - \frac{3(y_{n-1} - y_{n-2})}{7h}$$

and

$$\{|m_0|, |m_n|\} \le \frac{13}{7h} \cdot \varpi(f,h), \quad \{|m_1|, |m_{n-1}|\} \le \frac{16}{7h} \cdot \varpi(f,h).$$

Then,

$$|f(x) - s(x)| \le \begin{cases} \frac{11}{7} \cdot \varpi(f,h), & x \in [x_0, x_2] \cup [x_{n-2}, x_n] \\ \frac{5}{4} \cdot \varpi(f,h), & x \in [x_2, x_{n-2}] \end{cases}$$

and this error estimate is not a significant improvement of the Akima's method in comparison with (2.15) and (2.16).

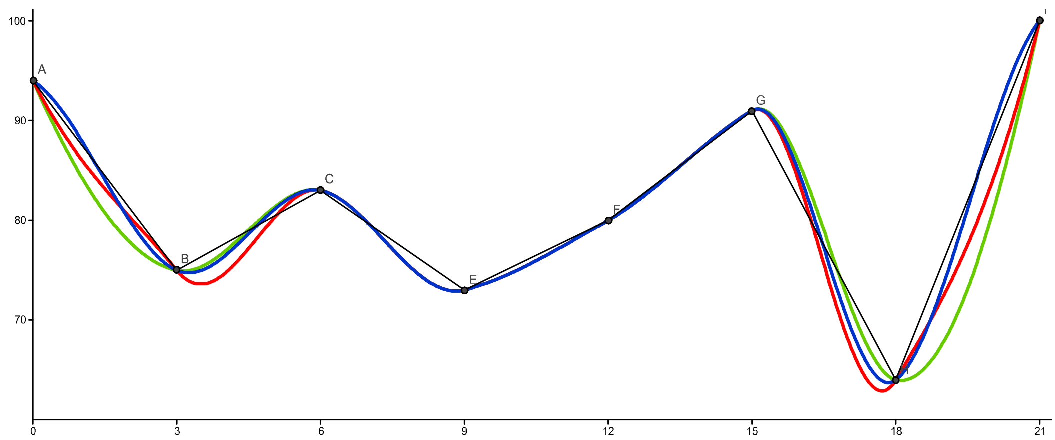

Figure 2. The graphs of the Akima cubic spline (green) and of its end optimizations: natural end-conditions (red) and minimal PQOA (blue)

2.4 Optimization at the end points of the Catmull-Rom's cubic spline

For cubic splines $s \in C^1[a,b]$, there are some ways to compute empirically the values m_i, $i = \overline{0,n}$. One of the simplest procedure is to consider the three-point finite difference

$$m_i = \frac{y_{i+1} - y_i}{2\,(x_{i+1} - x_i)} + \frac{y_i - y_{i-1}}{2\,(x_i - x_{i-1})}, \quad i = \overline{1, n-1}$$

and one-sided difference for the end-points, $m_0 = \frac{y_1 - y_0}{x_1 - x_0}$, $m_n = \frac{y_n - y_{n-1}}{x_n - x_{n-1}}$. Another ways generated by geometric reasons leads to the Kochanek-Bartels splines, and to the cardinal splines with $m_i = (1-c) \cdot \frac{y_{i+1} - y_{i-1}}{x_{i+1} - x_{i-1}}$, $i = \overline{1, n-1}$, and $c \in [0,1]$ be a tension parameter. Two remarkable particular cases of cardinal splines are the following: the Catmull-Rom spline (see [100]) (for $c = 0$) and the cubic spline with zero tangents (when $c = 1$, the maximal value of the tension parameter).

The Catmull-Rom's splines was introduced in [100]. Here, we consider the values of the parameters $m_1, m_2, ..., m_{n-1}$ be obtained by considering the following formula

$$m_i = \frac{y_{i+1} - y_{i-1}}{x_{i+1} - x_{i-1}}, \quad i = \overline{1, n-1} \tag{2.18}$$

inspired by the expressions of the tangents for the Catmull-Rom's cubic splines (see [100]), the parameters m_0 and m_n remaining free. In the case of periodic cubic splines the values m_0 and m_n are not free, being obtained by using the same formula (2.18). In the nonperiodic case we have obtained in [81], the values m_0 and m_n such that the PQOA of the Catmull-Rom's cubic spline to be minimized on the intervals $[x_0, x_1]$ and $[x_{n-1}, x_n]$, that corresponds to the subset $K = \{1, n\} \subset \{1, 2, ..., n\}$.

Proposition 51 *(see [81]) For given data* (x_i, y_i), $i = \overline{0,n}$, *and with the values* m_i, $i = \overline{1, n-1}$, *be computed using (2.18), there are uniquely determined the values* m_0 *and* m_n *such that the partial quadratic oscillation in average,* $\rho(K)(s)$ *is minimal. The corresponding error estimate for* $f \in C([a,b], \Delta, y)$ *is:*

$$|s(x) - f(x)| \leq \left(1 + \frac{h}{4\underline{h}}\right) \cdot \varpi(f, h), \quad \forall x \in [a,b].$$

For the set $K = \{1, n\}$, is considered the residual

$$R_K(s)(m_0, m_n) = \int_{x_0}^{x_1} [s(x) - \frac{x_i - x}{h_i} \cdot y_{i-1} - \frac{x - x_{i-1}}{h_i} \cdot y_i]^2 dx +$$

$$+ \int_{x_{n-1}}^{x_n} [s(x) - \frac{x_i - x}{h_i} \cdot y_{i-1} - \frac{x - x_{i-1}}{h_i} \cdot y_i]^2 dx$$

and the least squares method is applied. It obtains,

$$\begin{cases} m_0 = \frac{3}{4} \cdot m_1 + \frac{y_1 - y_0}{4h_1} = \frac{3}{4} \cdot \frac{y_2 - y_0}{h_1 + h_2} + \frac{y_1 - y_0}{4h_1} \\ m_n = \frac{3}{4} \cdot m_{n-1} + \frac{y_n - y_{n-1}}{4h_n} = \frac{3}{4} \cdot \frac{y_n - y_{n-2}}{h_n + h_{n-1}} + \frac{y_n - y_{n-1}}{4h_n} \end{cases}.$$

Remark 52 *(see [81]) We can consider another method to obtain the free values m_0 and m_n, observing that on the intervals $[x_0, x_1)$ and $(x_{n-1}, x_n]$, the Hermite type cubic spline s has continuous second derivative. We can impose natural end conditions $s''(a) = s''(b) = 0$, which lead to the equations*

$$\begin{cases} m_0 + \frac{1}{2} m_1 = \frac{3(y_1 - y_0)}{2h_1} \\ \frac{1}{2} m_{n-1} + m_n = \frac{3(y_n - y_{n-1})}{2h_n} \end{cases}$$

obtaining

$$\begin{cases} m_0 = -\frac{1}{2} m_1 + \frac{3(y_1 - y_0)}{2h_1} = -\frac{1}{2} \cdot \frac{y_2 - y_0}{h_1 + h_2} + \frac{3(y_1 - y_0)}{2h_1} \\ m_n = -\frac{1}{2} m_{n-1} + \frac{3(y_n - y_{n-1})}{2h_n} = -\frac{1}{2} \cdot \frac{y_n - y_{n-2}}{h_n + h_{n-1}} + \frac{3(y_n - y_{n-1})}{2h_n} \end{cases}$$

and

$$|m_0| \leq \frac{1}{2} \cdot |m_1| + \frac{3\varpi(f, h_1)}{2h_1} \leq \frac{2\varpi(f, h)}{\underline{h}}$$

$$|m_n| \leq \frac{1}{2} \cdot |m_{n-1}| + \frac{3\varpi(f, h_n)}{2h_n} \leq \frac{2\varpi(f, h)}{\underline{h}}.$$

So, the corresponding error estimate is

$$|s(x) - f(x)| \leq \begin{cases} \left(1 + \frac{h}{2\underline{h}}\right) \cdot \varpi(f, h), & x \in [x_0, x_1] \cup [x_{n-1}, x_n] \\ \left(1 + \frac{h}{4\underline{h}}\right) \cdot \varpi(f, h), & x \in [x_1, x_{n-1}] \end{cases}.$$

2.5 Splines generated by initial conditions

In this section we investigate what become the quadratic and cubic splines generated by initial conditions, when the free initial values of the first and of the second derivative, respectively, on the first knot, are determined such that the PQOA is minimized on the first subinterval for $K = \{1\}$.

Consider a function $f : [a, b] \to \mathbb{R}$, and the partition

$$\Delta : a = x_0 < x_1 < ... < x_{n-1} < x_n = b.$$

Let $y_0, y_1, ..., y_n$ be given such that $y_i = f(x_i)$, $\forall i = \overline{0, n}$ and denote $h_i = x_i - x_{i-1}$, $\forall i = \overline{1, n}$.

Quadratic spline generated by initial conditions

For a quadratic spline $s : [a, b] \longrightarrow \mathbb{R}$, $s \in C^1[a, b]$ interpolating the function $f : [a, b] \to \mathbb{R}$, on the knots of the partition Δ we use the notations $y_i = s(x_i)$, $m_i = s'(x_i)$,

$\forall i=\overline{0,n}$. The restrictions of s to the intervals $[x_{i-1},x_i]$, $i=\overline{1,n}$ will be denoted by s_i. These restrictions are determined solving the Cauchy problems:

$$\begin{cases} s_i'(x) = m_{i-1} + \frac{m_i-m_{i-1}}{h_i}\cdot(x-x_{i-1}), & x\in[x_{i-1},x_i] \\ s_i(x_{i-1}) = y_{i-1} \end{cases}, i=\overline{1,n} \tag{2.19}$$

and obtaining,

$$s_i(x) = \frac{m_i-m_{i-1}}{2h_i}\cdot(x-x_{i-1})^2 + m_{i-1}\cdot(x-x_{i-1}) + y_{i-1}, \quad \forall x\in[x_{i-1},x_i], \forall i=\overline{1,n}. \tag{2.20}$$

The continuity requirement $s\in C[a,b]$ lead to $s_i(x_i)=y_i, \forall i=\overline{1,n}$. We see that $s_i'(x_{i-1})=m_{i-1}$, and $s_i'(x_i)=m_i$, $\forall i=\overline{1,n}$, automatic holds. From the smoothness conditions it follows the relations $y_i-y_{i-1}=(m_i-m_{i-1})\cdot\frac{h_i}{2}+m_{i-1}h_i$, $i=\overline{1,n}$, which can be written in recurrent form as,

$$m_i = \frac{2}{h_i}\cdot(y_i-y_{i-1}) - m_{i-1}, \quad \forall i=\overline{1,n}. \tag{2.21}$$

From the recurrence relations (2.21) it follows that for given values $y_0, y_1, ..., y_n$ and m_0, there exists a unique quadratic spline $s\in C^1[a,b]$ such that $s(x_i)=y_i$, $\forall i=\overline{0,n}$ and $s'(x_0)=m_0$.

In that follows, the value m_0 will be determined in order to minimize the PQOA for $K=\{1\}$. So, let

$$R(m_0) = \int_{x_0}^{x_1}\left[s(x) - \frac{x_1-x}{h_1}\cdot y_0 - \frac{x-x_0}{h_1}\cdot y_1\right]^2 dx, \text{ and } \rho(K)(s) = \sqrt{R(m_0)}.$$

We see that

$$R(m_0) = \int_{x_0}^{x_1}\left[\left((x-x_0)-\frac{(x-x_0)^2}{h_1}\right)\cdot m_0 - \frac{y_1-y_0}{h_1}\cdot\left((x-x_0)-\frac{(x-x_0)^2}{h_1}\right)\right]^2 dx$$

because, $m_1=-m_0+\frac{2(y_1-y_0)}{h_1}$. Minimizing $R(m_0)$ we get the corresponding value of m_0 by $\frac{\partial R}{\partial m_0}=0$. Since

$$\frac{\partial R}{\partial m_0} = 2m_0\cdot\int_{x_0}^{x_1}\left((x-x_0)-\frac{(x-x_0)^2}{h_1}\right)^2 dx - \frac{2(y_1-y_0)}{h_1}\cdot\int_{x_0}^{x_1}\left((x-x_0)-\frac{(x-x_0)^2}{h_1}\right)^2 dx$$

we obtain $m_0=\frac{y_1-y_0}{h_1}$. We see that $m_1=\frac{2(y_1-y_0)}{h_1}-\frac{y_1-y_0}{h_1}=m_0$ and $s_1(x)=y_0+\frac{y_1-y_0}{h_1}\cdot(x-x_0)$, $x\in[x_0,x_1]$, becomes first order polynomial. Now, using the numerical differentiation formula we obtain the following result.

Theorem 53 *(see [84]) If $f:[a,b]\to\mathbb{R}$ is a continuous function such that $f(x_i)=y_i$, $\forall i=\overline{0,n}$ and if $s:[a,b]\longrightarrow\mathbb{R}$, $s\in C^1[a,b]$, is the quadratic spline interpolating f with $m_0=\frac{y_1-y_0}{h_1}$, then*

$$|s(x)-f(x)| \le \left(1+\frac{h}{\underline{h}}\right)\cdot\omega(f,h) + O(h^2), \quad \forall x\in[a,b]$$

where $h = \max\{h_i : i = \overline{1,n}\}$ *and* $\underline{h} = \min\{h_i : i = \overline{1,n}\}$. *For* $f \in C^1[a,b]$ *then*

$$|s(x) - f(x)| \le \left(h + \frac{h^2}{\underline{h}}\right) \cdot \|f'\|_\infty + O\left(h^2\right), \quad \forall x \in [a,b]$$

with $\|f'\|_\infty = \max\{|f'(x)| : x \in [a,b]\}$. *In the case of equidistant knots we have*

$$|s(x) - f(x)| \le \begin{cases} 2\omega(f,h) + O\left(h^2\right), & \forall x \in [a,b], \quad \text{if } f \in C[a,b] \\ \frac{2(b-a)}{n} \cdot \|f'\|_\infty + O\left(h^2\right), & \forall x \in [a,b], \quad \text{if } f \in C^1[a,b]. \end{cases}$$

In the proof of this theorem, by using the numerical differentiation formula for $i = \overline{1,n}$,

$$m_i = s_i'(x_i) = \frac{s(x_i) - s(x_{i-1})}{h_i} + O(h) = \frac{f(x_i) - f(x_{i-1})}{h_i} + O(h)$$

and

$$|m_i| \le \left|\frac{f(x_i) - f(x_{i-1})}{h_i}\right| + O(h) \le \frac{1}{\underline{h}} \cdot \omega(f,h) + O(h), \quad \forall i = \overline{1,n}.$$

So,

$$\|m\|_\infty = \max\{|m_i| : i = \overline{0,n}\} \le \frac{1}{\underline{h}} \cdot \omega(f,h) + O(h).$$

Thus,

$$|s(x) - f(x)| = \left|\frac{(x - x_{i-1})^2}{2h_i} \cdot m_i + \left[(x - x_{i-1}) - \frac{(x - x_{i-1})^2}{2h_i}\right] \cdot m_{i-1} + f(x_{i-1}) - f(x)\right| \le$$

$$\le \left[\left|\frac{(x - x_{i-1})^2}{2h_i}\right| + \left|(x - x_{i-1}) - \frac{(x - x_{i-1})^2}{2h_i}\right|\right] \cdot \|m\|_\infty + \omega(f,h) \le$$

$$= \left(1 + \frac{h}{\underline{h}}\right) \cdot \omega(f,h) + O\left(h^2\right), \quad \forall x \in [x_{i-1}, x_i],\ \forall i = \overline{1,n}.$$

In the case $f \in C^1[a,b]$ we have $\omega(f,h) \le \|f'\|_\infty \cdot h$, and

$$|s(x) - f(x)| \le \left(h + \frac{h^2}{\underline{h}}\right) \cdot \|f'\|_\infty + O\left(h^2\right), \quad \forall x \in [x_{i-1}, x_i],\ \forall i = \overline{1,n}.$$

For equidistant knots, $h_i = h = \underline{h} = \frac{b-a}{n}$, $\forall i = \overline{1,n}$ and

$$|s(x) - f(x)| \le \begin{cases} 2\omega(f,h) + O\left(h^2\right), & \forall x \in [a,b], \quad \text{if } f \in C[a,b] \\ 2h \cdot \|f'\|_\infty + O\left(h^2\right), & \forall x \in [a,b], \quad \text{if } f \in C^1[a,b]. \end{cases}$$

Cubic splines generated by initial conditions

In [145], Crăciun Iancu introduced the cubic spline $s \in C^2[a,b]$ generated by initial conditions, having the restrictions to the intervals $[x_{i-1}, x_i]$, $i = \overline{1,n}$

$$s_i(x) = \frac{1}{6h_i} \cdot (M_i - M_{i-1})(x - x_{i-1})^3 + \frac{M_{i-1}}{2} \cdot (x - x_{i-1})^2 + m_{i-1}(x - x_{i-1}) + y_{i-1} \quad (2.22)$$

where in traditional notations, $y_i = s(x_i)$, $m_i = s'(x_i)$, $M_i = s''(x_i)$, $i = \overline{0,n}$. For given $y = (y_0, y_1, ..., y_n) \in \mathbb{R}^{n+1}$, it is proved in [145] that for any choice of the values m_0 and M_0, the cubic spline generated by initial conditions is uniquely determined using the recurrent relations

$$\begin{cases} M_i = \frac{6}{h_i^2} \cdot (y_i - y_{i-1}) - \frac{6m_{i-1}}{h_i} - 2M_{i-1} \\ m_i = \frac{3}{h_i} \cdot (y_i - y_{i-1}) - 2m_{i-1} - \frac{M_{i-1}h_i}{2} \end{cases}, \ i = \overline{1,n}. \quad (2.23)$$

These relations are obtained by the smoothness conditions. In [84], the values m_0 and M_0 were determined such that the PQOA of s to be minimized on the first subinterval, for $K=\{1\}$.

Observing that (2.22) can be written as

$$s_i(x)=\frac{(x-x_{i-1})^3}{6h_i}\cdot M_i+\left[\frac{(x-x_{i-1})^2}{2}-\frac{(x-x_{i-1})^3}{6h_i}\right]\cdot M_{i-1}+(x-x_{i-1})\cdot m_{i-1}+y_{i-1}$$

and since $M_1=-2M_0-\frac{6m_0}{h_1}+\frac{6}{h_1^2}\cdot(y_1-y_0)$, it obtains

$$s_1(x)=\left[\frac{(x-x_0)^2}{2}-\frac{(x-x_0)^3}{2h_1}\right]\cdot M_0+\left[(x-x_0)-\frac{(x-x_0)^3}{h_1^2}\right]\cdot m_0+$$

$$+\left(\frac{x-x_0}{h_1}\right)^3\cdot y_1+\left[1-\left(\frac{x-x_0}{h_1}\right)^3\right]\cdot y_0=A(x)\cdot M_0+B(x)\cdot m_0+C(x)\cdot y_1+D(x)\cdot y_0$$

and therefore can be considered the residual

$$R(m_0,M_0)=\int_{x_0}^{x_1}[A(x)\cdot M_0+B(x)\cdot m_0+C(x)\cdot y_1+D(x)\cdot y_0-\frac{x_1-x}{h_1}\cdot y_0-\frac{x-x_0}{h_1}\cdot y_1]^2dx.$$

Applying the least squares method, the system of normal equations is

$$\begin{cases}\frac{\partial R}{\partial m_0}=0\\ \frac{\partial R}{\partial M_0}=0\end{cases}\iff\begin{cases}\frac{h_1^5}{420}\cdot M_0+\frac{11h_1^4}{840}\cdot m_0=\frac{11h_1^3}{840}\cdot(y_1-y_0)\\ \frac{11h_1^4}{840}\cdot M_0+\frac{8h_1^3}{105}\cdot m_0=\frac{8h_1^2}{105}\cdot(y_1-y_0)\end{cases}$$

having the solution $m_0=\frac{y_1-y_0}{h_1}$, $M_0=0$. Since

$$\Delta=\begin{vmatrix}\frac{h_1^5}{420} & \frac{11h_1^4}{840}\\ \frac{11h_1^4}{840} & \frac{8h_1^3}{105}\end{vmatrix}=\frac{h_1^8}{100800}>0\text{ and }\delta=\frac{h_1^5}{420}>0$$

we infer that $\left(\frac{y_1-y_0}{h_1},0\right)$ minimizes $R(m_0,M_0)$. Moreover, it obtains, $s_1(x)=\frac{x_1-x}{h_1}\cdot y_0+\frac{x-x_0}{h_1}\cdot y_1$, $x\in[x_0,x_1]$, that is first order polynomial. In the following, we present the error estimate of this cubic spline with $m_0=\frac{y_1-y_0}{h_1}$ and $M_0=0$, in terms of the modulus of continuity. The error estimate for the spline (2.22), with arbitrary m_0 and M_0, using divided differences is obtained in [146].

Theorem 54 *(see [84]) The error estimate of the cubic spline (2.22) with* $m_0=\frac{y_1-y_0}{h_1}$ *and* $M_0=0$, *interpolating a continuous function* $f:[a,b]\to\mathbb{R}$ *on the knots* x_i, $i=\overline{0,n}$, *is:*

$$|s(x)-f(x)|\le\begin{cases}\left(1+\frac{h}{\underline{h}}+\frac{6h^2}{\underline{h}^2}\right)\cdot\omega(f,h)+O\left(\frac{h^3}{\underline{h}}\right)+O\left(h^2\right), & x\in[x_1,x_n]\\ \omega(f,h), & x\in[x_0,x_1]\end{cases}\tag{2.24}$$

where $h=\max\{h_i:i=\overline{1,n}\}$ *and* $\underline{h}=\min\{h_i:i=\overline{1,n}\}$.

Sketch of proof: Using the numerical differentiation formula it obtains similarly, $|m_i|\le\frac{1}{\underline{h}}\cdot\omega(f,h)+O(h)$, for any $i=\overline{1,n}$, and $|m_0|\le\frac{1}{\underline{h}}\cdot\omega(f,h)$. From the recurrent relations (2.23) it follows

$$M_i=\frac{2m_{i-1}}{h_i}+\frac{4m_i}{h_i}-\frac{6}{h_i^2}\cdot(y_i-y_{i-1}),\quad i=\overline{1,n}$$

and $|M_i| \le \frac{6}{h_i} \cdot \max\{|m_{i-1}|, |m_i|\} + \frac{6}{h_i^2} \cdot \omega(f, h_i) \le \frac{12}{\underline{h}^2} \cdot \omega(f, h) + O\left(\frac{h}{\underline{h}}\right)$, $\forall i = \overline{1,n}$. So,

$$|s(x) - f(x)| \le |M_i| \cdot \left|\frac{(x - x_{i-1})^3}{6h_i}\right| + |M_{i-1}| \cdot \left|\frac{(x - x_{i-1})^2}{2} - \frac{(x - x_{i-1})^3}{6h_i}\right| +$$

$$+|m_{i-1}| \cdot (x - x_{i-1}) + |y_{i-1} - f(x)| \le \frac{h^2}{2} \cdot \left(\frac{12}{\underline{h}^2} \cdot \omega(f, h) + O\left(\frac{h}{\underline{h}}\right)\right) + \frac{h}{\underline{h}} \cdot \omega(f, h) +$$

$$+O\left(h^2\right) + \omega(f, h) \le \left(1 + \frac{h}{\underline{h}} + \frac{6h^2}{\underline{h}^2}\right) \cdot \omega(f, h) + O\left(\frac{h^3}{\underline{h}}\right) + O\left(h^2\right), \quad \forall x \in [x_1, x_n].$$

On $[x_1, x_n]$, for equidistant knots, the error estimate becomes

$$|s(x) - f(x)| \le 8\omega(f, h) + O\left(h^2\right), \quad \forall x \in [x_1, x_n].$$

Remark 55 *The quartic spline $s \in C^3[a, b]$ generated by initial conditions is obtained by solving on each interval $[x_{i-1}, x_i]$, $i = \overline{1,n}$, the initial value problems*

$$\begin{cases} s_i'''(x) = V_{i-1} + \frac{V_i - V_{i-1}}{h_i} \cdot (x - x_{i-1}), \quad x \in [x_{i-1}, x_i] \\ s_i(x_{i-1}) = y_{i-1}, \quad s_i'(x_{i-1}) = m_{i-1}, \quad s_i''(x_{i-1}) = M_{i-1}. \end{cases}$$

Thus,

$$s_i(x) = \frac{V_i - V_{i-1}}{24h_i} \cdot (x - x_{i-1})^4 + \frac{V_{i-1}}{6} \cdot (x - x_{i-1})^3 + \frac{M_{i-1}}{2} \cdot (x - x_{i-1})^2 + m_{i-1}(x - x_{i-1}) + y_{i-1} \tag{2.25}$$

and the smoothness requirements $s \in C[a, b]$, $s \in C^1[a, b]$, $s \in C^2[a, b]$ lead to the conditions $s_i(x_i) = y_i$, $s_i'(x_i) = m_i$, $s_i''(x_i) = M_i$, $i = \overline{1,n}$, obtaining the recurrent relations

$$\begin{cases} m_i = \frac{4(y_i - y_{i-1})}{h_i} - 3m_{i-1} - h_i M_{i-1} - \frac{h_i^2}{6} \cdot V_{i-1} \\ M_i = \frac{12(y_i - y_{i-1})}{h_i^2} - \frac{12}{h_i} \cdot m_{i-1} - 5M_{i-1} - h_i V_{i-1} \quad , \quad i = \overline{1,n}. \\ V_i = \frac{24(y_i - y_{i-1})}{h_i^3} - \frac{24m_{i-1}}{h_i^2} - \frac{12}{h_i} \cdot M_{i-1} - 3V_{i-1} \end{cases} \tag{2.26}$$

Inserting V_i from the third relation of (2.26) into (2.25) we get

$$s_i(x) = \left[\frac{(x - x_{i-1})^3}{6} - \frac{(x - x_{i-1})^4}{6h_i}\right] \cdot V_{i-1} + \left[\frac{(x - x_{i-1})^2}{2} - \frac{(x - x_{i-1})^4}{2h_i^2}\right] \cdot M_{i-1} +$$

$$+\left[(x - x_{i-1}) - \frac{(x - x_{i-1})^4}{h_i^3}\right] \cdot m_{i-1} + y_{i-1} + \frac{(x - x_{i-1})^4}{h_i^4} \cdot (y_i - y_{i-1}) = A_i(x) \cdot V_{i-1} +$$

$$+B_i(x) \cdot M_{i-1} + C_i(x) \cdot m_{i-1} + y_{i-1} + D_i(x) \cdot (y_i - y_{i-1}), \quad x \in [x_{i-1}, x_i], \ i = \overline{1,n}.$$

For the n+3 unknown m_i, M_i, V_i, $i = \overline{0,n}$, we have the 3n equations (2.26) remaining three free parameters m_0, M_0 and V_0. The values of m_0, M_0 and V_0 can be obtained such that the PQOA of s to be minimized for $K = \{1\}$. In this purpose, we consider the residual

$$R(m_0, M_0, V_0) = \int_{x_0}^{x_1} [A_1(x) \cdot V_0 + B_1(x) \cdot M_0 + C_1(x) \cdot m_0 + \left(D_0(x) - \frac{x - x_0}{h_1}\right)(y_1 - y_0)]^2 dx$$

and applying the least squares method to this residual it obtains $m_0 = \frac{y_1 - y_0}{h_1}$, $M_0 = V_0 = 0$. Since

$$\Delta = \begin{vmatrix} \frac{h_1^3}{9} & \frac{13h_1^4}{504} & \frac{7h_1^5}{2160} \\ \frac{13h_1^4}{504} & \frac{2h_1^5}{315} & \frac{5h_1^6}{6048} \\ \frac{7h_1^5}{2160} & \frac{5h_1^6}{6048} & \frac{h_1^7}{9072} \end{vmatrix} = \frac{h_1^{15}}{64012032000} > 0, \quad \delta = \begin{vmatrix} \frac{h_1^3}{9} & \frac{13h_1^4}{504} \\ \frac{13h_1^4}{504} & \frac{2h_1^5}{315} \end{vmatrix} = \frac{17h_1^8}{423360} > 0$$

and $\delta_0 = \frac{h_1^3}{9} > 0$, *we infer that* $\left(\frac{y_1 - y_0}{h_1}, 0, 0\right)$ *minimize* $R(m_0, M_0, V_0)$. *We see that in this case,* $s_1(x) = y_0 + \frac{y_1 - y_0}{h_1} \cdot (x - x_0)$, $x \in [x_0, x_1]$, *becoming first order polynomial. We conclude that minimizing partially the PQOA on the first subinterval for quadratic, cubic, and quartic spline generated by initial conditions, these splines become first order polynomial on this first interval* $[x_0, x_1]$, *remaining second, third, and quartic order polynomials on the other subintervals* $[x_{i-1}, x_i]$, $i = \overline{2, n}$. *But for other splines this is not true. For instance, in the previous sections we have observed that the Akima's Hermite type* C^1 *cubic spline interpolation procedure was modified on the knots* x_0, x_1, x_{n-1}, x_n *(that is* $K = \{1, 2, n-1, n\}$*) in order to minimize the corresponding PQOA, and the obtained cubic spline remains third order polynomial on each subinterval.*

Remark 56 *Similarly, the partial minimization of the PQOA on the set* $K = \{1, n\}$ *for the Hermite type cubic spline* $s \in C^2[a, b]$ *do not conduct to first order polynomials on the first and on the last subintervals. Indeed, since the smoothness requirements lead to the n-1 equations (2.4) for the unknown* $m_0, m_1, ..., m_n$, *and by minimizing the PQOA on the intervals* $[x_0, x_1]$ *and* $[x_{n-1}, x_n]$, *we obtain (from the normal equations* $\frac{\partial R}{\partial m_0} = 0$, $\frac{\partial R}{\partial m_n} = 0$*) another two equations*

$$\begin{cases} m_0 - \frac{3}{4} m_1 = \frac{1}{4h_1} \cdot (y_1 - y_0) \\ -\frac{3}{4} m_{n-1} + m_n = \frac{1}{4h_n} \cdot (y_n - y_{n-1}) . \end{cases}$$

The solution $(m_0, m_1, ..., m_n)$ *of the system composed by these n+1 equations leads to a unique cubic spline of Hermite type which has third order polynomials on each subinterval.*

Chapter 3

The method of successive interpolations

3.1 Introduction

In this chapter we present a synthesis of the results obtained in [83], [56], [69], [57]-[72], [74], [78], [79], [82], [85]-[87] concerning the method of successive interpolations recently introduced by the author for functional differential and integral equations. This method combines the technique of successive approximations with a suitable interpolation procedure activated at each iterative step. This interpolation procedure are usually quadratic or cubic splines. In the convergence analysis of this method we obtain the *apriori* and *a posteriori* error estimates and we have introduced a new concept of numerical stability: *the numerical stability with respect to the first iteration.*

Firstly, we present the development of this method for initial value problems associated to differential equations of first and second order with vanishing delay, investigating the convergence and the numerical stability with respect to the initial values (in this case, the numerical stability with respect to the first iteration it reduces to those with respect to the initial values). Secondly, the method of successive interpolations is applied to two-point boundary value problems associated to differential equations of second and fourth order with deviating argument. For these problems, the numerical stability with respect to the first iteration becomes the numerical stability with respect to the boundary values. As a generalization of these problems, we approach afterward, the cases of Hammerstein, Fredholm and Volterra functional integral equations.

3.2 The natural cubic spline interpolation procedure

In this first section, we present the natural cubic spline interpolation procedure used in [72], [71], [74], [87], and [79], in the construction of the method of successive interpolations.

In [145], C. Iancu had proposed a cubic spline of interpolation of given values $y_0, ..., y_n$ corresponding to the knots $x_0, ..., x_n \in [a, b]$, $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$. This cubic spline $s \in C^2[a, b]$ can be called as generated by initial conditions and has the expression:

$$s_i\left(x\right) = \frac{1}{6h_i} \cdot (M_i - M_{i-1})(x - x_{i-1})^3 + \frac{M_{i-1}}{2} \cdot (x - x_{i-1})^2 + m_{i-1}(x - x_{i-1}) + y_{i-1} \quad (3.1)$$

where $h_i = x_i - x_{i-1}$, $s_i = s\mid_{[x_{i-1},x_i]}$, $i = \overline{1, n}$, and $y_i = s\left(x_i\right), m_i = s'\left(x_i\right), M_i = s''\left(x_i\right),$

$i=\overline{0,n}$. The conditions $s_i(x_i)=y_i$, $i=\overline{1,n}$ lead to

$$m_{i-1}=\frac{y_i-y_{i-1}}{h_i}-\frac{h_i(M_i+2M_{i-1})}{6},\quad i=\overline{1,n}. \tag{3.2}$$

Replacing (3.2) in (3.1) we obtain the following equivalent form of (3.1) expressed in terms of the moments M_i, $i=\overline{0,n}$:

$$s_i(x)=\left[\frac{(x-x_{i-1})^2}{2}-\frac{(x-x_{i-1})^3}{6h_i}-\frac{h_i(x-x_{i-1})}{3}\right]\cdot M_{i-1}+\frac{x-x_{i-1}}{h_i}\cdot y_i+ \tag{3.3}$$
$$+\left[\frac{(x-x_{i-1})^3}{6h_i}-\frac{h_i(x-x_{i-1})}{6}\right]\cdot M_i+\frac{x_i-x}{h_i}\cdot y_{i-1},\quad x\in[x_{i-1},x_i].$$

C. Iancu obtained in [145] the existence and uniqueness of the cubic spline (3.1) for given $y_0,...,y_n$, m_0, M_0 and the following system of recurrent relations that gives the values m_i, M_i, $i=\overline{1,n}$ starting from $y_0,...,y_n$, m_0, M_0:

$$\begin{cases} M_i=\frac{6}{h_i^2}\cdot(y_i-y_{i-1})-\frac{6m_{i-1}}{h_i}-2M_{i-1} \\ m_i=\frac{3}{h_i}\cdot(y_i-y_{i-1})-2m_{i-1}-\frac{M_{i-1}h_i}{2}\end{cases},\ i=\overline{1,n}. \tag{3.4}$$

In [74], we have proposed an algorithm to compute the values M_i, $i=\overline{1,n-1}$, for this cubic spline with natural boundary conditions $s''(a)=s''(b)=0$ in a recurrent way obtaining the error estimate in the spline interpolation (3.3) of uniformly continuous functions. Since $s\in C^2[a,b]$, the requirement $s\in C^1[a,b]$ lead to the conditions $s_i'(x_i)=s_{i+1}'(x_i)$, $i=\overline{1,n-1}$, obtaining the following linear system:

$$\frac{h_i}{6}\cdot M_{i-1}+\frac{h_i+h_{i+1}}{3}\cdot M_i+\frac{h_{i+1}}{6}\cdot M_{i+1}=\frac{y_{i+1}-y_i}{h_{i+1}}-\frac{y_i-y_{i-1}}{h_i},\ i=\overline{1,n-1} \tag{3.5}$$

to be solved for M_i, $i=\overline{0,n}$. Assuming the natural boundary conditions $s''(a)=s''(b)=0$, we have $M_0=M_n=0$.

An algorithm (obtained applying the method presented in [8], pages 14-15) in recurrent form, which gives the solutions of the system (3.5), is the following:

Firstly, let $a_i=1$, $i=\overline{1,n-1}$, $b_1=0$, $c_1=\frac{h_2}{2(h_1+h_2)}$, $b_i=\frac{h_i}{2(h_i+h_{i+1})}$, $c_i=\frac{1}{2}-b_i$, $i=\overline{2,n-2}$, $b_{n-1}=\frac{h_{n-1}}{2(h_{n-1}+h_n)}$, $c_{n-1}=0$ and $d_i=\frac{3(y_{i+1}-y_i)}{h_{i+1}(h_i+h_{i+1})}-\frac{3(y_i-y_{i-1})}{h_i(h_i+h_{i+1})}$, $i=\overline{1,n-1}$. Recurrently, it computes

$$\alpha_1=\frac{c_1}{a_1},\quad \omega_i=a_i-\alpha_{i-1}\cdot b_i,\quad \alpha_i=\frac{c_i}{\omega_i},\quad i=\overline{2,n-2},$$

and

$$\omega_{n-1}=a_{n-1}-\alpha_{n-2}\cdot b_{n-1}, z_1=\frac{d_1}{2},\quad z_i=\frac{d_i-b_i\cdot z_{i-1}}{\omega_i},\quad i=\overline{2,n-1}.$$

Finally, with backward recurrence we obtain the moments:

$$M_0=M_n=0,\quad M_{n-1}=z_{n-1},\quad M_i=z_i-\alpha_i\cdot M_{i+1},\quad i=\overline{n-2,1}. \tag{3.6}$$

Theorem 57 *(see [74]) Let $h=\max\{h_i: i=\overline{1,n}\}$, $\overline{h}=\min\{h_i: i=\overline{1,n}\}$ and $\beta\geq 1$ be such that $\frac{h}{\overline{h}}\leq\beta$. If $f:[a,b]\to\mathbb{R}$ is an uniformly continuous function and $s\in C^2[a,b]$ is cubic spline of interpolation generated by initial conditions, with natural boundary conditions $s''(a)=s''(b)=0$, such that $s(x_i)=f(x_i)=y_i$, $\forall i=\overline{0,n}$, then the following error estimation holds:*

$$\max_{x\in[a,b]}|s(x)-f(x)|\leq\frac{3\beta^2}{4}\cdot\omega(f,h)+\omega(f,h) \tag{3.7}$$

where $\omega(f,h)=\sup\{|f(x)-f(x')|: x,x'\in[a,b], |x-x'|\leq h\}$ is the uniform modulus of continuity.

Sketch of proof: The system (3.5) with $M_0 = M_n = 0$ can be written in the diagonally dominant form:

$$\begin{cases} \frac{h_1+h_2}{3} \cdot M_1 + \frac{h_2}{6} \cdot M_2 = \frac{y_2-y_1}{h_2} - \frac{y_1-y_0}{h_1} \\ \frac{h_i}{6} \cdot M_{i-1} + \frac{h_i+h_{i+1}}{3} \cdot M_i + \frac{h_{i+1}}{6} \cdot M_{i+1} = \frac{y_{i+1}-y_i}{h_{i+1}} - \frac{y_i-y_{i-1}}{h_i},\ i = \overline{2, n-2} \\ \frac{h_{n-1}}{6} \cdot M_{n-2} + \frac{h_{n-1}+h_n}{3} \cdot M_{n-1} = \frac{y_n-y_{n-1}}{h_n} - \frac{y_{n-1}-y_{n-2}}{h_{n-1}} \end{cases}$$

and after division by $\frac{h_i+h_{i+1}}{3}$, $i = \overline{1, n-1}$ in each equation we obtain the following form $G \cdot m = d$ of this system with $G = I + A$:

$$\begin{cases} M_1 + \frac{h_2}{2(h_1+h_2)} \cdot M_2 = \frac{3(y_2-y_1)}{h_2(h_1+h_2)} - \frac{3(y_1-y_0)}{h_1(h_1+h_2)} = d_1 \\ \frac{h_i}{2(h_i+h_{i+1})} \cdot M_{i-1} + M_i + \frac{h_{i+1}}{2(h_i+h_{i+1})} \cdot M_{i+1} = \frac{3(y_{i+1}-y_i)}{h_{i+1}(h_i+h_{i+1})} - \frac{3(y_i-y_{i-1})}{h_i(h_i+h_{i+1})} = d_i,\ i = \overline{2, n-2} \\ \frac{h_{n-1}}{2(h_{n-1}+h_n)} \cdot M_{n-2} + M_{n-1} = \frac{3(y_n-y_{n-1})}{h_n(h_{n-1}+h_n)} - \frac{3(y_{n-1}-y_{n-2})}{h_{n-1}(h_{n-1}+h_n)} = d_{n-1}. \end{cases}$$

Since $\|A\|_\infty = \frac{1}{2} < 1$ we infer that the matrix $I + A$ is invertible with

$$\left\|G^{-1}\right\|_\infty = \left\|(I + A)^{-1}\right\|_\infty \leq \frac{1}{1 - \|A\|_\infty} = 2$$

and $m = G^{-1} \cdot d$, where $m = (M_1, ..., M_{n-1})$, $d = (d_1, ..., d_{n-1})$. It is easy to see that

$$\|d\|_\infty = \max\{|d_1|, ..., |d_{n-1}|\} \leq \frac{3\omega(f, h)}{\overline{h}^2}$$

and

$$\|m\|_\infty = \max\{|M_1|, ..., |M_{n-1}|\} \leq \left\|G^{-1}\right\|_\infty \cdot \|d\|_\infty \leq \frac{6\omega(f, h)}{\overline{h}^2}.$$

Then,

$$|s(x) - f(x)| \leq |M_{i-1}| \cdot \left|\frac{(x - x_{i-1})^2}{2} - \frac{(x - x_{i-1})^3}{6h_i} - \frac{h_i(x - x_{i-1})}{3}\right| +$$

$$+ |M_i| \cdot \left|\frac{(x - x_{i-1})^3}{6h_i} - \frac{h_i(x - x_{i-1})}{6}\right| + \left|\frac{x - x_{i-1}}{h_i} \cdot y_i + \frac{x_i - x}{h_i} \cdot y_{i-1} - f(x)\right| \leq$$

$$\leq \frac{6\omega(f, h)}{\overline{h}^2} \cdot \frac{(x - x_{i-1})(x_i - x)}{2} + \omega(f, h), \quad \forall x \in [x_{i-1}, x_i],\ i = \overline{1, n}.$$

So,

$$|s(x) - f(x)| \leq \frac{6\omega(f, h)}{\overline{h}^2} \cdot \frac{h^2}{8} + \omega(f, h) \leq \frac{3\beta^2}{4} \cdot \omega(f, h) + \omega(f, h), \quad \forall x \in [x_{i-1}, x_i],\ i = \overline{1, n}.$$

For uniform partitions it follows that $h = \overline{h} = \frac{b-a}{n}$ and $\beta = 1$. Then in this case,

$$\max_{x \in [a,b]} |s(x) - f(x)| \leq \frac{7}{4} \cdot \omega(f, h). \tag{3.8}$$

3.3 Initial value problems with deviating argument

For first order delay differential equations there are various numerical methods involved in the approximation of the solution, such as: Runge-Kutta techniques (see [35], [38], [93], and [126]), multistep methods (see [153]), collocation methods (see [36], [37], [92], [93], [94], [151], [172], and [185]), spline functions methods (see for instance, [124], [125], [185], and [206]), Adomian decomposition method (see [127]), variational iteration method (see [105]), rational approximation method (see [150]), Bernstein series based on polynomial interpolation (see [152]). The interest for delay and functional differential equations with vanishing delay is motivated by their applications in biomathematics (see [188]) and in electrodynamics, such as the pantograph type equation modelling the current collection of electric locomotives (see [193]).

Consider the initial value problem

$$\begin{cases} x'(t) = f(t, x(t), x(\varphi(t))), & t \in [a,b] \\ x(a) = x_0 \end{cases} \tag{3.9}$$

under the following conditions:

(i) $f \in C([a,b] \times \mathbb{R} \times \mathbb{R})$ and $\varphi \in C[a,b]$ with $a \leq \varphi(t) \leq b$ and $\varphi(t) \leq t$ for all $t \in [a,b]$,

(ii) exist $L_1, L_2 > 0$ such that $|f(s,u,v) - f(s,u',v')| \leq L_1 |u-u'| + L_2 |v - v'|$, for all $s \in [a,b], u, u', v, v' \in \mathbb{R}$;

(iii) $(b-a)(L_1 + L_2) < 1$,

(iv) exist $\alpha, \gamma > 0$ such that $|\varphi(s) - \varphi(s')| \leq \alpha |s - s'|$, for all $s, s' \in [a,b]$ and $|f(s,u,v) - f(s',u,v)| \leq \gamma |s - s'|$, for all $s, s' \in [a,b], u, v \in \mathbb{R}$.

Since f is continuous we infer that there is $M_0 > 0$ such that $|f_0(s)| \leq M_0$, for all $s \in [a,b]$, where $f_0 : [a,b] \to \mathbb{R}$, $f_0(s) = f(s, x_0, x_0)$, $\forall s \in [a,b]$.

Applying the fixed point technique (based on the Picard-Banach principle) to the operator $A : C[a,b] \to C[a,b]$, given by $A(x(t)) = x_0 + \int_a^t f(s, x(s), x(\varphi(s)))\,ds$, $t \in [a,b]$, under the conditions (i)-(iii), we obtain the convergence of the sequence of successive approximations: $x_0(t) = x_0$, $t \in [a,b]$,

$$x_m(t) = x_0 + \int_a^t f(s, x_{m-1}(s), x_{m-1}(\varphi(s)))\,ds, \quad t \in [a,b],\ m \in \mathbb{N}^* \tag{3.10}$$

to the unique solution $x^* \in C[a,b]$, of the initial value problem (3.9). So, $\lim_{m\to\infty} x_m(t) = x^*(t)$ uniformly on $[a,b]$ and the following error estimations holds:

$$|x_m(t) - x^*(t)| \leq \frac{(b-a)(L_1+L_2)}{1-(b-a)(L_1+L_2)} \cdot |x_m(t) - x_{m-1}(t)| \leq \tag{3.11}$$

$$\leq \frac{(b-a)(L_1+L_2)}{1-(b-a)(L_1+L_2)} \cdot \|x_m - x_{m-1}\|_C, \quad \text{for all } t \in [a,b],\ m \in \mathbb{N}^*,$$

$|x_m(t) - x^*(t)| \leq \frac{(b-a)^m(L_1+L_2)^m}{1-(b-a)(L_1+L_2)} \cdot \|x_1 - x_0\|_C$, for all $t \in [a,b]$, $m \in \mathbb{N}^*$. Moreover, since $f \in C([a,b] \times \mathbb{R} \times \mathbb{R})$ we infer that $x^* \in C^1[a,b]$. Because $|x_1(t) - x_0(t)| \leq \int_a^t M_0 ds$, it follows that

$$|x_m(t) - x^*(t)| \leq \frac{(b-a)^m(L_1+L_2)^m}{1-(b-a)(L_1+L_2)} \cdot (b-a) M_0, \quad \text{for all } t \in [a,b],\ m \in \mathbb{N}^*. \tag{3.12}$$

Define the functions $F_m : [a,b] \to \mathbb{R}$, $F_m(s) = f(s, x_m(s), x_m(\varphi(s)))$, for all $s \in [a,b]$, $m \in \mathbb{N}$.

We observe that $|x_m(t) - x_{m-1}(t)| \leq \int_a^t (L_1 + L_2) \|x_{m-1} - x_{m-2}\|_C ds \leq (b-a)(L_1+L_2) \cdot \|x_{m-1} - x_{m-2}\|_C$, and by induction it follows that,

$$|x_m(t) - x_{m-1}(t)| \leq (b-a)^{m-1} (L_1+L_2)^{m-1} \cdot \|x_1 - x_0\|_C,$$

for all $t \in [a,b]$. So, $|x_m(t) - x_0(t)| \leq \sum_{j=1}^{m} |x_j(t) - x_{j-1}(t)| \leq \sum_{j=0}^{m-1} (b-a)^j (L_1+L_2)^j \cdot \|x_1 - x_0\|_C \leq \frac{(b-a)M_0}{1-(b-a)(L_1+L_2)}$ and

$$|x_m(t)| \leq |x_m(t) - x_0(t)| + |x_0(t)| \leq |x_0| + \frac{(b-a)M_0}{1-(b-a)(L_1+L_2)} = R, \forall t \in [a,b], m \in \mathbb{N}^*.$$

Let $M \geq 0$ be

$$M = \max\left(M_0, \max\{|f(t,u,v)| : t \in [a,b], u, v \in [-R,R]\}\right). \tag{3.13}$$

For arbitrary $t, t' \in [a,b]$ we have

$$\left|x_m(t) - x_m(t')\right| \leq \int_t^{t'} |f(s, x_{m-1}(s), x_{m-1}(\varphi(s)))| ds \leq M \left|t - t'\right|,$$

for all $m \in \mathbb{N}^*$, and

$$\left|F_m(t) - F_m(t')\right| \leq \gamma \left|t - t'\right| + L_1 \left|x_m(t) - x_m(t')\right| +$$

$$+ L_2 \left|x_m(\varphi(t)) - x_m(\varphi(t'))\right| \leq [\gamma + (L_1 + \alpha L_2) M] \cdot \left|t - t'\right|,$$

for all $t, t' \in [a,b], m \in \mathbb{N}$. Moreover, we get,

$$\left|x'_m(t) - x'_m(t')\right| \leq \gamma \left|t - t'\right| + L_1 \left|x_{m-1}(t) - x_{m-1}(t')\right| +$$

$$+ L_2 \left|x_{m-1}(\varphi(t)) - x_{m-1}(\varphi(t'))\right| \leq (\gamma + L_1 M + L_2 \alpha M) \left|t - t'\right|,$$

and $|x'_m(t)| \leq M$ for all $t, t' \in [a,b]$, $m \in \mathbb{N}^*$. In this way it obtains:

Theorem 58 *(see [79]) Under the conditions (i)-(iv), the terms of the sequence of successive approximations (3.10) are uniformly bounded and the functions F_m, $m \in \mathbb{N}$, are uniformly bounded and Lipschitzian with the same Lipschitz constant $L = \gamma + (L_1 + \alpha L_2) M$, where $M \geq 0$ is given in (3.13).*

The algorithm

In order to compute the integrals from (3.13) we consider an uniform grid of $[a,b]$ given by the knots $t_i = a + \frac{i(b-a)}{n}$, $i = \overline{0,n}$ and the sequence $(x_m(t_i))_{m \in \mathbb{N}}$, $i = \overline{0,n}$, constructed in the following algorithm: $x_0(t_i) = x_0$, $\forall i = \overline{0,n}$, and $x_m(t_0) = \overline{x_m(t_0)} = x_0$, $\quad \forall m \in \mathbb{N}^*$,

$$x_1(t_i) = x_0 + \frac{(b-a)}{2n} \cdot \sum_{j=1}^{i} [f(t_{j-1}, x_0(t_{j-1}), x_0(\varphi(t_{j-1}))) + \tag{3.14}$$

$$+ f(t_j, x_0(t_j), x_0(\varphi(t_j)))] + R_{1,i} = \overline{x_1(t_i)} + R_{1,i} = \overline{x_1(t_i)} + \overline{R_{1,i}}, \quad i = \overline{1,n}$$

and by induction for $m \in \mathbb{N}^*$, $m \geq 2$, $i = \overline{1,n}$:

$$x_m(t_i) = x_0 + \frac{(b-a)}{2n} \cdot \sum_{j=1}^{i} [f\left(t_{j-1}, \overline{x_{m-1}(t_{j-1})} + \overline{R_{m,j-1}}, x_{m-1}(\varphi(t_{j-1}))\right) + \tag{3.15}$$

$$+f\left(t_j, \overline{x_{m-1}(t_j)} + \overline{R_{m-1,j}}, x_{m-1}(\varphi(t_j))\right)] + R_{m,i} = x_0 + \frac{(b-a)}{2n} \cdot \sum_{j=1}^{i} [f(t_{j-1}, \overline{x_{m-1}(t_{j-1})},$$

$$, s_{m-1}(\varphi(t_{j-1}))) + f\left(t_j, \overline{x_{m-1}(t_j)}, s_{m-1}(\varphi(t_j))\right)] + \overline{R_{m,i}} = \overline{x_m(t_i)} + \overline{R_{m,i}}.$$

In (3.15), $s_{m-1} : [a,b] \to \mathbb{R}$ is the cubic spline generated by initial conditions inspired from (3.3) interpolating the values $\overline{x_{m-1}(t_i)}$, $i = \overline{0,n}$. The restrictions $s_{m-1}^{(i)}$ of s_{m-1} to the intervals $[t_{i-1}, t_i]$, $i = \overline{1,n}$ are:

$$s_{m-1}^{(i)}(t) = \left[\frac{(t-t_{i-1})^2}{2} - \frac{(t-t_{i-1})^3}{6h} - \frac{h(t-t_{i-1})}{3}\right] \cdot M_{m-1}^{(i-1)} + \frac{t-t_{i-1}}{h} \cdot \overline{x_{m-1}(t_i)} + \tag{3.16}$$

$$+\left[\frac{(t-t_{i-1})^3}{6h} - \frac{h(t-t_{i-1})}{6}\right] \cdot M_{m-1}^{(i)} + \frac{t_i - t}{h} \cdot \overline{x_{m-1}(t_{i-1})}, \quad t \in [t_{i-1}, t_i], \ i = \overline{1,n}$$

where $h = \frac{b-a}{n}$ and the values $M_{m-1}^{(i)}$, $i = \overline{0,n}$ are computed using the recurrent algorithm presented in (3.6) with $a_i = 1$, $i = \overline{1,n-1}$, $b_i = c_i = \frac{1}{4}$, $i = \overline{2,n-2}$, $b_1 = c_{n-1} = 0$, $c_1 = b_{n-1} = \frac{1}{4}$ and $d_i = \frac{3}{2h^2} \cdot \left[\overline{x_{m-1}(t_{i+1})} - 2\overline{x_{m-1}(t_i)} + \overline{x_{m-1}(t_{i-1})}\right]$, $i = \overline{1,n-1}$.

The effective computed approximations are $\overline{x_m(t_i)}$, $i = \overline{1,n}$, $m \in \mathbb{N}^*$, and a practical stopping criterion of this algorithm is the following: for given $\varepsilon' > 0$ and given $n \in \mathbb{N}^*$, previously chosen, find the first natural number $m \in \mathbb{N}^*$ such that $\left|\overline{x_m(t_i)} - \overline{x_{m-1}(t_i)}\right| < \varepsilon'$, $\forall i = \overline{1,n}$. We stop at this step m, retaining the approximations $\overline{x_m(t_i)}$, $i = \overline{1,n}$, of the solution.

The convergence analysis

In order to define the numerical stability with respect to the initial value we consider a small perturbation in the initial condition: $x(a) = y_0$. Suppose that for small $\varepsilon > 0$ we have $|x_0 - y_0| < \varepsilon$. Now, applying the above presented algorithm to the initial value problem

$$\begin{cases} x'(t) = f(t, x(t), x(\varphi(t))), & t \in [a,b] \\ x(a) = y_0. \end{cases} \tag{3.17}$$

we obtain the sequence of successive approximations $y_0(t) = y_0$, $t \in [a,b]$,

$$y_m(t) = y_0 + \int_a^t f(s, y_{m-1}(s), y_{m-1}(\varphi(s)))\, ds, \quad t \in [a,b], \ m \in \mathbb{N}^* \tag{3.18}$$

and the effective computed values $\overline{y_m(t_i)}$, $i = \overline{1,n}$, $m \in \mathbb{N}^*$ with $y_m(t_i) = \overline{y_m(t_i)} + \overline{R'_{m,i}}$, $i = \overline{1,n}$, $m \in \mathbb{N}^*$.

Definition 59 *(see [79]) We say that the algorithm of successive interpolations applied to the initial value problem (3.9) is numerically stable with respect to the initial value if there exist $p \in \mathbb{N}^*$, a sequence of continuous functions $\mu_m : [0, b-a] \to [0, \infty)$, $m \in \mathbb{N}^*$ with the property $\lim_{h \to 0} \mu_m(h) = 0$, $\forall m \in \mathbb{N}^*$ and the constants $K_1, K_2, K_3 > 0$ which not depend on h, such that*

$$\left|\overline{x_m(t_i)} - \overline{y_m(t_i)}\right| \leq K_1 \varepsilon + K_2 \cdot h^p + K_3 \cdot \mu_m(h),$$

for all $i = \overline{1,n}$, $m \in \mathbb{N}^*$.

Theorem 60 *(see [79]) Under the conditions (i)-(iv), the solution* $x^* \in C^1[a,b]$ *of the initial value problem (3.9) is approximated on the knots* $t_i = a + \frac{i(b-a)}{n}$, $i = \overline{1,n}$, *by the sequence* $\left(\overline{x_m(t_i)}\right)_{m\in\mathbb{N}^*}$ *computed in (3.14)-(3.16) with the apriori error estimate*

$$\left|x^*(t_i) - \overline{x_m(t_i)}\right| \leq \frac{(b-a)^m (L_1+L_2)^m}{1-(b-a)(L_1+L_2)} \cdot (b-a) M_0 + \tag{3.19}$$

$$+\frac{L(b-a)^2}{4n[1-(b-a)(L_1+L_2)]} + \frac{7(b-a)L_2 \cdot \varpi(V_m,h)}{4[1-(b-a)(L_1+L_2)]}, \quad \forall i = \overline{1,n},\ \forall m \in \mathbb{N}^*$$

and the algorithm presented in (3.14)-(3.16) is numerically stable with respect to the initial value.

Sketch of proof: Firstly, $\left|x^*(t) - \overline{x_m(t_i)}\right| \leq |x^*(t) - x_m(t_i)| + \left|x_m(t_i) - \overline{x_m(t_i)}\right| \leq \frac{(b-a)^m(L_1+L_2)^m}{1-(b-a)(L_1+L_2)} \cdot (b-a) M_0 + \left|\overline{R_{m,i}}\right|$, $\forall i = \overline{1,n}$, $\forall m \in \mathbb{N}^*$, and therefore we search for an upper bound of $\left|\overline{R_{m,i}}\right|$. Define the function V_m, $m \in \mathbb{N}^*$, $V_m : [a,b] \to \mathbb{R}$ given by,

$$V_m(t) = x_m(t) + \frac{t_i - t}{h} \cdot [\overline{x_m(t_{i-1})} - x_m(t_{i-1})] + \frac{t - t_{i-1}}{h} \cdot [\overline{x_m(t_i)} - x_m(t_i)], \quad t \in [t_{i-1}, t_i],$$

for all $i = \overline{1,n}$. We see that $V_m(t_i) = \overline{x_m(t_i)}$, for all $i = \overline{0,n}$, $m \in \mathbb{N}^*$. So, the cubic splines s_m interpolates V_m on the knots t_i, $i = \overline{0,n}$ and $V_m \in C[a,b]$. It is easy to see that, $|x_m(t) - V_m(t)| \leq \max\{\left|\overline{R_{m,i}}\right| : i = \overline{0,n}\}$, $\forall t \in [t_{i-1}, t_i]$, $\forall i = \overline{1,n}$, $\forall m \in \mathbb{N}^*$, and $|R_{m,i}| \leq \frac{L(b-a)^2}{4n}$, $\forall i = \overline{1,n}$, $\forall m \in \mathbb{N}^*$. So, using Theorem 57 we infer that, $|x_1(t) - s_1(t)| \leq |x_1(t) - V_1(t)| + |V_1(t) - s_1(t)| \leq \frac{L(b-a)^2}{4n} + \frac{7\omega(V_1,h)}{4}$. By induction for $m \geq 2$ it obtains:

$$\left|x_m(t_i) - \overline{x_m(t_i)}\right| = \left|\overline{R_{m,i}}\right| \leq |R_{m,i}| + \frac{(b-a)}{2n} \cdot \sum_{j=1}^{i} [L_1 \left(\left|\overline{R_{m-1,j-1}}\right| + \left|\overline{R_{m-1,j}}\right|\right) +$$

$$+L_2 |x_{m-1}(\varphi(t_{j-1})) - s_{m-1}(\varphi(t_{j-1}))| + |\ x_{m-1}(\varphi(t_j)) - s_{m-1}(\varphi(t_j))\ |] \leq$$

$$\leq \sum_{j=0}^{m-1} (b-a)^j (L_1+L_2)^j \cdot \frac{L(b-a)^2}{4n} + (b-a) L_2 \cdot \sum_{j=0}^{m-2} (b-a)^j (L_1+L_2)^j \cdot \frac{7\omega(V_m,h)}{4},$$

$\forall i = \overline{1,n}$, and the inequality (3.19) follows, where
$\omega(V_m,h) = \max\{\omega(V_k,h) : k = \overline{1,m-1}\}$ and $\lim\limits_{h\to 0} \omega(V_m,h) = 0$.
On the other hand, $\left|\overline{x_m(t_i)} - \overline{y_m(t_i)}\right| \leq \left|\overline{x_m(t_i)} - x_m(t_i)\right| + |x_m(t_i) - y_m(t_i)| + \left|y_m(t_i) - \overline{y_m(t_i)}\right| \leq |x_m(t_i) - y_m(t_i)| + \left|\overline{R_{m,i}}\right| + \left|\overline{R'_{m,i}}\right|$, $\forall i = \overline{1,n}, \forall m \in \mathbb{N}^*$, and as above,

$$\left|\overline{R'_{m,i}}\right| \leq \frac{L(b-a)^2}{4n[1-(b-a)(L_1+L_2)]} + \frac{7(b-a)L_2 \cdot \omega(V_m,h)}{4[1-(b-a)(L_1+L_2)]}, \quad \forall i = \overline{1,n}, \forall m \in \mathbb{N}^*. \tag{3.20}$$

In inductive manner according to the condition $(b-a)(L_1+L_2) < 1$, we get: $|x_0(t) - y_0(t)| = |x_0 - y_0| < \varepsilon$, $\forall t \in [a,b]$ and $|x_m(t_0) - y_m(t_0)| = |x_0 - y_0| < \varepsilon$, $\forall m \in \mathbb{N}$. Thus,

$$|x_1(t) - y_1(t)| \leq |x_0 - y_0| + \int_0^t |f(s, x_0(s), x_0(\varphi(s))) - f(s, y_0(s), y_0(\varphi(s)))|\, ds \leq$$

$$\leq [1 + (L_1 + L_2)(b - a)] \cdot |x_0 - y_0| \leq [1 + (L_1 + L_2)(b - a)] \cdot \varepsilon, \forall t \in [a, b],$$

and

$$|x_m(t) - y_m(t)| \leq [1 + (L_1 + L_2)(b - a) + ... + (L_1 + L_2)^m (b - a)^m] \cdot$$
$$\cdot |x_0 - y_0| \leq \frac{\varepsilon}{1 - (L_1 + L_2)(b - a)}, \quad \forall t \in [a, b], \ \forall m \in \mathbb{N}^*.$$

Consequently,

$$|x_m(t_i) - y_m(t_i)| \leq \frac{\varepsilon}{1 - (L_1 + L_2)(b - a)}, \quad \forall i = \overline{1, n}, \quad \forall m \in \mathbb{N}^*. \tag{3.21}$$

From (3.20) and (3.21) we get,

$$\left|\overline{x_m(t_i)} - \overline{y_m(t_i)}\right| \leq \frac{\varepsilon}{1 - (L_1 + L_2)(b - a)} + \frac{L(b - a)^2}{2n[1 - (b - a)(L_1 + L_2)]} + \frac{7(b - a) L_2 \cdot \omega(V_m, h)}{2[1 - (b - a)(L_1 + L_2)]}$$
$$= \varepsilon K_1 + K_2 \cdot h + K_3 \cdot \omega(V_m, h), \quad \forall i = \overline{1, n}, \quad \forall m \in \mathbb{N}^*,$$

with $p = 1$ and $K_1 = \frac{1}{1-(L_1+L_2)(b-a)}$, $K_2 = \frac{L(b-a)}{2[1-(b-a)(L_1+L_2)]}$, $K_3 = \frac{7(b-a)L_2}{2[1-(b-a)(L_1+L_2)]}$. Here, $\mu_m(h) = \omega(V_m, h)$. We conclude that the numerical method is convergent. Of course, under the same conditions the continuous dependence by the data f, φ of the solution can be obtained by using an analogous technique.

Numerical experiments

In order to test the convergence and the numerical stability of the method, theoretically obtained in the previous theorem, and to illustrate the accuracy of the algorithm, we present two numerical examples, one linear and the other one, nonlinear.

Example 1: The initial value problem

$$\begin{cases} x'(t) = \frac{2}{3}x(t) + \frac{1}{3}x\left(\frac{t}{2}\right) \cdot e^{\frac{t}{2}}, & t \in [0, \frac{1}{2}] \\ x(0) = 1 \end{cases}$$

has the solution $x^*(t) = e^t$ and applying the above presented algorithm for $\varepsilon' = 10^{-15}$ and $n = 10$, $n = 100$, $n = 1000$ we obtain $m = 13$ and the results are in Table 1, where $e_i = \left|x^*(t_i) - \overline{x_m(t_i)}\right|$, $i = \overline{0, n}$ and $d_i = \left|\overline{x_m(t_i)} - \overline{y_m(t_i)}\right|$, $i = \overline{0, n}$. The order of error is $O(10^{-5} \div 10^{-4})$ for $n = 10$, $O(10^{-7} \div 10^{-6})$ for $n = 100$, $O(10^{-9} \div 10^{-8})$ for $n = 1000$, that confirms the convergence of the method. In order to test the numerical stability we consider $|x_0 - y_0| = 10^{-1}$ and we observe that $d_i \in [1 \times 10^{-1}, 1.647237 \times 10^{-1}]$.

t_i	e_i, for $n = 10$	e_i, for $n = 100$	e_i, for $n = 1000$	d_i, for $n = 10$
0.00	0.000000e+000	0.000000e+000	0.000000e+000	1.000000e-001
0.05	1.089385e-005	1.090510e-007	1.090530e-009	1.051173e-001
0.10	2.281095e-005	2.283393e-007	2.283411e-009	1.104966e-001
0.15	3.584230e-005	3.585894e-007	3.585909e-009	1.161512e-001
0.20	5.004745e-005	5.005739e-007	5.005752e-009	1.220952e-001
0.25	6.550351e-005	6.551148e-007	6.551157e-009	1.283436e-001
0.30	8.230297e-005	8.230856e-007	8.230862e-009	1.349118e-001
0.35	1.005402e-004	1.005415e-006	1.005415e-008	1.418163e-001
0.40	1.203126e-004	1.203090e-006	1.203090e-008	1.490742e-001
0.45	1.417244e-004	1.417159e-006	1.417158e-008	1.567037e-001
0.50	1.648876e-004	1.648736e-006	1.648735e-008	1.647237e-001

Table 1

Example 2: The initial value problem

$$\begin{cases} x'(t) = -\frac{1}{2}\left(x(t) + [x(t)]^2\right) \cdot x\left(\frac{t}{2}\right), & t \in [0, \frac{1}{2}] \\ x(0) = 1 \end{cases}$$

has the exact solution $x^*(t) = \frac{1}{t+1}$ and for $\varepsilon' = 10^{-15}$ and $n = 10$ we obtain $m = 16$. For $n = 100$ and $n = 1000$ the number of iterations is $m = 15$. The results are in Table 2.

t_i	e_i, for $n = 10$	e_i, for $n = 100$	e_i, for $n = 1000$	d_i, for $n = 10$
0.00	0.000000e+000	0.000000e+000	0.000000e+000	1.000000e-001
0.05	5.431000e-005	5.398341e-007	5.397887e-009	8.750842e-002
0.10	9.443514e-005	9.384402e-007	9.383925e-009	7.699325e-002
0.15	1.236071e-004	1.230716e-006	1.230668e-008	6.807342e-002
0.20	1.446994e-004	1.442358e-006	1.442312e-008	6.045330e-002
0.25	1.597022e-004	1.592511e-006	1.592466e-008	5.390183e-002
0.30	1.699913e-004	1.695569e-006	1.695527e-008	4.823649e-002
0.35	1.766468e-004	1.762448e-006	1.762409e-008	4.331136e-002
0.40	1.805189e-004	1.801489e-006	1.801452e-008	3.900876e-002
0.45	1.822535e-004	1.819102e-006	1.819068e-008	3.523304e-002
0.50	1.823421e-004	1.820248e-006	1.820217e-008	3.190586e-002

Table 2.

3.4 Second order initial value problems with deviating argument

The numerical methods involved in the approximation of the solution of initial value problems for second order functional differential equations can be summarized as: Runge-Kutta-Nyström methods (see [35]), variational iterations (see [105] and [246]), methods based on power series (see [219] and [241]), spline functions (see [98], [124], and [185]), Adomian decomposition (see [127]), θ-methods (see [242]), pseudospectral tau method based on Chebyshev polynomials (see [229]), and collocation methods (see [92], [98], and [185]).

We present here the application of the method of successive interpolations to second order initial value problems with deviating argument (the results were obtained in [69] and [78]).

The sequence of successive approximations

Consider the initial value problem

$$\begin{cases} x''(t) = f(t, x(t), x(\varphi(t))), & t \in [0, a] \\ x(0) = x_0, \quad x'(0) = v_0 \end{cases} \tag{3.22}$$

under the following conditions:

(i) $f \in C([0,a] \times \mathbb{R} \times \mathbb{R})$, $\varphi \in C[0,a]$ and $0 \leq \varphi(t) \leq a$ for all $t \in [0,a]$

(ii) exist $\alpha, \beta > 0$ such that

$$\left|f(s,u,v) - f(s,u',v')\right| \leq \alpha\left|u - u'\right| + \beta\left|v - v'\right|, \quad \text{for all } s \in [0,a], u, u', v, v' \in \mathbb{R}$$

(iii) $\frac{a^2}{2}(\alpha + \beta) < 1$

(iv) exist $\gamma, \delta > 0$ such that

$$\left|f(s,u,v) - f(s',u,v)\right| \le \gamma \left|s - s'\right|, \quad \text{for all } s, s' \in [0,a], u, v \in \mathbb{R}$$

and

$$\left|\varphi(s) - \varphi(s')\right| \le \delta \left|s - s'\right|, \quad \text{for all } s \in [0,a].$$

Let the function $x_0 : [0.a] \to \mathbb{R}$ be given by $x_0(t) = x_0 + v_0 t$, $t \in [0,a]$. Since f and φ are continuous, there exists $M_0 \ge 0$ such that

$$M_0 = \max\{|f(s, x_0(s), x_0(\varphi(s)))| : s \in [0,a]\}.$$

The initial value problem (3.22) is equivalent in $C[0,a]$ with the following Volterra integral equation

$$x(t) = x_0 + v_0 t + \int_0^t (t-s) \cdot f(s, x(s), x(\varphi(s)))\, ds, \quad t \in [0.a]. \tag{3.23}$$

Applying the Banach's fixed point theorem to the operator $A : C[0,a] \to C[0,a]$,

$$A(x)(t) = x_0 + v_0 t + \int_0^t (t-s) \cdot f(s, x(s), x(\varphi(s)))\, ds, \quad t \in [0.a]$$

and under the conditions (i), (ii), (iii), it obtains the existence and uniqueness of the solution $x^* \in C[0,a]$ of the integral equation (3.23) and its approximation by the terms of the sequence of successive approximations given by

$$x_0(t) = x_0 + v_0 t, \quad t \in [0.a]$$

$$x_k(t) = x_0 + v_0 t + \int_0^t (t-s) \cdot f(s, x_{k-1}(s), x_{k-1}(\varphi(s)))\, ds, \quad t \in [0.a], k \in \mathbb{N}^*. \tag{3.24}$$

Moreover, it is proved that $x^* \in C^2[0,a]$ and the following *apriori* and *a posteriori* error estimates hold:

$$|x_k(t) - x^*(t)| \le \frac{\left(a^2/2\right)^k (\alpha+\beta)^k}{1 - (a^2/2)(\alpha+\beta)} \cdot \frac{M_0 a^2}{2}, \qquad \text{for all } t \in [0,a],\ k \in \mathbb{N}^* \tag{3.25}$$

$$|x_k(t) - x^*(t)| \le \frac{(a^2/2)(\alpha+\beta)}{1 - (a^2/2)(\alpha+\beta)} \cdot |x_k(t) - x_{k-1}(t)| \le$$

$$\le \frac{(a^2/2)(\alpha+\beta)}{1 - (a^2/2)(\alpha+\beta)} \cdot \|x_k - x_{k-1}\|_C, \qquad \text{for all } t \in [0,a],\ k \in \mathbb{N}^*. \tag{3.26}$$

Similarly, as in the previous section, the uniformly boundedness of the terms of the sequence of successive approximations is obtained:

$$|x_k(t)| \le |x_k(t) - x_0(t)| + |x_0(t)| \le \frac{M_0 a^2}{2[1 - (a^2/2)(\alpha+\beta)]} + R_0 = R, \quad \forall t \in [0,a]$$

where $R_0 = |x_0| + a|v_0|$. Then, $-R \le |x_k(t)| \le R$ for all $t \in [0,a]$, $k \in \mathbb{N}^*$, and considering the functions $F_k : [0.a] \to \mathbb{R}$ given by $F_k(s) = f(s, x_k(s), x_k(\varphi(s)))$, $s \in [0,a]$, $k \in \mathbb{N}$ it follows that $|F_k(s)| \leq M$, where $M = \max\{|f(s,u,v)| : s \in [0,a], u, v \in [-R,R]\}$. It is easy to see that $-R \leq |x^*(t)| \leq R$ for all $t \in [0,a]$ and $|x_k''(t)| \leq M, \quad \forall t \in [0,a],\ k \in \mathbb{N}^*$.

The interpolation procedure

Now, we present the procedure of piecewise Birkhoff interpolation used in [78], in the iterative steps of the algorithm. The piecewise Birkhoff interpolation function $s : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and its restrictions to the subintervals $[t_{i-1}, t_i]$, $i = \overline{1,n}$ of a partition Δ of $[a,b]$

$$\Delta : a = t_0 < t_1 < ... < t_{n-1} < t_n = b$$

are s_i, $i = \overline{1,n}$. Considering the following interpolation conditions:

$$\begin{cases} s_i(t_{i-1}) = y_{i-1},\ s_i(t_i) = y_i \\ s_i''(t_{i-1}) = y_{i-1}'',\ s_i''(t_i) = y_i'' \end{cases},\ i = \overline{1,n}, \tag{3.27}$$

for given values y_i, y_i'', $i = \overline{0,n}$, it obtains

$$s_i(t) = \frac{t_i - t}{h_i} \cdot y_{i-1} + \frac{t - t_{i-1}}{h_i} \cdot y_i - \frac{(t - t_{i-1})(t_i - t)[(t_i - t) + h_i]}{6h_i} \cdot y_{i-1}'' -$$

$$- \frac{(t - t_{i-1})(t_i - t)[(t - t_{i-1}) + h_i]}{6h_i} \cdot y_i'', \quad \forall t \in [t_{i-1}, t_i], \quad i = \overline{1,n} \tag{3.28}$$

where $h_i = t_i - t_{i-1}$, $i = \overline{1,n}$.

In regard to the error of the interpolation procedure, the following result for lacunary Birkhoff polynomial interpolation of two times differentiable functions on open intervals holds:

Lemma (see [70]): Let $f : [a,b] \to \mathbb{R}$ be a two times differentiable function on (a,b), $f \in C[a,b]$ and $f', f'' \in C(a,b)$ with finite limits $\lim\limits_{x \to a, x > a} f''(x) \overset{notation}{=} f''(a)$, $\lim\limits_{x \to b, x < b} f''(x) \overset{notation}{=} f''(b)$. The interpolation conditions

$$P(a) = f(a),\ P(b) = f(b),\ P''(a) = f''(a),\ P''(b) = f''(b)$$

uniquely determine the Birkhoff interpolation polynomial $P : [a,b] \to \mathbb{R}$ and the error estimation in the interpolation formula $f(x) = P(x) + R(x)$, $x \in [a,b]$ is given in the inequality

$$|R(x)| \leq \frac{(b-a)^2}{4} \cdot \|f''\|_C, \quad \forall x \in [a,b]$$

where $\|f''\|_C = \max\{|f''(a)|, |f''(b)|, \sup\limits_{x \in (a,b)} |f''(x)|\}$.

The algorithm

In order to compute the terms of the sequence of successive approximations consider the uniform partition of $[0,a]$ given by the knots $t_i = \frac{i \cdot a}{n}$, $i = \overline{0,n}$. Let $h = \frac{a}{n}$. On these knots, the relations (3.24) become

$$x_k(t_i) = x_0 + v_0 t_i + \int_0^{t_i} (t_i - s) \cdot f(s, x_{k-1}(s), x_{k-1}(\varphi(s)))\, ds, \quad i = \overline{0,n}, k \in \mathbb{N}^* \tag{3.29}$$

and the following functions $F_{k,i} : [0,a] \to \mathbb{R}$, $i = \overline{0,n}$, $k \in \mathbb{N}$, given by $F_{k,i}(s) = (t_i - s) \cdot f(s, x_k(s), x_k(\varphi(s)))$., can be defined. Elementary calculus lead to the following uniform Lipschitz properties:

$$\left|x_k(t) - x_k(t')\right| \leq (|v_0| + 2aM)\left|t - t'\right|, \quad \forall k \in \mathbb{N}^*$$

and

$$\left|F_k(t) - F_k(t')\right| \leq [\gamma + (\alpha + \delta\beta)(|v_0| + 2aM)] \cdot |t - t'| = \overline{L} \cdot |t - t'|, \quad \forall k \in \mathbb{N}^*.$$

So,

$$\left|F_{k,i}(t) - F_{k,i}(t')\right| \leq M|t - t'| + a\overline{L} \cdot |t - t'| = L \cdot |t - t'|, \quad \forall k \in \mathbb{N}^*, \forall t, t' \in [0, a].$$

Applying the trapezoidal quadrature rule for Lipschitzian functions to the integrals from (3.29) it obtains the following numerical method:

$$x_0(t_i) = x_0 + v_0 t_i, \quad \text{for all } i = \overline{0, n} \tag{3.30}$$

$$x_k(t_0) = x_0, \quad x_k(t_i) = x_0 + v_0 t_i + \int_0^{t_i} F_{k-1,i}(s)ds =$$

$$= x_0 + v_0 t_i + \frac{a}{2n} \cdot \sum_{j=1}^{i} [(t_i - t_{j-1}) \cdot f(t_{j-1}, x_{k-1}(t_{j-1}), x_{k-1}(\varphi(t_{j-1}))) +$$

$$+ (t_i - t_j) \cdot f(t_j, x_{k-1}(t_j), x_{k-1}(\varphi(t_j)))] + R_{k,i}, \quad \text{for all } i = \overline{1, n}, \ k \in \mathbb{N}^*. \tag{3.31}$$

Since the functions $F_{k.i}$, $i = \overline{0, n}$, $k \in \mathbb{N}$ are Lipschitzian with the same constant L, for the remainder estimation in (3.31) we have

$$|R_{k,i}| \leq \frac{La^2}{4n}, \quad \text{for all } i = \overline{1, n}, \quad k \in \mathbb{N}^*. \tag{3.32}$$

The relations (3.30)-(3.31) lead to the following algorithm:

$$x_0(t_i) = x_0 + v_0 t_i, \quad \text{for all } i = \overline{0, n} \text{ and } x_k(t_0) = x_0, \quad k \in \mathbb{N}^* \tag{3.33}$$

$$x_1(t_i) = x_0 + v_0 t_i + \frac{a}{2n} \cdot \sum_{j=1}^{i} [(t_i - t_{j-1}) \cdot f(t_{j-1}, x_0(t_{j-1}), x_0(\varphi(t_{j-1}))) +$$

$$+ (t_i - t_j) \cdot f(t_j, x_0(t_j), x_0(\varphi(t_j)))] + R_{1,i} = \overline{x_1(t_i)} + R_{1,i}, \quad \text{for all } i = \overline{1, n} \tag{3.34}$$

and

$$x_2(t_i) = x_0 + v_0 t_i + \frac{a}{2n} \cdot \sum_{j=1}^{i} [(t_i - t_{j-1}) \cdot f(t_{j-1}, \overline{x_1(t_{j-1})} + R_{1,j-1}, x_1(\varphi(t_{j-1}))) +$$

$$+ (t_i - t_j) \cdot f(t_j, \overline{x_1(t_j)} + R_{1,j}, x_1(\varphi(t_j)))] + R_{2,i} =$$

$$= x_0 + v_0 t_i + \frac{a}{2n} \cdot \sum_{j=1}^{i} [(t_i - t_{j-1}) \cdot f(t_{j-1}, \overline{x_1(t_{j-1})}, s_1(\varphi(t_{j-1}))) +$$

$$+ (t_i - t_j) \cdot f(t_j, \overline{x_1(t_j)}, s_1(\varphi(t_j)))] + \overline{R_{2,i}} = \overline{x_2(t_i)} + \overline{R_{2,i}}, \quad \text{for all } i = \overline{1, n}, \tag{3.35}$$

where $s_1 : [0, a] \to \mathbb{R}$ is the piecewise Birkhoff interpolation function inspired by the construction in (3.27), (3.28). The function s_1 interpolates the values $\overline{x_1(t_i)}$, $i = \overline{0, n}$ and has the restrictions to the intervals $[t_{i-1}, t_i]$, $i = \overline{1, n}$,

$$s_1^{(i)}(t) = \frac{t_i - t}{h} \cdot \overline{x_1(t_{i-1})} + \frac{t - t_{i-1}}{h} \cdot \overline{x_1(t_i)} -$$

$$-\frac{(t-t_{i-1})(t_i-t)[(t-t_{i-1})+h]}{6h}\cdot f(t_i,x_0(t_i),x_0(\varphi(t_i)))-$$

$$-\frac{(t-t_{i-1})(t_i-t)[(t_i-t)+h]}{6h}\cdot f(t_{i-1},x_0(t_{i-1}),x_0(\varphi(t_{i-1}))), \qquad (3.36)$$

$t\in[t_{i-1},t_i],\ i=\overline{1,n}$. By induction, for $k\geq 3$ it obtains:

$$x_k(t_i)=x_0+v_0t_i+\frac{a}{2n}\cdot$$

$$\cdot\sum_{j=1}^{i}[(t_i-t_{j-1})\cdot f(t_{j-1},\overline{x_{k-1}(t_{j-1})+R_{k-1,j-1}},x_{k-1}(\varphi(t_{j-1})))+$$

$$+(t_i-t_j)\cdot f(t_j,\overline{x_{k-1}(t_j)+R_{k-1,j}},x_{k-1}(\varphi(t_j)))]+R_{k,i}=$$

$$=x_0+v_0t_i+\frac{a}{2n}\cdot\sum_{j=1}^{i}[(t_i-t_{j-1})\cdot f(t_{j-1},\overline{x_{k-1}(t_{j-1})},s_{k-1}(\varphi(t_{j-1})))+$$

$$+(t_i-t_j)\cdot f(t_j,\overline{x_{k-1}(t_j)},s_{k-1}(\varphi(t_j)))]+\overline{R_{k,i}}=\overline{x_k(t_i)}+\overline{R_{k,i}},\quad \forall i=\overline{1,n} \qquad (3.37)$$

where $s_{k-1}:[0,a]\to\mathbb{R}$ is the piecewise Birkhoff interpolation function inspired by the construction in (3.27), (3.28). The function s_{k-1} interpolates the values $\overline{x_{k-1}(t_i)}$, $i=\overline{0,n}$ and has the restrictions to the intervals $[t_{i-1},t_i]$, $i=\overline{1,n}$:

$$s_{k-1}^{(i)}(t)=\frac{t_i-t}{h}\cdot\overline{x_{k-1}(t_{i-1})}+\frac{t-t_{i-1}}{h}\cdot\overline{x_{k-1}(t_i)}-\frac{(t-t_{i-1})(t_i-t)[(t_i-t)+h]}{6h}\cdot$$

$$\cdot\left(\overline{x_{k-1}(t_{i-1})}\right)''-\frac{(t-t_{i-1})(t_i-t)[(t-t_{i-1})+h]}{6h}\cdot\left(\overline{x_{k-1}(t_i)}\right)'', \qquad (3.38)$$

$t\in[t_{i-1},t_i],\quad i=\overline{1,n}$, where

$$\left(\overline{x_{k-1}(t_i)}\right)''=f\left(t_i,\ \overline{x_{k-2}(t_i)},s_{m-2}(\varphi(t_i))\right),\quad i=\overline{0,n}. \qquad (3.39)$$

The convergence analysis

Theorem 61 *(see [78]) Under the conditions (i)-(iv), if $a^2(\alpha+\beta)<1$, then the values of the unique solution x^* of the initial value problem (3.22) are approximated on the knots $t_i=\frac{i\cdot a}{n}$, $i=\overline{0,n}$ by the sequence $(\overline{x_k(t_i)})_{k\in\mathbb{N}^*}$. The apriori error estimate is:*

$$\left|x^*(t_i)-\overline{x_k(t_i)}\right|\leq\frac{\left(a^2/2\right)^k(\alpha+\beta)^k}{1-(a^2/2)(\alpha+\beta)}\cdot\frac{M_0a^2}{2}+\frac{a^2L}{4n[1-a^2(\alpha+\beta)]}+$$

$$+\frac{2a^2\beta Mh^2+\left(5M+3\overline{M}\right)h^2}{8\left[1-a^2(\alpha+\beta)\right]},\quad\forall i=\overline{1,n},\forall k\in\mathbb{N}^*,k\geq 2 \qquad (3.40)$$

where the constant $\overline{M}$ is given in (3.41).

Sketch of proof: Firstly,

$$\left|x^*(t_i)-\overline{x_k(t_i)}\right|\leq|x^*(t_i)-x_k(t_i)|+$$

$$+\left|x_k(t_i)-\overline{x_k(t_i)}\right|=|x^*(t_i)-x_k(t_i)|+\left|\overline{R_{k,i}}\right|,\quad\forall k\in\mathbb{N}^*,i=\overline{1,n}$$

remaining to estimate $\left|\overline{R_{k,i}}\right|$. Since

$$\left|x_1(t_i) - \overline{x_1(t_i)}\right| = |R_{1,i}| \le \frac{La^2}{4n}, \qquad \forall i = \overline{1,n}$$

we get

$$\left|\overline{x_1(t_i)}\right| \le \left|x_1(t_i) - \overline{x_1(t_i)}\right| + |x_1(t_i)| \le R + \frac{La^2}{4n}, \qquad \forall i = \overline{1,n}.$$

Because $x_k(t_i) \ne \overline{x_k(t_i)}$, $\forall k \in \mathbb{N}^*, i = \overline{1,n}$ we infer that s_k interpolates the values $\overline{x_k(t_i)}$, $i = \overline{0,n}$, but not the function x_k. Therefore we define for any k the function V_{k-1}, $k \in \mathbb{N}^*$, $V_{k-1} : [0,a] \to \mathbb{R}$ given by its restrictions to the subintervals $[t_{i-1}, t_i]$, $i = \overline{1,n}$ as follows:

$$V_{k-1}(t) = x_{k-1}(t) + [\overline{x_{k-1}(t_i)} - x_{k-1}(t_i)] \cdot \frac{t - t_{i-1}}{h} + [\overline{x_{k-1}(t_{i-1})} - x_{k-1}(t_{i-1})] \cdot \frac{t_i - t}{h} -,$$

$$-\frac{(t - t_{i-1})(t_i - t)[(t_i - t) + h]}{6h} \cdot [f\left(t_{i-1}, \ \overline{x_{k-2}(t_{i-1})}, s_{m-2}(\varphi(t_{i-1}))\right) - x''_{k-1}(t_{i-1})] -$$

$$-\frac{(t - t_{i-1})(t_i - t)[(t - t_{i-1}) + h]}{6h} \cdot [f\left(t_i, \ \overline{x_{k-2}(t_i)}, s_{m-2}(\varphi(t_i))\right) - x''_{k-1}(t_i)].$$

On the one hand, we see that $V_{k-1}(t_i) = \overline{x_{k-1}(t_i)}$, $\forall i = \overline{0,n}$, that is V_{k-1} interpolates the values $\overline{x_{k-1}(t_i)}$, $i = \overline{0,n}$ and it is continuous. So, s_{k-1} interpolates the function V_{k-1} for any $k \in \mathbb{N}^*$ and $V_{k-1} \in C[t_{i-1}, t_i] \cap C^2(t_{i-1}, t_i)$ for any $i = \overline{1,n}$, $V''_{k-1}(t_i) = f\left(t_i, \ \overline{x_{k-2}(t_i)}, s_{m-2}(\varphi(t_i))\right) = (s_{k-1})''(t_i)$, $\forall i = \overline{0,n}$. Consequently, V_{k-1} and s_{k-1} are in the context of the previous Lemma on each interval $[t_{i-1}, t_i]$, $i = \overline{1,n}$.
In inductive manner it obtains

$$\left|\overline{R_{k,i}}\right| = \left|x_k(t_i) - \overline{x_k(t_i)}\right| \le |R_{k,i}| + \frac{a}{2n} \cdot \sum_{j=1}^{i} [(t_i - t_{j-1}) \cdot$$

$$\cdot \left(\alpha\left|\overline{R_{k-1,j-1}}\right| + \beta\left|x_{k-1}(\varphi(t_{j-1})) - s_{k-1}(\varphi(t_{j-1}))\right|\right) + (t_i - t_j) \cdot$$

$$\cdot \left(\alpha\left|\overline{R_{k-1,j}}\right| + \beta\left|x_{k-1}(\varphi(t_j)) - s_{k-1}(\varphi(t_j))\right|\right)], \quad \forall i = \overline{1,n}, \ k \ge 2.$$

These suggest the necessity to estimate $|x_{k-1}(t) - s_{k-1}(t)|$ for $t \in [0,a]$ and $k \ge 2$. Recurrently, it follows that

$$|x_1(t) - s_1(t)| \le |x_1(t) - V_1(t)| + |V_1(t) - s_1(t)| \le \frac{t - t_{i-1}}{h} \cdot |R_{1,i}| + \frac{t_i - t}{h} \cdot |R_{1,i-1}| +$$

$$+\left|\frac{(t - t_{i-1})(t_i - t)[(t_i - t) + h]}{6h}\right| \cdot \left|f(t_{i-1}, x_0(t_{i-1}), x_0(\varphi(t_{i-1}))) - x''_1(t_{i-1})\right| +$$

$$+\left|\frac{(t - t_{i-1})(t_i - t)[(t - t_{i-1}) + h]}{6h}\right| \cdot \left|f(t_i, x_0(t_i), x_0(\varphi(t_i))) - x''_1(t_i)\right| +$$

$$+\frac{h^2}{4} \cdot \max\{[\left|x''_1(t)\right| + \frac{t_i - t}{h} \cdot \left|f(t_{i-1}, x_0(t_{i-1}), x_0(\varphi(t_{i-1}))) - x''_1(t_{i-1})\right| +$$

$$+\frac{t - t_{i-1}}{h} \cdot \left|f(t_i, x_0(t_i), x_0(\varphi(t_i))) - x''_1(t_i)\right|], \left|f(t_i, x_0(t_i), x_0(\varphi(t_i)))\right|,$$

$$, \left|f(t_{i-1}, x_0(t_{i-1}), x_0(\varphi(t_{i-1})))\right|\} \le \frac{La^2}{4n} + \frac{Mh^2}{4}, \quad \forall t \in [0,a],$$

$$|s_1(t)| \le |x_1(t) - s_1(t)| + |x_1(t)| \le R + \frac{La^2}{4n} + \frac{Mh^2}{4}, \quad \forall t \in [0,a],$$

$$\left|\overline{R_{2,i}}\right| \leq [1+a^2(\alpha+\beta)]\cdot\frac{La^2}{4n}+\frac{a^2\beta Mh^2}{4}, \quad \forall i=\overline{1,n}$$

$$\left|\overline{x_2(t_i)}\right| \leq R+[1+a^2(\alpha+\beta)]\cdot\frac{La^2}{4n}+\frac{a^2\beta Mh^2}{4}, \quad \forall i=\overline{1,n}$$

and

$$|x_2(t)-s_2(t)| \leq |x_2(t)-V_2(t)|+|V_2(t)-s_2(t)| \leq \max\{\left|\overline{R_{2,i}}\right| : i=\overline{0,n}\}+$$
$$+\left|\frac{(t-t_{i-1})(t_i-t)[(t_i-t)+h]}{6h}\right|\cdot\left|f\left(t_{i-1},\ \overline{x_1(t_{i-1})}, s_1(\varphi(t_{i-1}))\right)-x_2''(t_{i-1})\right|+$$
$$+\left|\frac{(t-t_{i-1})(t_i-t)[(t-t_{i-1})+h]}{6h}\right|\cdot\left|f\left(t_i,\ \overline{x_1(t_i)}, s_1(\varphi(t_i))\right)-x_2''(t_i)\right|+$$
$$+\frac{h^2}{4}\cdot\max\{[\left|x_2''(t)\right|+\frac{t_i-t}{h}\cdot\left|f\left(t_{i-1},\ \overline{x_1(t_{i-1})}, s_1(\varphi(t_{i-1}))\right)-x_2''(t_{i-1})\right|+$$
$$+\frac{t-t_{i-1}}{h}\cdot\left|f\left(t_i,\ \overline{x_1(t_i)}, s_1(\varphi(t_i))\right)-x_2''(t_i)\right|],\left|f\left(t_i,\ \overline{x_1(t_i)}, s_1(\varphi(t_i))\right)\right|,$$
$$,\left|f\left(t_{i-1},\ \overline{x_1(t_{i-1})}, s_1(\varphi(t_{i-1}))\right)\right|\} \leq [1+a^2(\alpha+\beta)]\cdot\frac{La^2}{4n}+\frac{a^2\beta Mh^2}{4}+\frac{(5M+3M_1)h^2}{8}$$

where

$$M_1=\max\{|f(t,u,v)| : t\in[0,a], u,v\in[-\left(R+\frac{La^2}{4n}+\frac{Mh^2}{4}\right), R+\frac{La^2}{4n}+\frac{Mh^2}{4}]\}.$$

Moreover,

$$|s_2(t)| \leq R+\frac{La^2}{4n[1-a^2(\alpha+\beta)]}+\frac{a^2\beta Mh^2}{4}+\frac{(5M+3M_1)h^2}{8}, \quad \forall t\in[0,a],$$

$$\left|\overline{R_{3,i}}\right| \leq \frac{La^2}{4n[1-a^2(\alpha+\beta)]}+[1+a^2(\alpha+\beta)]\cdot\frac{a^2\beta Mh^2}{4}+\frac{a^2\beta(5M+3M_1)h^2}{8}, \quad \forall i=\overline{1,n}$$

and

$$\left|\overline{x_3(t_i)}\right| \leq R+\frac{La^2}{4n[1-a^2(\alpha+\beta)]}+$$
$$+[1+a^2(\alpha+\beta)]\cdot\frac{a^2\beta Mh^2}{4}+\frac{a^2\beta(5M+3M_1)h^2}{8}, \quad \forall i=\overline{1,n}.$$

Let

$$R'=R+\frac{La^2}{4n[1-a^2(\alpha+\beta)]}+\frac{a^2\beta Mh^2}{4}+\frac{(5M+3M_1)h^2}{8}$$

and

$$M'=\max\{M, M_1, \max\{|f(t,u,v)| : t\in[0,a], u,v\in[-R', R']\}\}.$$

Similarly,

$$|x_3(t)-s_3(t)| \leq \max\{\left|\overline{R_{3,i}}\right| : i=\overline{0,n}\}+\left(M+M'\right)\cdot\frac{h^2}{8}+$$
$$+\frac{h^2}{4}\cdot\max\{2M+M', M', M'\} \leq \max\{\left|\overline{R_{3,i}}\right| : i=\overline{0,n}\}+\frac{(5M+3M')h^2}{8} \leq$$
$$\frac{La^2}{4n[1-a^2(\alpha+\beta)]}+[1+a^2(\alpha+\beta)]\cdot\frac{a^2\beta Mh^2}{4}+[1+a^2\beta]\cdot\frac{(5M+3M')h^2}{8}, \quad \forall t\in[0,a]$$

and

$$|s_3(t)| \le R + \frac{La^2}{4n[1-a^2(\alpha+\beta)]} + \frac{2a^2\beta M h^2 + (5M+3M')h^2}{8[1-a^2(\alpha+\beta)]}, \quad \forall t \in [0,a].$$

Let

$$\overline{R} = R + \frac{La^2}{4n[1-a^2(\alpha+\beta)]} + \frac{2a^2\beta M h^2 + (5M+3M')h^2}{8[1-a^2(\alpha+\beta)]}$$

and by induction for $k \ge 3$ we obtain

$$\left|\overline{R_{k,i}}\right| \le \frac{La^2}{4n[1-a^2(\alpha+\beta)]} + \frac{2a^2\beta M h^2 + \left(5M+3\overline{M}\right)h^2}{8[1-a^2(\alpha+\beta)]}, \quad \forall i = \overline{1,n}, \forall k \in \mathbb{N}^*,\ k \ge 2$$

and

$$\left|\overline{x_k(t_i)}\right| \le R + \frac{La^2}{4n[1-a^2(\alpha+\beta)]} + \frac{2a^2\beta M h^2 + \left(5M+3\overline{M}\right)h^2}{8[1-a^2(\alpha+\beta)]}, \quad \forall i = \overline{1,n},\ k \ge 2$$

$$|x_{k-1}(t) - s_{k-1}(t)| \le \frac{La^2}{4n[1-a^2(\alpha+\beta)]} + \frac{2a^2\beta M h^2 + \left(5M+3\overline{M}\right)h^2}{8[1-a^2(\alpha+\beta)]} +$$

$$+ \frac{\left(5M+3\overline{M}\right)h^2}{8}, \quad \forall t \in [0,a], \forall k \in \mathbb{N}^*, \forall k \ge 2$$

$$|s_{k-1}(t)| \le R + \frac{La^2}{4n[1-a^2(\alpha+\beta)]} + \frac{\left(5M+3\overline{M}\right)h^2}{8} +$$

$$+ \frac{2a^2\beta M h^2 + \left(5M+3\overline{M}\right)h^2}{8[1-a^2(\alpha+\beta)]}, \quad \forall t \in [0,a],, \forall k \in \mathbb{N}^*, \forall k \ge 2$$

where

$$\overline{M} = \max\{M', \max\{|f(t,u,v)| : t \in [0,a], u,v \in [-\overline{R},\overline{R}]\}\}. \tag{3.41}$$

Remark 62 *From the estimate (3.40) it follows the consistency of the method. The conditions in Theorem 61 differ from the conditions for the existence and uniqueness of the solution only by the inequality $a^2(\alpha+\beta) < 1$ and by the Lipschitz requirements (iv). Supplementary boundedness and smoothness conditions are not necessary.*

Remark 63 *Under the conditions of Theorem 61 we obtain similar continuous approximation of the solution interpolating the computed values $\overline{x_k(t_i)}$, $i = \overline{0,n}$, by the same procedure as in (3.38)-(3.39). So, we obtain the continuous approximation of the solution, $s_k : [0,a] \to \mathbb{R}$ given by its restrictions to the intervals $[t_{i-1}, t_i]$, $i = \overline{1,n}$:*

$$s_k^{(i)}(t) = \frac{t_i - t}{h} \cdot \overline{x_k(t_{i-1})} + \frac{t - t_{i-1}}{h} \cdot \overline{x_k(t_i)} - \frac{(t-t_{i-1})(t_i - t)[(t_i - t) + h]}{6h} \cdot$$

$$\cdot \left(\overline{x_k(t_{i-1})}\right)'' - \frac{(t-t_{i-1})(t_i-t)[(t-t_{i-1})+h]}{6h} \cdot \left(\overline{x_k(t_i)}\right)'', \ \forall t \in [t_{i-1}, t_i],\ i = \overline{1,n}, \tag{3.42}$$

with

$$\left(\overline{x_k(t_i)}\right)'' = f\left(t_i,\ \overline{x_{k-1}(t_i)}, s_{m-1}(\varphi(t_i))\right), \quad i = \overline{0,n}. \tag{3.43}$$

Moreover, the approximations of the second derivative on the knots $t_i = \frac{i \cdot a}{n}$, $i = \overline{0,n}$, are computed in (3.43).

Corollary (see [78]): The error estimates in (3.42) and (3.43) are:

$$|x^*(t) - s_k(t)| \le \frac{\left(a^2/2\right)^k (\alpha+\beta)^k}{1-(a^2/2)(\alpha+\beta)} \cdot \frac{M_0 a^2}{2} + \frac{La^2}{4n\left[1-a^2(\alpha+\beta)\right]} +$$

$$+\frac{2a^2\beta M h^2 + \left(5M + 3\overline{M}\right)h^2}{8\left[1-a^2(\alpha+\beta)\right]} + \frac{\left(5M+3\overline{M}\right)h^2}{8}, \quad \forall t \in [0,a], \quad \forall k \in \mathbb{N}^*, \qquad (3.44)$$

$$\left|(x^*(t_i))'' - \left(\overline{x_k(t_i)}\right)''\right| \le \alpha\left[\frac{\left(a^2/2\right)^k (\alpha+\beta)^k}{1-(a^2/2)(\alpha+\beta)} \cdot \frac{M_0 a^2}{2} +\right.$$

$$+\frac{La^2}{4n\left[1-a^2(\alpha+\beta)\right]} + \frac{2a^2\beta M h^2 + \left(5M+3\overline{M}\right)h^2}{8\left[1-a^2(\alpha+\beta)\right]}\Big] + \beta\Big[\frac{\left(a^2/2\right)^k(\alpha+\beta)^k}{1-(a^2/2)(\alpha+\beta)} \cdot \frac{M_0 a^2}{2} +$$

$$\left.+\frac{La^2}{4n\left[1-a^2(\alpha+\beta)\right]} + \frac{2a^2\beta M h^2 + \left(5M+3\overline{M}\right)h^2}{8\left[1-a^2(\alpha+\beta)\right]} + \frac{\left(5M+3\overline{M}\right)h^2}{8}\right] \qquad (3.45)$$

for all $i = \overline{0,n}$ and $k \in \mathbb{N}^*$.

Sketch of proof: For $k \in \mathbb{N}^*$ we have

$$|x^*(t) - s_k(t)| \le |x^*(t) - x_k(t)| + |x_k(t) - s_k(t)|$$

and according to the inequality (3.25) and by Theorem 61, the estimate (3.44) follows. In addition to this,

$$\left|(x^*(t_i))'' - \left(\overline{x_k(t_i)}\right)''\right| = \left|f(t_i, x^*(t_i), x^*(\varphi(t_i))) - f\left(t_i,\ \overline{x_{k-1}(t_i)}, s_{m-1}(\varphi(t_i))\right)\right| \le$$

$$\le \alpha\left|x^*(t_i) - \overline{x_{k-1}(t_i)}\right| + \beta\left|x^*(\varphi(t_i)) - s_{m-1}(\varphi(t_i))\right|, \quad \forall i = \overline{0,n}, \quad \forall k \in \mathbb{N}^*$$

and from (3.25), (3.40) and (3.44) the estimate (3.45) follows.

Remark 64 *(see [78]) We see that the 'a posteriori' (3.26) and 'a priori' (3.40) estimates can offer a practical stopping criterion of the algorithm. This can be stated as follows: for given $\varepsilon' > 0$ and $n \in \mathbb{N}^*$ (previously chosen) we determine the first natural number $k \in \mathbb{N}^*$ for which,*

$$\left|\overline{x_k(t_i)} - \overline{x_{k-1}(t_i)}\right| < \varepsilon' \qquad \text{for all } i = \overline{1,n}$$

and we stop to this k, retaining the approximations $\overline{x_k(t_i)}$, $i = \overline{0,n}$ of the solution. The demonstration of this criterion is the following.
We denote:

$$\Omega = \frac{La^2}{4n\left[1-a^2(\alpha+\beta)\right]} + \frac{2a^2\beta M h^2 + \left(5M+3\overline{M}\right)h^2}{8\left[1-a^2(\alpha+\beta)\right]}.$$

For each $i = \overline{1,n}$ we have

$$\left|x^*(t_i) - \overline{x_k(t_i)}\right| \le |x^*(t_i) - x_k(t_i)| + \left|x_k(t_i) - \overline{x_k(t_i)}\right| \le$$

$$\le \frac{(a^2/2)(\alpha+\beta)}{1-(a^2/2)(\alpha+\beta)} \cdot |x_k(t_i) - x_{k-1}(t_i)| + \left|\overline{R_{k,i}}\right|$$

and

$$|x_k(t_i) - x_{k-1}(t_i)| \le \left|x_k(t_i) - \overline{x_k(t_i)}\right| + \left|\overline{x_k(t_i)} - \overline{x_{k-1}(t_i)}\right| +$$

$$+\left|\overline{x_{k-1}(t_i)} - x_{k-1}(t_i)\right| = \left|\overline{R_{k,i}}\right| + \left|\overline{R_{k-1,i}}\right| + \left|\overline{x_k(t_i)} - \overline{x_{k-1}(t_i)}\right|.$$

So,

$$\left|x^*(t_i) - \overline{x_k(t_i)}\right| \le \left|\overline{R_{k,i}}\right| + \frac{(a^2/2)(\alpha+\beta)}{1-(a^2/2)(\alpha+\beta)} \cdot \left|\overline{x_k(t_i)} - \overline{x_{k-1}(t_i)}\right| +$$

$$+\frac{(a^2/2)(\alpha+\beta)}{1-(a^2/2)(\alpha+\beta)} \cdot \left(\left|\overline{R_{k,i}}\right| + \left|\overline{R_{k-1,i}}\right|\right).$$

Then

$$\left|x^*(t_i) - \overline{x_k(t_i)}\right| \le \Omega \cdot \frac{1+(a^2/2)(\alpha+\beta)}{1-(a^2/2)(\alpha+\beta)} + \frac{(a^2/2)(\alpha+\beta)}{1-(a^2/2)(\alpha+\beta)} \cdot \left|\overline{x_k(t_i)} - \overline{x_{k-1}(t_i)}\right|.$$

For given $\varepsilon > 0$ *we require*

$$\Omega \cdot \frac{1+(a^2/2)(\alpha+\beta)}{1-(a^2/2)(\alpha+\beta)} < \frac{\varepsilon}{2} \tag{3.46}$$

and

$$\frac{(a^2/2)(\alpha+\beta)}{1-(a^2/2)(\alpha+\beta)} \cdot \left|\overline{x_k(t_i)} - \overline{x_{k-1}(t_i)}\right| < \frac{\varepsilon}{2}.$$

Since

$$\Omega \le \frac{La^2}{4n[1-a^2(\alpha+\beta)]} + \frac{2a^4\beta M + \left(5M + 3\overline{M}\right)a^2}{8n[1-a^2(\alpha+\beta)]}$$

we can chose the smallest natural number n,

$$n > \frac{[1+(a^2/2)(\alpha+\beta)] \cdot [2La^2 + 2a^4\beta M + \left(5M+3\overline{M}\right)a^2]}{4\varepsilon[1-(a^2/2)(\alpha+\beta)] \cdot [1-a^2(\alpha+\beta)]}$$

for which the inequality (3.46) holds. Afterwards we find the smallest natural number k *for which*

$$\left|\overline{x_k(t_i)} - \overline{x_{k-1}(t_i)}\right| < \frac{\varepsilon}{2} \cdot \frac{1-(a^2/2)(\alpha+\beta)}{(a^2/2)(\alpha+\beta)} \overset{notation}{=} \varepsilon'$$

for all $i = \overline{1,n}$. *This is the last iterative step to be made. With these we obtain* $\left|x^*(t_i) - \overline{x_k(t_i)}\right| < \varepsilon$, *for all* $i = \overline{1,n}$.

Remark 65 *From the error estimate (3.25) we see that the sequence of successive approximations* $(x_k)_{k\in\mathbb{N}^*}$ *uniformly converges on* $[0,a]$ *to the solution* x^*. *From the error estimates (3.40) it follows that*

$$\left|x_k(t_i) - \overline{x_k(t_i)}\right| = O(h) + O\left(h^2\right), \quad \forall i = \overline{1,n-1}, \quad \forall k \in \mathbb{N}^*$$

and these terms go to zero when $h = \frac{a}{n}$ *does. These lead to the convergence of the computed values* $\overline{x_k(t_i)}$ *to the values of the exact solution,* $x^*(t_i)$, $i = \overline{1,n}$.

The numerical stability

We consider the initial value problem with the same second order differential equation, but with modified initial conditions:

$$\begin{cases} x''(t) = f(t, x(t), x(\varphi(t))), & t \in [0,a] \\ x(0) = x_0', \quad x'(0) = v_0' & \end{cases} \tag{3.47}$$

such that $|x_0 - x_0'| < \epsilon$ and $|v_0 - v_0'| < \varepsilon$ for small $\epsilon > 0, \varepsilon > 0$.

For the initial value problem (3.47) the sequence of successive approximations on the knots $t_i = \frac{i \cdot a}{n}$, $i = \overline{0,n}$ is:

$$y_0(t_i) = x_0' + v_0' t_i, \quad i = \overline{0,n}, \qquad y_k(t_0) = x_0'$$

$$y_k(t_i) = x_0' + v_0' t + \int_0^{t_i} (t_i - s) \cdot f(s, y_{k-1}(s), y_{k-1}(\varphi(s)))\, ds, \quad i = \overline{1,n}, k \in \mathbb{N}^*.$$

The effective computed values are

$$y_0(t_i) = x_0' + v_0' t_i, \quad i = \overline{0,n}, \qquad y_k(t_0) = x_0'$$

and $\overline{y_k(t_i)}, i = \overline{1,n}, k \in \mathbb{N}^*$ with $y_k(t_i) = \overline{y_k(t_i)} + \overline{R_{k,i}'}, \quad \forall i = \overline{1,n}, k \in \mathbb{N}^*$. We see that

$$|x_0(t) - y_0(t)| \le |x_0 - x_0'| + |v_0 - v_0'|\, a < \epsilon + a\varepsilon, \quad \forall t \in [0,a].$$

Definition 66 *(see [78]) We say that the method of successive interpolations applied to the initial value problem (3.22) is numerically stable with respect to the initial values, if there exist $p, q \in \mathbb{N}^*$ and the constants $K_1, K_2, K_3, K_4 > 0$, such that*

$$\left|\overline{x_k(t_i)} - \overline{y_k(t_i)}\right| \le K_1\epsilon + K_2\varepsilon + K_3 \cdot h^p + K_4 \cdot h^q = K_1\epsilon + K_2\varepsilon + O(h^p) + O(h^q)$$

for all $i = \overline{1,n}$, $k \in \mathbb{N}^*$.

Theorem 67 *(see [78]) Under the conditions of Theorem 61, the method of successive interpolations applied to the initial value problem (3.22) is numerically stable with respect to the initial values.*

The proof follows the same technique as in the proof of Theorem 60, obtaining

$$\left|\overline{x_k(t_i)} - \overline{y_k(t_i)}\right| \le |x_k(t_i) - y_k(t_i)| + \frac{La^2}{4n\left[1 - a^2(\alpha+\beta)\right]} +$$

$$+ \frac{2a^4\beta M + \left(5M + 3\overline{M}\right)a^2}{8n^2\left[1 - a^2(\alpha+\beta)\right]} + \frac{L'a^2}{4n\left[1 - a^2(\alpha+\beta)\right]} + \frac{2a^4\beta M + \left(5M + 3\overline{M}\right)a^2}{8n^2\left[1 - a^2(\alpha+\beta)\right]} \le$$

$$\le \frac{(\epsilon + a\varepsilon)}{1 - (\alpha+\beta)\frac{a^2}{2}} + \frac{La + L'a}{4[1 - a^2(\alpha+\beta)]} \cdot \left(\frac{a}{n}\right) + \frac{2a^2\beta M + \left(5M + 3\overline{M}\right)}{4\left[1 - a^2(\alpha+\beta)\right]} \cdot \left(\frac{a}{n}\right)^2 =$$

$$= K_1\epsilon + K_2\varepsilon + K_3 \cdot h + K_4 \cdot h^2 = K_1\epsilon + K_2\varepsilon + O(h) + O\left(h^2\right), \ \forall i = \overline{1, n-1}, \ \forall k \in \mathbb{N}^*$$

where

$$L' = M + a \cdot \left[\gamma + (\alpha + \delta\beta)\left(|v_0'| + 2aM\right)\right].$$

Corollary (see [78]): The proposed method of successive interpolations for the initial value problem associated to functional differential equations of second order is convergent.

This result follows from Theorem 61, Remarks 64 and 65, and from Theorem 67.

Numerical experiment

In order to test the obtained theoretical result and to illustrate the accuracy of the method we present the following numerical example.

Example: Consider the initial value problem:

$$\begin{cases} x''(t) = 1 + 2\left(1 + t^2/8\right)\cos(t/2) - 2\cos(t/2) \cdot x(t/2), & t \in [0, \frac{\pi}{4}] \\ x(0) = 1, \quad x'(0) = 1 \end{cases}.$$

Here, $a = \frac{\pi}{4}$, $\varphi(t) = \lambda t$ with $\lambda = \frac{1}{2}$ and $f(t,u,v) = 1 + 2\left(1 + \frac{t^2}{8}\right)\cos\left(\frac{t}{2}\right) - 2\cos\left(\frac{t}{2}\right)\cdot v$. The exact solution is $x^*(t) = \frac{t^2}{2} + \sin t + 1$, $t \in [0, \frac{\pi}{4}]$. Applying the above presented algorithm with $n = 10$ and $\varepsilon' = 10^{-15}$ we get $k = 7$ (the number of iterations). The values of the errors $e_i = \left|x^*(t_i) - \overline{x_7(t_i)}\right|$, $i = \overline{0,10}$ are in the second column of Table 1. The order of effective error is $O\left(10^{-4}\right)$.

In order to test the numerical stability of the method we consider $\epsilon = \varepsilon = 0.1$ and we represent the differences between the effective computed values $d_i = \left|\overline{x_k(t_i)} - \overline{y_k(t_i)}\right|$, $i = \overline{0,10}$ in the fifth column. So as to illustrate and to test the convergence we put $n = 100$, $\varepsilon' = 10^{-15}$ and we can see how decrease e_i, $i = \overline{0,n}$ when h decreases. The number of iterations is $k = 7$. The results are in the third column of Table 1, with the knots and the corresponding values being selected by tens, such that the knots are the same as in the first column. It can be observed that the order of effective error becomes $O\left(10^{-6}\right)$. For $n = 1000$, $\varepsilon' = 10^{-15}$ we have $k = 7$ iterations and the order of effective error is $O\left(10^{-8}\right)$, the errors $e_i = \left|x^*(t_i) - \overline{x_7(t_i)}\right|$ for $i = \overline{0,1000}$, $i = 100\cdot k$, $k = \overline{1,9}$, being presented in the fourth column of Table 1 (on the same knots). The results presented in Table 1 confirm the convergence of the algorithm, that is $e_i \to 0$ when $h \to 0$. In the implementation of the algorithm we have used Visual C^{++} and the data were considered with 10^{-20} precision in the computational process.

t_i	e_i, for $n = 10$	e_i, for $n = 100$	e_i, for $n = 1000$	d_i
0	0	0	0	0.1
0.07853981634	8.072061e-005	8.062203e-007	8.062104e-009	0.105
0.15707963268	1.606948e-004	1.604982e-006	1.604962e-008	0.220
0.23561944901	2.391803e-004	2.388876e-006	2.388846e-008	0.330
0.31415926535	3.154464e-004	3.150592e-006	3.150553e-008	0.437
0.39269908169	3.887748e-004	3.882969e-006	3.882921e-008	0.536
0.47123889804	4.584701e-004	4.579041e-006	4.578985e-008	0.622
0.54977871438	5.238569e-004	5.232084e-006	5.232019e-008	0.759
0.62831853072	5.842935e-004	5.835661e-006	5.835589e-008	0.796
0.70685834706	6.391655e-004	6.383665e-006	6.383585e-008	0.812
0.78539816340	6.879020e-004	6.870357e-006	6.870270e-008	0.853

Table 1

Remark 68 *In [85], the interpolation procedure of complete-natural cubic spline is applied to initial value problems for first, second, and third order functional differential equations, in a similar manner as above. The initial value problem for third order equation*

$$\begin{cases} x'''(t) = f(t, x(t), x(\varphi(t))), & t \in [a,b] \\ x(a) = x_0, \quad x'(a) = v_0, \quad x''(a) = q_0 \end{cases}$$

is written in the equivalent form of Volterra functional integral equation

$$x(t) = x_0 + v_0(t-a) + q_0 \cdot \frac{(t-a)^2}{2} + \int_a^t \frac{(t-a)^2}{2} \cdot f(s, x(s), x(\varphi(s)))\, ds.$$

For the complete-natural cubic spline $s \in C^2[a,b]$ the end-conditions are $s'(a) = v_0$, $s''(b) = 0$.

3.5 Two-point boundary value problems of second order

The performing methods for the numerical solution of two-points boundary value problems associated to second order differential equations with deviating argument are frequently focused on: finite differences (see [5] and [111]), shooting techniques (see [27], [107], and [190]), Pade approximations (see [101]), Richardson extrapolation (see [209]), and spline collocation (see [98], [185], [204], and [217]). In this section we present the application of the method of successive interpolations to second order two-point boundary value problems with deviating argument. The results in this topic were obtained in [70] (by using the Birkhoff lacunary cubic spline of interpolation presented in the previous section) and in [71] (using the interpolation procedure of natural cubic spline).

Consider the two-point boundary value problem:

$$\begin{cases} x''(t) = f(t, x(t), x(\varphi(t))), & t \in [0,a] \\ x(0) = b, \quad x(a) = c \end{cases} \tag{3.48}$$

under the following conditions:

(i) $f \in C([0,a] \times \mathbb{R} \times \mathbb{R})$, $\varphi \in C[0,a]$ and $0 \leq \varphi(t) \leq a$ for all $t \in [0,a]$

(ii) there exist $\alpha, \beta > 0$ such that

$$\left|f(s,u,v) - f(s,u',v')\right| \leq \alpha\left|u-u'\right| + \beta\left|v-v'\right|, \quad \text{for all } s \in [0,a],\ u,u',v,v' \in \mathbb{R}$$

(iii) $\frac{a^2}{8}(\alpha+\beta) < 1$

(iv) there exist $\gamma, \delta > 0$ such that

$$\left|f(s,u,v) - f(s',u,v)\right| \leq \gamma\left|s-s'\right|, \quad \text{for all } s,s' \in [0,a],\ u,v \in \mathbb{R}$$

and

$$\left|\varphi(s) - \varphi(s')\right| \leq \delta\left|s-s'\right|, \quad \text{for all } s,s' \in [0,a],$$

where $a > 0$, $b, c \in \mathbb{R}$ and $\varphi : [0,a] \to \mathbb{R}$. Let $d = \max\{|b|, |c|\}$. Then $|x_0(s)| \leq d$, for any $s \in [0,a]$. Since f is continuous, there exists $M_0 \geq 0$ such that

$$M_0 = \max\{|f(s,u,v)| : s \in [0,a],\ u,v \in [-d,d]\}$$

and therefore $|f(s, x_0(s), x_0(\varphi(s)))| \leq M_0$ for any $s \in [0,a]$. The two-point boundary value problem (3.48) is equivalent with the functional integral equation

$$x(t) = \frac{ct}{a} + \frac{(a-t)b}{a} - \int_0^a G(t,s) \cdot f(s, x(s), x(\varphi(s)))ds,$$

where $G : [0,a] \times [0,a] \to \mathbb{R}$ is the well-known Green function:

$$G(t.s) = \begin{cases} \frac{s}{a}(a-t), & s \leq t \\ \frac{t}{a}(a-s), & s \geq t. \end{cases}$$

The above integral equation can be written in the form

$$x(t) = \frac{ct}{a} + \frac{(a-t)b}{a} - \int_0^t \frac{s(a-t)}{a} \cdot f(s, x(s), x(\varphi(s)))ds -$$

$$- \int_t^a \frac{t(a-s)}{a} \cdot f(s, x(s), x(\varphi(s)))ds, \qquad t \in [0,a]. \tag{3.49}$$

On $C[0,a]$ we apply the fixed point technique based on the Picard-Banach's principle to the operator $A : C[0,a] \to C[0,a]$, given by

$$A(x)(t) = \frac{ct}{a} + \frac{(a-t)\,b}{a} - \int_0^a G(t,s)\cdot f(s, x(s), x(\varphi(s)))ds$$

and under the conditions (i)-(iii) we obtain the convergence of the sequence of successive approximations:

$$x_0(t) = \frac{ct}{a} + \frac{(a-t)\,b}{a}, \qquad t \in [0,a],$$

$$x_k(t) = \frac{ct}{a} + \frac{(a-t)\,b}{a} - \int_0^t \frac{s\,(a-t)}{a} \cdot f(s, x_{k-1}(s), x_{k-1}(\varphi(s)))ds -$$

$$-\int_t^a \frac{t\,(a-s)}{a} \cdot f(s, x_{k-1}(s), x_{k-1}(\varphi(s)))ds, \qquad t \in [0,a],\ k \in \mathbb{N}^* \tag{3.50}$$

to the unique solution x^*, of the boundary value problem (3.48). Moreover, the following *apriori* and *a posteriori* error estimates are obtained:

$$|x_k(t) - x^*(t)| \le \frac{\left(a^2/8\right)^k (\alpha+\beta)^k}{1-(a^2/8)(\alpha+\beta)} \cdot \frac{M_0 a^2}{4}, \qquad \text{for all } t \in [0,a],\ k \in \mathbb{N}^*, \tag{3.51}$$

$$|x_k(t) - x^*(t)| \le \frac{(a^2/8)(\alpha+\beta)}{1-(a^2/8)(\alpha+\beta)} \cdot |x_k(t) - x_{k-1}(t)| \le$$

$$\le \frac{(a^2/8)(\alpha+\beta)}{1-(a^2/8)(\alpha+\beta)} \cdot \|x_k - x_{k-1}\|_C, \qquad \text{for all } t \in [0,a],\ k \in \mathbb{N}^*. \tag{3.52}$$

We can observe that the terms of the sequence of successive approximations (3.50) are uniformly bounded. Indeed, using the fixed point technique and the conditions (ii) and (iii), we get

$$|x_k(t) - x_{k-1}(t)| \le \int_0^a G(t,s)\cdot\left(\alpha\,|x_{k-1}(s) - x_{k-2}(s)| + \beta\,|x_{k-1}(\varphi(s)) - x_{k-2}(\varphi(s))|\right)ds \le$$

$$\le \int_0^a G(t,s)\cdot(\alpha+\beta)\,ds \cdot \|x_{k-1} - x_{k-2}\|_\infty \le \frac{a^2(\alpha+\beta)}{8} \cdot \|x_{k-1} - x_{k-2}\|_\infty \le \dots \le$$

$$\le \left[\frac{a^2(\alpha+\beta)}{8}\right]^{k-1} \cdot \|x_1 - x_0\|_\infty, \quad \forall t \in [0,a],\ k \in \mathbb{N}^*$$

and $\|x_1 - x_0\|_\infty \le \frac{M_0 a^2}{4}$. So,

$$|x_k(t) - x_0(t)| \le |x_k(t) - x_{k-1}(t)| + |x_{k-1}(t) - x_{k-2}(t)| + \dots + |x_1(t) - x_0(t)| \le$$

$$\le [1 + \frac{a^2(\alpha+\beta)}{8} + \dots + \left(\frac{a^2(\alpha+\beta)}{8}\right)^{k-2} + \left(\frac{a^2(\alpha+\beta)}{8}\right)^{k-1}] \cdot \frac{M_0 a^2}{4} \le \frac{M_0 a^2}{4[1-\frac{a^2(\alpha+\beta)}{8}]}$$

and

$$|x_k(t)| \le |x_k(t) - x_0(t)| + |x_0(t)| \le d + \frac{M_0 a^2}{4[1-\frac{a^2(\alpha+\beta)}{8}]} \overset{notation}{=} R, \quad \forall t \in [0,a],\ k \in \mathbb{N}^*.$$

Based on the continuity condition $f \in C([0,a] \times \mathbb{R} \times \mathbb{R})$ we infer that on the compact $[0,a] \times [-R,R] \times [-R,R]$ there exist $M \geq 0$ such that

$$M = \max\{|f(s,u,v)| : s \in [0,a], u,v \in [-R,R]\}.$$

Consequently, for $F_k : [0,a] \to \mathbb{R}$ we have,

$$|F_k(t)| = |f(t, x_k(t), x_k(\varphi(t)))| \leq M, \quad \forall t \in [0,a],\ k \in \mathbb{N}^*$$

and $|x_k''(t)| \leq M, \quad \forall t \in [0,a],\ k \in \mathbb{N}^*$.

In order to compute the terms of the sequence of successive approximations we consider the uniform partition of $[0,a]$ given by the knots $t_i = \frac{i \cdot a}{n},\ i = \overline{0,n}$. Let $h = \frac{a}{n}$. On these knots the relations (3.50) became

$$x_k(t_i) = \frac{ct_i}{a} + \frac{(a-t_i)b}{a} - \int_0^{t_i} \frac{s(a-t_i)}{a} \cdot f(s, x_{k-1}(s), x_{k-1}(\varphi(s)))ds$$

$$-\int_{t_i}^{a} \frac{t_i(a-s)}{a} \cdot f(s, x_{k-1}(s), x_{k-1}(\varphi(s)))ds, \quad i = \overline{0,n}, \quad k \in \mathbb{N}^*. \tag{3.53}$$

Define the functions $F_{k,i} : [0,a] \to \mathbb{R},\ \ i = \overline{0,n},\ k \in \mathbb{N}$, given by $F_{k,i}(s) = G(t_i,s) \cdot f(s, x_k(s), x_k(\varphi(s)))$. It obtains:

Theorem 69 *(see [71]) Under the conditions (i)-(v) the functions x_k'' and F_k, $k \in \mathbb{N}^*$ are Lipschitzian with the same Lipschitz constant $\overline{L} = \gamma + \alpha\left(3Ma + \frac{|c|+|b|}{a}\right) + \beta\delta\left(3Ma + \frac{|c|+|b|}{a}\right)$. Moreover, the functions $F_{k,.i}$, $i = \overline{0,n}$, $k \in \mathbb{N}$, are Lipschitzian with the constant $L = M + \frac{a}{4}\overline{L}$.*

Sketch of proof: After elementary calculus it obtains,

$$|G(t_i,s)| \leq \frac{a}{4} \text{ and } \left|G(t_i,s) - G(t_i,s')\right| \leq \left|s - s'\right|, \quad \forall i = \overline{0,n}.$$

Consequently,

$$\left|F_{0,i}(s) - F_{0,i}(s')\right| \leq [M + \frac{a}{4}\left(\gamma + \alpha\frac{|c|+|b|}{a} + \beta\delta\frac{|c|+|b|}{a}\right)] \cdot \left|s - s'\right|, \quad \forall i = \overline{0,n},$$

$$\left|F_{k,i}(s) - F_{k,i}(s')\right| \leq |f(s, x_k(s), x_k(\varphi(s)))| \cdot \left|G(t_i,s) - G(t_i,s')\right| +$$
$$+ \left|G(t_i,s')\right| \cdot \left|f(s, x_k(s), x_k(\varphi(s))) - f(s', x_k(s'), x_k(\varphi(s')))\right| \leq$$
$$\leq M\left|s - s'\right| + \frac{a}{4}[\gamma\left|s - s'\right| + \alpha\left|x_k(s) - x_k(s')\right| + \beta\left|x_k(\varphi(s)) - x_k(\varphi(s'))\right|],$$

and

$$\left|x_k(s) - x_k(s')\right| \leq [3Ma + \frac{|c|+|b|}{a}] \cdot \left|s - s'\right| = L_0\left|s - s'\right|, \quad \forall k \in \mathbb{N}^*.$$

So,

$$\left|F_k(s) - F_k(s')\right| \leq [\gamma + \alpha\left(3Ma + \frac{|c|+|b|}{a}\right) + \beta\delta\left(3Ma + \frac{|c|+|b|}{a}\right)] \cdot \left|s - s'\right| = \overline{L}\left|s - s'\right|$$

and

$$\left|F_{k,i}(s) - F_{k,i}(s')\right| \leq [M + \frac{a}{4}\left(\gamma + \alpha\left(3Ma + \frac{|c|+|b|}{a}\right) + \beta\delta\left(3Ma + \frac{|c|+|b|}{a}\right)\right)] \cdot$$

$$\cdot \left|s-s'\right| = L\left|s-s'\right|, \quad \forall s, s' \in [0,a], \quad \forall i=\overline{0,n},\ \forall k \in \mathbb{N}.$$

The algorithm

Applying the trapezoidal quadrature rule to the integrals from (3.53) it obtains,

$$x_0(t_i) = \frac{ct_i}{a} + \frac{(a-t_i)\,b}{a}, \quad \text{for all } i=\overline{0,n}$$

$$x_k(t_i) = \frac{ct_i}{a} + \frac{(a-t_i)\,b}{a} - \int_0^{t_i} \frac{s\,(a-t_i)}{a}\cdot f(s, x_{k-1}(s), x_{k-1}(\varphi(s)))ds -$$

$$-\int_{t_i}^{a} \frac{t_i\,(a-s)}{a}\cdot f(s, x_{k-1}(s), x_{k-1}(\varphi(s)))ds = \frac{ct_i}{a} + \frac{(a-t_i)\,b}{a} -$$

$$-\frac{a}{2n}\cdot\sum_{j=1}^{i}[\frac{t_{j-1}\,(a-t_i)}{a}\cdot f(t_{j-1}, x_{k-1}(t_{j-1}), x_{k-1}(\varphi(t_{j-1}))) +$$

$$+\frac{t_j\,(a-t_i)}{a}\cdot f(t_j, x_{k-1}(t_j), x_{k-1}(\varphi(t_j)))] - \frac{a}{2n}\cdot\sum_{j=i+1}^{n}[\frac{t_i\,(a-t_{j-1})}{a}\cdot$$

$$\cdot f(t_{j-1}, x_{k-1}(t_{j-1}), x_{k-1}(\varphi(t_{j-1}))) + \frac{t_i\,(a-t_j)}{a}\cdot f(t_j, x_{k-1}(t_j), x_{k-1}(\varphi(t_j)))] + R_{k,i}, \quad (3.54)$$

for all $i=\overline{0,n}$ and $k \in \mathbb{N}^*$.

Since the functions $F_{k,i}$, $i=\overline{0,n}$, $k\in\mathbb{N}$, are Lipschitzian with the same constant L, for the remainder estimation in (3.54) we have

$$|R_{k,i}| \le \frac{La^2}{4n}, \quad \text{for all } i=\overline{1,n}, \quad k\in\mathbb{N}^*. \quad (3.55)$$

Using the natural cubic spline interpolation procedure presented in Section 3.2, the following algorithm is obtained:

$$x_k(t_0) = \overline{x_k(t_0)} = b, \qquad x_k(t_n) = \overline{x_k(t_n)} = c, \qquad \forall k \in \mathbb{N} \quad (3.56)$$

$$x_0(t_i) = \frac{ct_i}{a} + \frac{(a-t_i)\,b}{a}, \quad \text{for all } i=\overline{1,n-1} \quad (3.57)$$

$$x_1(t_i) = \frac{ct_i}{a} + \frac{(a-t_i)\,b}{a} - \frac{a}{2n}\cdot\sum_{j=1}^{i}[\frac{t_{j-1}\,(a-t_i)}{a}\cdot f(t_{j-1}, x_0(t_{j-1}), x_0(\varphi(t_{j-1}))) +$$

$$+\frac{t_j\,(a-t_i)}{a}\cdot f(t_j, x_0(t_j), x_0(\varphi(t_j)))] - \frac{a}{2n}\cdot\sum_{j=i+1}^{n}[\frac{t_i\,(a-t_{j-1})}{a}\cdot f(t_{j-1}, x_0(t_{j-1}), x_0(\varphi(t_{j-1}))) +$$

$$+\frac{t_i\,(a-t_j)}{a}\cdot f(t_j, x_0(t_j), x_0(\varphi(t_j)))] + R_{1,i} = \overline{x_1(t_i)} + R_{1,i}, \quad \text{for all } i=\overline{1,n-1}. \quad (3.58)$$

By induction, for $k\ge 2$, we obtain:

$$x_k(t_i) = \frac{ct_i}{a} + \frac{(a-t_i)\,b}{a} - \frac{a}{2n}\cdot\sum_{j=1}^{i}[\frac{t_{j-1}\,(a-t_i)}{a}\cdot f(t_{j-1}, \overline{x_{k-1}(t_{j-1})} + \overline{R_{k-1,j-1}}, x_{k-1}(\varphi(t_{j-1}))) +$$

$$+\frac{t_j\,(a-t_i)}{a}\cdot f(t_j, \overline{x_{k-1}(t_j)} + \overline{R_{k-1,j}}, x_{k-1}(\varphi(t_j)))] - \frac{a}{2n}\cdot\sum_{j=i+1}^{n}[\frac{t_i\,(a-t_{j-1})}{a}\cdot$$

$$\cdot f(t_{j-1}, \overline{x_{k-1}(t_{j-1})} + \overline{R_{k-1,j-1}}, x_{k-1}(\varphi(t_{j-1}))) + \frac{t_i\,(a-t_j)}{a} \cdot f(t_j, \overline{x_{k-1}(t_j)} + \overline{R_{k-1,j}}, x_{k-1}(\varphi(t_j)))] +$$

$$+R_{k,i} = \frac{ct_i}{a} + \frac{(a-t_i)\,b}{a} - \frac{a}{2n} \cdot \sum_{j=1}^{i} [\frac{t_{j-1}\,(a-t_i)}{a} \cdot f(t_{j-1}, \overline{x_{k-1}(t_{j-1})}, s_{k-1}(\varphi(t_{j-1}))) + \frac{t_j\,(a-t_i)}{a} \cdot$$

$$\cdot f(t_j, \overline{x_{k-1}(t_j)} + \overline{R_{k-1,j}}, s_{k-1}(\varphi(t_j)))] - \frac{a}{2n} \cdot \sum_{j=i+1}^{n} [\frac{t_i\,(a-t_{j-1})}{a} \cdot f(t_{j-1}, \overline{x_{k-1}(t_{j-1})}, s_{k-1}(\varphi(t_{j-1}))) +$$

$$+\frac{t_i\,(a-t_j)}{a} \cdot f(t_j, \overline{x_{k-1}(t_j)}, s_{k-1}(\varphi(t_j)))] + \overline{R_{k,i}} = \overline{x_k(t_i)} + \overline{R_{k,i}}, \quad \forall i = \overline{1, n-1} \quad (3.59)$$

where $s_{k-1} : [0, a] \to \mathbb{R}$, is the natural cubic spline of interpolation as in (3.3), interpolating the values b, $\overline{x_{k-1}(t_i)}, i = \overline{1, n-1}, c$ and having the restrictions to the intervals $[t_{i-1}, t_i]$, $i = \overline{1, n}$:

$$s_{k-1}^{(i)}(t) = \frac{(t-t_{i-1})^3 \cdot M_{k-1}^{(i)} + (t_i - t)^3 \cdot M_{k-1}^{(i-1)}}{6h} + \frac{t_i - t}{h} \cdot \overline{x_{k-1}(t_{i-1})} +$$

$$+\frac{t - t_{i-1}}{h} \cdot \overline{x_{k-1}(t_i)} - \frac{hM_{k-1}^{(i-1)}}{6}\,(t_i - t) - \frac{hM_{k-1}^{(i)}}{6}\,(t - t_{i-1}), \quad t \in [t_{i-1}, t_i]. \quad (3.60)$$

The values $M_{k-1}^{(i)}$, $i = \overline{0, n}$ are obtained in the following recurrent way: for $i = \overline{1, n-1}$ let

$$a_i = 2, \quad b_i = c_i = \frac{1}{2}, \quad d_i = \frac{3}{h^2} \cdot [\overline{x_{k-1}(t_{i+1})} - 2\overline{x_{k-1}(t_i)} + \overline{x_{k-1}(t_{i-1})}].$$

Now, $\alpha_1 = \frac{c_1}{a_1}$ and for $i = \overline{2, n-2}$,

$$\omega_i = a_i - \alpha_{i-1} \cdot b_i, \qquad \alpha_i = \frac{c_i}{\omega_i}, \quad \omega_{n-1} = a_{n-1} - \alpha_{n-2} \cdot b_{n-1}.$$

Let $z_1 = \frac{d_1}{2}$ and for $i = \overline{2, n-1}$, it computes $z_i = \frac{d_i - b_i \cdot z_{i-1}}{\omega_i}$. Finally, $M_{k-1}^{(0)} = M_{k-1}^{(n)} = 0$, $M_{k-1}^{(n-1)} = z_{n-1}$ and for $i = \overline{n-2, 1}$:

$$M_{k-1}^{(i)} = z_i - \alpha_i \cdot M_{k-1}^{(i+1)}.$$

This algorithm has a practical stopping criterion below presented in Remark 73.

The convergence analysis

Theorem 70 *(see [71]) Under the conditions (i)-(iv), if* $\frac{a^2}{4}(\alpha + \beta) < 1$, *then the unique solution* x^*, *of the boundary value problem (3.48), is approximated on the knots* $t_i = \frac{i \cdot a}{n}$, $i = \overline{0, n}$ *by the sequence* $(\overline{x_k\,(t_i)})_{k \in \mathbb{N}^*}$ *given in (3.56)-(3.59), (3.60) and the apriori error estimation is:*

$$\left| x^*\,(t_i) - \overline{x_k\,(t_i)} \right| \leq \frac{\left(a^2/8\right)^k (\alpha+\beta)^k}{1 - (a^2/8)\,(\alpha+\beta)} \cdot \frac{M_0 a^2}{4} +$$

$$+\frac{La^2}{4n[1 - \frac{a^2}{4}(\alpha+\beta)]} + \frac{7\beta a^2}{4[1 - \frac{a^2}{4}(\alpha+\beta)]} \cdot \omega\,(V_{k-1}, h)\,, \quad \forall i = \overline{1, n-1}, \forall k \in \mathbb{N}^*, \quad (3.61)$$

where V_{k-1} *is defined bellow in (3.62)*

Sketch of proof: Since

$$\left|x^*(t_i) - \overline{x_k(t_i)}\right| \leq |x^*(t_i) - x_k(t_i)| +$$

$$+\left|x_k(t_i) - \overline{x_k(t_i)}\right| = |x^*(t_i) - x_k(t_i)| + \left|\overline{R_{k,i}}\right|, \quad \forall k \in \mathbb{N}^*, i = \overline{1, n-1}$$

remains to estimate $\left|\overline{R_{k,i}}\right|$.
Because $x_k(t_i) \neq \overline{x_k(t_i)}$, $\forall k \in \mathbb{N}^*, i = \overline{1, n-1}$, we infer that s_k interpolates the values $\overline{x_k(t_i)}$, $i = \overline{0, n}$, but not the function x_k. Therefore we define for any k the auxiliary function V_k, $k \in \mathbb{N}^*$, $V_k : [0, a] \to \mathbb{R}$ given by its restrictions to the subintervals $[t_{i-1}, t_i]$, $i = \overline{1, n}$, as follows:

$$V_k(t) = x_k(t) + [\overline{x_k(t_i)} - x_k(t_i)] \cdot \frac{t - t_{i-1}}{h} + [\overline{x_k(t_{i-1})} - x_k(t_{i-1})] \cdot \frac{t_i - t}{h}, \quad t \in [t_{i-1}, t_i]. \tag{3.62}$$

We see that $V_k(t_i) = \overline{x_k(t_i)}$, $\forall i = \overline{0, n}$, and V_k interpolates the values $\overline{x_k(t_i)}$, $i = \overline{0, n}$ being continuous. So, s_k interpolates the function V_k for any $k \in \mathbb{N}^*$ and V_k is uniformly continuous on the compact $[0, a]$.
Recurrently, by induction, it obtains

$$\left|\overline{R_{k,i}}\right| = \left|x_k(t_i) - \overline{x_k(t_i)}\right| \leq |R_{k,i}| + \frac{a}{2n} \cdot \sum_{j=1}^{i} [\frac{t_{j-1}(a - t_i)}{a} \cdot$$

$$\cdot \left(\alpha \left|\overline{R_{k-1,j-1}}\right| + \beta \left|x_{k-1}(\varphi(t_{j-1})) - s_{k-1}(\varphi(t_{j-1}))\right|\right) + \frac{t_j(a - t_i)}{a} \cdot$$

$$\cdot \left(\alpha \left|\overline{R_{k-1,j}}\right| + \beta \left|x_{k-1}(\varphi(t_j)) - s_{k-1}(\varphi(t_j))\right|\right)] + \frac{a}{2n} \cdot \sum_{j=i+1}^{n} [\frac{t_i(a - t_{j-1})}{a} \cdot$$

$$\cdot \left(\alpha \left|\overline{R_{k-1,j-1}}\right| + \beta \left|x_{k-1}(\varphi(t_{j-1})) - s_{k-1}(\varphi(t_{j-1}))\right|\right) +$$

$$+\frac{t_i(a - t_j)}{a} \cdot \left(\alpha \left|\overline{R_{k-1,j}}\right| + \beta \left|x_{k-1}(\varphi(t_j)) - s_{k-1}(\varphi(t_j))\right|\right)]., \quad \forall i = \overline{1, n-1}$$

These lead us to the necessity to estimate $|x_{k-1}(t) - s_{k-1}(t)|$ for $t \in [0, a]$ and $k \geq 2$. In this purpose we have for $k \geq 2$ and $i = \overline{1, n}$,

$$|x_{k-1}(t) - s_{k-1}(t)| \leq |x_{k-1}(t) - V_{k-1}(t)| + |V_{k-1}(t) - s_{k-1}(t)| \leq \left|\frac{t - t_{i-1}}{h}\right| \cdot \left|\overline{R_{k-1,i}}\right| +$$

$$+\left|\frac{t_i - t}{h}\right| \cdot \left|\overline{R_{k-1,i-1}}\right| + \frac{7}{4} \cdot \omega(V_{k-1}, h) \leq \max\left(\left|\overline{R_{k-1,i-1}}\right|, \left|\overline{R_{k-1,i}}\right|\right) + \frac{7}{4} \cdot \omega(V_{k-1}, h), \ \forall t \in [t_{i-1}, t_i].$$

So, by induction, for $k \geq 2$ it obtains:

$$\left|\overline{R_{k,i}}\right| \leq [1 + \frac{a^2}{4}(\alpha + \beta) + ... + \left(a^2/4\right)^{k-1} (\alpha + \beta)^{k-1}] \cdot \frac{La^2}{4n} +$$

$$+\beta a^2 \cdot [1 + \frac{a^2}{4}(\alpha + \beta) + ... + \left(a^2/4\right)^{k-2} (\alpha + \beta)^{k-2}] \cdot \frac{7}{4} \omega(V_{k-1}, h) =$$

$$= \frac{1 - \left(a^2/4\right)^k (\alpha + \beta)^k}{1 - \frac{a^2}{4}(\alpha + \beta)} \cdot \frac{La^2}{4n} + \frac{1 - \left(a^2/4\right)^{k-1} (\alpha + \beta)^{k-1}}{1 - \frac{a^2}{4}(\alpha + \beta)} \cdot \frac{7\beta a^2}{4} \cdot \omega(V_{k-1}, h),$$

$\forall i = \overline{1, n-1}$. According to the condition $\frac{a^2}{4}(\alpha + \beta) < 1$, the inequality (3.61) can be derived.

Remark 71 *From the error estimate (3.61), since* $\lim_{h\to 0} \omega(V_{k-1}, h) = 0$, *we see on the one hand that for* $k \to \infty$, $n \to \infty$, *it follows that* $\left| x^*(t_i) - \overline{x_k(t_i)} \right| \to 0$ *for any* $i = \overline{1, n-1}$. *This is the convergence of the proposed algorithm. On the other hand, the differences between the conditions for the existence and uniqueness of the solution and the conditions from Theorem 70 are: the contraction condition* $\frac{a^2}{8}(\alpha+\beta) < 1$ *is replaced by the convergence condition* $\frac{a^2}{4}(\alpha+\beta) < 1$ *and the supplementary Lipschitz condition (iv) is included. We observe that in order to obtain the result of Theorem 70 no smoothness or boundedness conditions are needed.*

Remark 72 *Under the conditions of Theorem 70 we can obtain continuous approximation of the solution. This is obtained interpolating the computed values* $\overline{x_k(t_i)}$, $i = \overline{0, n}$ *by the same procedure as in (3.60). Consequently, the continuous approximation of the solution,* $s_k : [0, c] \to \mathbb{R}$ *is given by its restrictions to the intervals* $[t_{i-1}, t_i]$, $i = \overline{1, n}$:

$$s_k^{(i)}(t) = \frac{(t - t_{i-1})^3 \cdot M_k^{(i)} + (t_i - t)^3 \cdot M_k^{(i-1)}}{6h} + \frac{t_i - t}{h} \cdot \overline{x_k(t_{i-1})} +$$

$$+ \frac{t - t_{i-1}}{h} \cdot \overline{x_k(t_i)} - \frac{h M_k^{(i-1)}}{6}(t_i - t) - \frac{h M_k^{(i)}}{6}(t - t_{i-1}), \ \forall t \in [t_{i-1}, t_i], \ i = \overline{1, n} \quad (3.63)$$

Moreover, the approximations of the second derivative on the knots $t_i = \frac{i \cdot a}{n}$, $i = \overline{0, n}$ *can be computed in:*

$$\left(\overline{x_k(t_i)}\right)'' = f\left(t_i, \ \overline{x_{k-1}(t_i)}, s_{k-1}(\varphi(t_i))\right), \quad i = \overline{0, n}. \quad (3.64)$$

Corollary (see [71]): The error estimates in (3.63) and (3.64) are:

$$|x^*(t) - s_k(t)| \le \frac{(a^2/8)^k (\alpha+\beta)^k}{1 - (a^2/8)(\alpha+\beta)} \cdot \frac{M_0 a^2}{4} + \frac{L a^2}{4n[1 - \frac{a^2}{4}(\alpha+\beta)]} +$$

$$+ \frac{7\beta a^2}{4[1 - \frac{a^2}{4}(\alpha+\beta)]} \cdot \omega(V_{k-1}, h), \quad \forall t \in [0, a], \quad \forall k \in \mathbb{N}^*, \quad (3.65)$$

$$\left| (x^*(t_i))'' - \left(\overline{x_k(t_i)}\right)'' \right| \le \alpha[\frac{(a^2/8)^k (\alpha+\beta)^k}{1 - (a^2/8)(\alpha+\beta)} \cdot \frac{M_0 a^2}{4} +$$

$$+ \frac{L a^2}{4n[1 - \frac{a^2}{4}(\alpha+\beta)]} + \frac{7\beta a^2}{4[1 - \frac{a^2}{4}(\alpha+\beta)]} \cdot \omega(V_{k-1}, h)] + \beta[\frac{(a^2/8)^k (\alpha+\beta)^k}{1 - (a^2/8)(\alpha+\beta)} \cdot$$

$$\cdot \frac{M_0 a^2}{4} + \frac{L a^2}{4n[1 - \frac{a^2}{4}(\alpha+\beta)]} + \frac{7\beta a^2}{4[1 - \frac{a^2}{4}(\alpha+\beta)]} \cdot \omega(V_{k-1}, h)] \quad (3.66)$$

for all $i = \overline{0, n}$ and $k \in \mathbb{N}^*$.

The proof is based on the inequalities

$$|x^*(t) - s_k(t)| \le |x^*(t) - x_k(t)| + |x_k(t) - s_k(t)|$$

and

$$\left| (x^*(t_i))'' - \left(\overline{x_k(t_i)}\right)'' \right| = \left| f(t_i, x^*(t_i), x^*(\varphi(t_i))) - f\left(t_i, \ \overline{x_{k-1}(t_i)}, s_{m-1}(\varphi(t_i))\right) \right| \le$$

$$\le \alpha \left| x^*(t_i) - \overline{x_{k-1}(t_i)} \right| + \beta |x^*(\varphi(t_i)) - s_{m-1}(\varphi(t_i))|, \quad \forall i = \overline{0, n}, \quad \forall k \in \mathbb{N}^*.$$

Remark 73 *(see [71]) Now, we can see that the 'a posteriori' (3.52) and 'apriori' (3.61) error estimates can give a practical stopping criterion of the algorithm. This can be stated as follows: For given $\varepsilon' > 0$ and given $n \in \mathbb{N}^*$ (previously chosen) it determines the first natural number $k \in \mathbb{N}^*$ for which,*

$$\left|\overline{x_k(t_i)} - \overline{x_{k-1}(t_i)}\right| < \varepsilon' \qquad \text{for all } i = \overline{1, n-1}$$

and we stop to this k, retaining the approximations $\overline{x_k(t_i)}$, $i = \overline{0, n}$, of the solution. A demonstration of this criterion is the following:
We denote:

$$\Omega = \frac{La^2}{4n[1 - \frac{a^2}{4}(\alpha + \beta)]} + \frac{7\beta a^2}{4[1 - \frac{a^2}{4}(\alpha + \beta)]} \cdot \omega(V_{k-1}, h).$$

For each $i = \overline{1, n-1}$ we have

$$\left|x^*(t_i) - \overline{x_k(t_i)}\right| \le |x^*(t_i) - x_k(t_i)| + \left|x_k(t_i) - \overline{x_k(t_i)}\right| \le$$

$$\le \frac{(a^2/8)(\alpha + \beta)}{1 - (a^2/8)(\alpha + \beta)} \cdot |x_k(t_i) - x_{k-1}(t_i)| + \left|\overline{R_{k,i}}\right|$$

and

$$|x_k(t_i) - x_{k-1}(t_i)| \le \left|x_k(t_i) - \overline{x_k(t_i)}\right| + \left|\overline{x_k(t_i)} - \overline{x_{k-1}(t_i)}\right| +$$

$$+ \left|\overline{x_{k-1}(t_i)} - x_{k-1}(t_i)\right| = \left|\overline{R_{k,i}}\right| + \left|\overline{R_{k-1,i}}\right| + \left|\overline{x_k(t_i)} - \overline{x_{k-1}(t_i)}\right|.$$

So,

$$\left|x^*(t_i) - \overline{x_k(t_i)}\right| \le \left|\overline{R_{k,i}}\right| + \frac{(a^2/8)(\alpha + \beta)}{1 - (a^2/8)(\alpha + \beta)} \cdot \left|\overline{x_k(t_i)} - \overline{x_{k-1}(t_i)}\right| +$$

$$+ \frac{(a^2/8)(\alpha + \beta)}{1 - (a^2/8)(\alpha + \beta)} \cdot \left(\left|\overline{R_{k,i}}\right| + \left|\overline{R_{k-1,i}}\right|\right).$$

Then

$$\left|x^*(t_i) - \overline{x_k(t_i)}\right| \le \Omega \cdot \frac{1 + (a^2/8)(\alpha + \beta)}{1 - (a^2/8)(\alpha + \beta)} + \frac{(a^2/8)(\alpha + \beta)}{1 - (a^2/8)(\alpha + \beta)} \cdot \left|\overline{x_k(t_i)} - \overline{x_{k-1}(t_i)}\right|.$$

For given $\varepsilon > 0$ we require

$$\Omega \cdot \frac{1 + (a^2/8)(\alpha + \beta)}{1 - (a^2/8)(\alpha + \beta)} < \frac{\varepsilon}{2} \tag{3.67}$$

and

$$\frac{(a^2/8)(\alpha + \beta)}{1 - (a^2/8)(\alpha + \beta)} \cdot \left|\overline{x_k(t_i)} - \overline{x_{k-1}(t_i)}\right| < \frac{\varepsilon}{2}.$$

According to the value of Ω, and since $\lim_{h \to 0} \omega(V_{k-1}, h) = 0$, we can chose the smallest natural number n, for which the inequality (3.67) holds. Afterwards we find the smallest natural number k (this is the last iterative step to be made) for which

$$\left|\overline{x_k(t_i)} - \overline{x_{k-1}(t_i)}\right| < \frac{\varepsilon}{2} \cdot \frac{1 - (a^2/8)(\alpha + \beta)}{(a^2/8)(\alpha + \beta)} = \varepsilon'$$

for all $i = \overline{1, n-1}$. With these we obtain $\left|x^*(t_i) - \overline{x_k(t_i)}\right| < \varepsilon$ for all $i = \overline{1, n-1}$.

The numerical stability with respect to the boundary values

Consider the two-point boundary value problem with the same second order differential equation, but with modified boundary values:

$$\begin{cases} x''(t) = f(t, x(t), x(\varphi(t))), & t \in [0,a] \\ x(0) = b', \quad x(a) = c' \end{cases} \tag{3.68}$$

such that $|b - b'| < \epsilon$ and $|c - c'| < \varepsilon$ for small $\epsilon > 0, \varepsilon > 0$.

Applying the above presented numerical method to the boundary value problem (3.68) we obtain the sequence of successive approximations on the knots $t_i = \frac{i \cdot a}{n}$, $i = \overline{0,n}$:

$$y_0(t_i) = \frac{c' t_i}{a} + \frac{(a - t_i) b'}{a}, \qquad i = \overline{0,n},$$

$$y_k(t_0) = b', \quad y_k(t_n) = c'$$

$$y_k(t_i) = \frac{c' t_i}{a} + \frac{(a - t_i) b'}{a} - \int_0^{t_i} \frac{s(a - t_i)}{a} \cdot f(s, y_{k-1}(s), y_{k-1}(\varphi(s)))ds$$

$$- \int_{t_i}^{a} \frac{t_i(a - s)}{a} \cdot f(s, y_{k-1}(s), y_{k-1}(\varphi(s)))ds, \quad i = \overline{1, n-1}, k \in \mathbb{N}^*.$$

The effective computed values are

$$y_0(t_i) = \frac{c' t_i}{a} + \frac{(a - t_i) b'}{a}, \qquad i = \overline{0,n},$$

$\overline{y_k(t_0)} = b'$, $\overline{y_k(t_n)} = c'$ and $\overline{y_k(t_i)}, i = \overline{1, n-1}, k \in \mathbb{N}^*$. The values $\overline{y_k(t_i)}, i = \overline{1, n-1}, k \in \mathbb{N}^*$ are computed in the same way as in (3.56)-(3.59), (3.60) and $y_k(t_i) = \overline{y_k(t_i)} + \overline{R'_{k,i}}$, $\forall i = \overline{1, n-1}$, $k \in \mathbb{N}^*$. We see that

$$|x_0(t) - y_0(t)| \le |b - b'| + |c - c'| < \epsilon + \varepsilon, \quad \forall t \in [0,a].$$

Definition 74 *(see [71]) We say that the successive interpolations method applied to the problem (3.48) is numerically stable with respect to the boundary values if there exist* $p \in \mathbb{N}^*$, *a sequence of continuous functions* $\mu_k : [0,a] \to [0,\infty)$, $k \in \mathbb{N}^*$ *with the property* $\lim_{h \to 0} \mu_k(h) = 0$, $\forall k \in \mathbb{N}^*$ *and the constants* $K_1, K_2, K_3, K_4 > 0$ *which not depend on* h, *such that*

$$\left| \overline{x_k(t_i)} - \overline{y_k(t_i)} \right| \le K_1 \epsilon + K_2 \varepsilon + K_3 \cdot h^p + K_4 \cdot \mu_k(h),$$

for all $i = \overline{1, n-1}, k \in \mathbb{N}^*$.

Theorem 75 *(see [71]) Under the conditions of Theorem 70, the successive interpolations method applied to the boundary value problem (3.48) is numerically stable with respect to the boundary values.*

Sketch of proof: Since

$$\left| \overline{x_k(t_i)} - \overline{y_k(t_i)} \right| \le \left| \overline{x_k(t_i)} - x_k(t_i) \right| + |x_k(t_i) - y_k(t_i)| + \left| y_k(t_i) - \overline{y_k(t_i)} \right| \le$$

$$\le |x_k(t_i) - y_k(t_i)| + \left| \overline{R_{k,i}} \right| + \left| \overline{R'_{k,i}} \right|, \quad \forall i = \overline{1, n-1}, \forall k \in \mathbb{N}^*$$

and

$$\left|\overline{R_{k,i}}\right|, \left|\overline{R'_{k,i}}\right| \leq \frac{La^2}{4n[1-\frac{a^2}{4}(\alpha+\beta)]} + \frac{7\beta a^2}{4[1-\frac{a^2}{4}(\alpha+\beta)]} \cdot \omega\left(V_{k-1},h\right), \ \forall i=\overline{1,n-1}, \forall k \in \mathbb{N}^*$$

in inductive manner, it obtains

$$|x_0(t) - y_0(t)| < \epsilon + \varepsilon, \quad \forall t \in [0,a]$$

$$|x_k(t_0) - y_k(t_0)| \leq \left|b - b'\right| < \epsilon, \quad \forall k \in \mathbb{N}^*,$$

$$|x_k(t_n) - y_k(t_n)| \leq \left|c - c'\right| < \varepsilon, \quad \forall k \in \mathbb{N}^*,$$

$$|x_1(t) - y_1(t)| \leq [1 + (\alpha+\beta)\frac{a^2}{8}] \cdot (\epsilon+\varepsilon), \quad \forall t \in [0,a],$$

and for $k \geq 2$,

$$|x_k(t) - y_k(t)| \leq \frac{(\epsilon+\varepsilon)}{1-(\alpha+\beta)\frac{a^2}{8}}, \quad \forall t \in [0,a], \quad \forall k \in \mathbb{N}^*.$$

So, for any $k \in \mathbb{N}^*$, we it follows that

$$\left|\overline{x_k(t_i)} - \overline{y_k(t_i)}\right| \leq \frac{(\epsilon+\varepsilon)}{1-(\alpha+\beta)\frac{a^2}{8}} + \frac{La}{2[1-\frac{a^2}{4}(\alpha+\beta)]} \cdot \left(\frac{a}{n}\right) +$$

$$+\frac{7\beta a^2}{4[1-\frac{a^2}{4}(\alpha+\beta)]} \cdot \omega\left(V_{k-1},h\right) = K_1\epsilon + K_2\varepsilon + K_3 \cdot h^p + K_4 \cdot \mu_k(h), \ \forall i = \overline{1,n-1}$$

with $p=1$, $K_1 = K_2 = \frac{1}{1-(\alpha+\beta)\frac{a^2}{8}}$, $K_3 = \frac{La}{2[1-\frac{a^2}{4}(\alpha+\beta)]}$, $K_4 = \frac{7\beta a^2}{4[1-\frac{a^2}{4}(\alpha+\beta)]}$ and $\mu_k(h) = \omega\left(V_{k-1},h\right)$.

Numerical experiments

In order to test the theoretical results and to show the accuracy of the method, we present below three numerical examples. The numerical results in Examples 2 and 3 present a comparison between the use of the Birkhoff interpolation procedure and of the natural cubic spline.

Example 1: Firstly, we present the following test two-point boundary value problem:

$$\begin{cases} x''(t) = \frac{2}{3}x(t) + \frac{1}{3}e^{\frac{t}{2}}x(\frac{t}{2}), t \in [0, \frac{1}{2}] \\ x(0) = 1, x\left(\frac{1}{2}\right) = \sqrt{e} \end{cases}.$$

Here, $a = \frac{1}{2}, b = 1, c = \sqrt{e}$, $\varphi(t) = \lambda t$ with $\lambda = \frac{1}{2}$ and $f(t,u,v) = \frac{2}{3}u + \frac{1}{3}e^{\frac{t}{2}}v$. The exact solution is $x^*(t) = e^t$, $t \in [0, \frac{1}{2}]$. Applying the above presented algorithm with $n = 10$, $n = 100$, $n = 1000$ and $\varepsilon' = 10^{-12}$, we obtain $k = 10$ (the number of iterations to be made). In the numerical calculations we use a precision of 22 decimals. The errors $e_i = \left|x^*(t_i) - \overline{x_{10}(t_i)}\right|$, $i = \overline{0,n}$ can be found in the second, third, and fourth column. We see that the accuracy is $O\left(10^{-6}\right)$ for $n = 10$, $O\left(10^{-8}\right)$ for $n = 100$, and $O\left(10^{-10}\right)$ for $n = 1000$, that confirms the convergence of the algorithm. In order to test the numerical stability of the method we consider $\epsilon = \varepsilon = 0.0001$ and the differences between the effective computed values $d_i = \left|\overline{x_{10}(t_i)} - \overline{y_{10}(t_i)}\right|$, $i = \overline{0,n}$ for $n = 10$, are placed in the fifth column.

t_i	e_i, $n = 10$	e_i, $n = 100$	e_i, $n = 1000$	d_i
0	0	0	0	1.000e-004
0,05	2.699071e-006	2.770881e-008	2.771270e-010	8.347e-004
0,10	4.953936e-006	5.000326e-008	5.000675e-010	8.423e-004
0,15	6.643763e-006	6.664720e-008	6.665031e-010	8.533e-004
0,20	7.715454e-006	7.738440e-008	7.738710e-010	8.683e-004
0,25	8.168366e-006	8.193291e-008	8.193519e-010	8.978e-004
0,30	7.979337e-006	7.998374e-008	7.998557e-010	8.825e-004
0,35	7.106930e-006	7.119941e-008	7.120082e-010	8.532e-004
0,40	5.512279e-006	5.521239e-008	5.521335e-010	8.307e-004
0,45	3.157611e-006	3.162347e-008	3.162395e-010	8.260e-004
0,50	0	0	0	1.000e-004

Table: numerical results for the test Example 1

Example 2: The two-point boundary value problem

$$\begin{cases} x''(t) = -2e^{-t} + \frac{x(t)}{2} + e^{-\frac{t}{2}} \cdot x\left(\frac{t}{2}\right), & t \in [0,1] \\ x(0) = 0, \quad x(1) = e^{-1} \end{cases} .$$

has exact solution $x^*(t) = te^{-t}$ and applying the above presented algorithm with $n = 10$ and $\varepsilon' = 10^{-15}$ we get $m = 19$ iterations. In order to test the numerical stability we consider $\varepsilon = 0.1$ that is

$$|c - \overline{c}| = \left|d - \overline{d}\right| = \varepsilon.$$

The results are in Table 1. For comparison, the computed errors obtained using the Birkhoff interpolation procedure are in Table 2.

t_i	e_i, $n = 10$	e_i, $n = 100$	e_i, $n = 1000$	d_i
0.0	0	0	0	1.000000e-001
0.1	8.347134e-005	8.987888e-007	8.991558e-009	2.222297e-001
0.2	1.470296e-004	1.512476e-006	1.512797e-008	1.786435e-001
0.3	1.861387e-004	1.882085e-006	1.882362e-008	1.379872e-001
0.4	2.022286e-004	2.042901e-006	2.043135e-008	1.000010e-001
0.5	2.004134e-004	2.024831e-006	2.025024e-008	6.399758e-002
0.6	1.837264e-004	1.853078e-006	1.853231e-008	2.950494e-002
0.7	1.537791e-004	1.548741e-006	1.548855e-008	3.802391e-003
0.8	1.121958e-004	1.129326e-006	1.129402e-008	3.626476e-002
0.9	6.054034e-005	6.092045e-007	6.092420e-009	6.822615e-002
1.0	0	0	0	1.000000e-001

Table 1, using natural cubic spline interpolation

t_i	e_i, $n=10$	e_i, $n=100$	e_i, $n=1000$
0.0	0	0	0
0.1	5.107399970e-005	5.9062251e-007	5.61975e-009
0.2	9.506771118e-005	9.4956851e-007	9.45501e-009
0.3	1.1405923019e-004	1.16371794e-006	1.176478e-008
0.4	1.2836330898e-004	1.25380213e-006	1.276962e-008
0.5	1.2396752180e-004	1.23741408e-006	1.265642e-008
0.6	1.1644682209e-004	1.12940094e-006	1.158271e-008
0.7	9.479543085e-005	9.4220758e-007	9.68035e-009
0.8	7.095277201e-005	6.8617669e-007	7.05877e-009
0.9	3.63367244e-005	3.6981074e-007	3.80777e-009
1.0	0	0	0

Table 2, using Birkhoff interpolation

Example 3: The two-point boundary value problem

$$\begin{cases} x''(t) = 1 + 2\left(1 + t^2/8\right)\cos(t/2) - 2\cos(t/2)\cdot x(t/2), & t \in [0, \frac{\pi}{4}] \\ x(0) = 1, \qquad x(\pi/4) = 1 + \frac{\sqrt{2}}{2} + \pi^2/32 \end{cases}.$$

has the exact solution $x^*(t) = \frac{t^2}{2} + \sin t + 1$, $t \in [0, \frac{\pi}{4}]$. For $n = 10$, $\varepsilon' = 10^{-15}$ we obtain the number of iterations $m = 17$ with Birkhoff interpolations and $m = 15$ with natural cubic splines. In Table 3 we present the errors $e_i = \left| x^*(t_i) - \overline{x_m(t_i)} \right|$, $i = \overline{0,n}$ obtained by using the natural cubic spline interpolation procedure, while in Table 4 there are those obtained with the Birkhoff interpolation.

t_i	e_i, $n=10$	e_i, $n=100$	e_i, $n=1000$
0	0	0	0
0.0785	7.080501e-006	4.405040e-008	4.390106e-010
0.1571	1.038505e-005	8.518104e-008	8.504168e-010
0.2356	1.310659e-005	1.208132e-007	1.206855e-009
0.3142	1.590362e-005	1.482504e-007	1.481361e-009
0.3927	1.759446e-005	1.648578e-007	1.647587e-009
0.4712	1.768575e-005	1.680825e-007	1.680002e-009
0.5498	1.618056e-005	1.554724e-007	1.554086e-009
0.6283	1.291040e-005	1.246943e-007	1.246504e-009
0.7069	7.590222e-006	7.355158e-008	7.352896e-010
0.7854	0	0	0

Table 3

t_i	e_i, $n = 10$	e_i, $n = 100$	e_i, $n = 1000$
0	0	0	0
0.0785	2.632026e-005	2.4115e-007	2.41e-009
0.1571	4.315370e-005	4.3261e-007	4.33e-009
0.2356	5.916750e-005	5.7246e-007	5.72e-009
0.3142	6.573055e-005	6.5906e-007	6.59e-009
0.3927	7.078812e-005	6.9097e-007	6.91e-009
0.4712	6.654985e-005	6.6707e-007	6.67e-009
0.5498	6.016255e-005	5.8648e-007	5.86e-009
0.6283	4.474781e-005	4.4860e-007	4.49e-009
0.7069	2.669843e-005	2.5312e-007	2.53e-009
0.7854	0	0	0

Table 4

3.6 Two-point boundary value problems of fourth order

In the investigation of the deformations of elastic beams, the study of appropriate mathematical models involves the beam equation which begins with Euler and Bernoulli in the 18th century revealing the phenomenon of curvature of elastic beams. The modern theory of the beam equation starts from 1921 and four theories were developed in the last one hundred years based on the following models: the Euler-Bernoulli model, the Rayleigh model (since 1877), the Timoshenko model (see [102]) and the shear model (see [227]). We present here the approximation of the solution of a nonlinear variant with deviating argument of the Euler-Bernoulli differential beam equation under clamped ends. The corresponding two-point boundary value problem is:

$$\begin{cases} x^{IV}(t) = f(t, x(t), x(\varphi(t))), & t \in [a,b] \\ x(a) = c, & x(b) = d \\ x'(a) = w, & x'(b) = r \end{cases} \tag{3.69}$$

with $f \in C([a,b] \times \mathbb{R} \times \mathbb{R})$, $\varphi \in C[a,b]$, $a \leq \varphi(t) \leq b$, $\forall t \in [a,b]$ and $c, d, w, r \in \mathbb{R}$. In the study of this boundary value problem we consider the equivalent Hammerstein-Fredholm integral equation:

$$x(t) = g(t) + \int_a^b G(t,s) \cdot f(s, x(s), x(\varphi(s)))\, ds, \quad t \in [a,b] \tag{3.70}$$

where

$$g(t) = \frac{(b-t)^2 [2(t-a) + (b-a)]}{(b-a)^3} \cdot c + \frac{(t-a)^2 [2(b-t) + (b-a)]}{(b-a)^3} \cdot d + \tag{3.71}$$

$$+ \frac{(b-t)^2 (t-a)}{(b-a)^2} \cdot w - \frac{(t-a)^2 (b-t)}{(b-a)^2} \cdot r, \quad t \in [a,b].$$

Here

$$G(t,s) = \tag{3.72}$$

$$= \begin{cases} H(t,s) = \frac{1}{6}\left(\frac{s-a}{b-a}\right)^2 \left(1 - \frac{t-a}{b-a}\right)^2 \cdot \left[\left(\frac{t-a}{b-a} - \frac{s-a}{b-a}\right) + 2\left(1 - \frac{s-a}{b-a}\right)\left(\frac{t-a}{b-a}\right)\right], & s \leq t \\ K(t,s) = \frac{1}{6}\left(\frac{t-a}{b-a}\right)^2 \left(1 - \frac{s-a}{b-a}\right)^2 \cdot \left[\left(\frac{s-a}{b-a} - \frac{t-a}{b-a}\right) + 2\left(1 - \frac{t-a}{b-a}\right)\left(\frac{s-a}{b-a}\right)\right], & s \geq t \end{cases}$$

is the corresponding Green's function. In order to prove the convergence of the method, we require only Lipschitz conditions, without any smoothness or boundedness conditions. As we can see, in [224] and [225] first order partial differentiability conditions for f are required and the boundedness of the first order partial derivatives is requested.

In the study of the boundary value problem (3.69) few papers are published. More studied is the ordinary differential equation variant

$$\begin{cases} x^{IV}(t) = f(t, x(t)), \quad t \in [a,b] \\ x(a) = c, \qquad x(b) = d \\ x'(a) = w, \qquad x'(b) = r \end{cases} \tag{3.73}$$

from the existence of positive solutions point of view and for constructing numerical methods. The existence results for the problem (3.73) can be found in [7], [157], [235], and [243]. The numerical methods for (3.73) are based on: Picard's and quasilinear iterative methods (see [6]), spline functions methods (see [185], [231], and [232]), finite differences methods (see [230]), Galerkin methods (see [102] and [240]), shooting methods (see [11]), homotopy perturbation methods (see [244]), and collocation methods (see [226]).

The study of functional beam equations is realized in [155], [224], [225], and [234]. In [155], the existence of multiple positive solutions of the equation

$$x^{IV}(t) = h(t) \cdot f(t, x(t), x(\alpha(t))), \quad t \in [0,1]$$

is obtained using the fixed point theorem of Avery and Peterson (see [17]). In [224] and [225] it is approached the boundary value problem with retardation and anticipation:

$$\begin{cases} x^{(4)}(t) - c_1 x^{(4)}(t-\tau_1) - c_2 x^{(4)}(t+\tau_2) = f(t, \overline{x}(t), \overline{x}(t-\tau_1), \overline{x}(t+\tau_2)) \\ x(t) = 0, \quad t \in [a-\tau_1, a] \cup [b, b+\tau_2] \\ x'(a+0) = x'(b-0) = 0 \end{cases}$$

where $\overline{x}(t) = (x(t), x'(t), x''(t), x'''(t)), t \in (a,b)$, using the quasilinear iterative method and the Picard's iterative method, respectively. The authors of [224] and [225] obtain the existence and uniqueness of the solution and a convergent numerical method.

The sequence of successive approximations for the beam equation

Here we present the application of the method of successive interpolations to the two-point boundary value problem (3.69) written in the equivalent form of integral equation (3.70).

Consider the following conditions:

(i) $f \in C([a,b] \times \mathbb{R} \times \mathbb{R})$ and $\varphi \in C[a,b]$, with $a \le \varphi(t) \le b$, $\forall t \in [a,b]$

(ii) there exist $\alpha, \beta \ge 0$ such that

$$\left| f(t,u,v) - f\left(t,u',v'\right) \right| \le \alpha \left| u - u' \right| + \beta \left| v - v' \right|, \quad \text{for all } t \in [a,b],\ u,v,u',v' \in \mathbb{R}$$

(iii) $\frac{(b-a)(\alpha+\beta)}{192} < 1$

(iv) there exist $\gamma, \eta \ge 0$ such that

$$\left| f(t,u,v) - f\left(t',u,v\right) \right| \le \gamma \left| t - t' \right|$$

$$\left| \varphi(t) - \varphi\left(t'\right) \right| \le \eta \left| t - t' \right|$$

for all $t, t' \in [a,b]$, $u, v \in \mathbb{R}$.

Let $f_0 : [a,b] \to \mathbb{R}$, be given by $f_0(s) = f(s, g(s), g(\varphi(s)))$, $s \in [a,b]$. Since f, g, φ are continuous we infer that f_0 is continuous on $[a,b]$ and therefore there exists $M_0 \ge 0$ such that

$$|f_0(s)| = |f(s, g(s), g(\varphi(s)))| \le M_0, \text{ for all } s \in [a,b].$$

Applying the Banach's fixed point theorem to the operator $A : C[a,b] \to C[a,b]$

$$A(x)(t) = g(t) + \int_a^b G(t,s) \cdot f(s, x(s), x(\varphi(s)))\, ds, \quad x \in C[a,b], \quad t \in [a,b]$$

under the conditions (i)-(iii), it obtains the existence and uniqueness of the solution $x^* \in C[a,b]$ of (3.70). Moreover, in the approximation of x^* by the terms of the sequence of successive approximations

$$x_0(t) = g(t), \quad t \in [a,b] \tag{3.74}$$

$$x_m(t) = g(t) + \int_a^b G(t,s) \cdot f(s, x_{m-1}(s), x_{m-1}(\varphi(s)))\, ds, \quad m \in \mathbb{N}^*, \quad t \in [a,b] \tag{3.75}$$

the following *apriori* and *a posteriori* error estimates hold:

$$|x^*(t) - x_m(t)| \le \frac{\frac{(b-a)^m (\alpha+\beta)^m}{(192)^m}}{1 - \frac{(b-a)(\alpha+\beta)}{192}} \cdot \|x_1 - x_0\|_\infty \le \tag{3.76}$$

$$\le \frac{\frac{(b-a)^m (\alpha+\beta)^m}{(192)^m}}{1 - \frac{(b-a)(\alpha+\beta)}{192}} \cdot \frac{(b-a)M_0}{192}, \quad \forall m \in \mathbb{N}^*, \quad \forall t \in [a,b]$$

$$|x_m(t) - x^*(t)| \le \frac{\frac{(b-a)(\alpha+\beta)}{192}}{1 - \frac{(b-a)(\alpha+\beta)}{192}} \cdot \|x_m - x_{m-1}\|_\infty, \quad \forall m \in \mathbb{N}^*, \quad \forall t \in [a,b]. \tag{3.77}$$

After elementary calculus with four time differentiation in the relation

$$x^*(t) = g(t) + \int_a^b G(t,s) \cdot f(s, x^*(s), x^*(\varphi(s)))\, ds, \quad t \in [a,b]$$

we infer that $x^* \in C^4[a,b]$ is the unique solution of the boundary value problem (3.69).

The hypotheses (i)-(iii) lead to the uniformly boundedness of the sequence of successive approximations,

$$|x_m(t)| \le M_g + \frac{(b-a)M_0}{192\left(1 - \frac{(b-a)(\alpha+\beta)}{192}\right)} \overset{notation}{=} R$$

for all $t \in [a,b]$ and $m \in \mathbb{N}^*$, where $M_g \ge 0$ is such that

$$|g(t)| \le \max\{|c|, |d|\} + \frac{b-a}{4} \cdot \max\{|w|, |r|\} \overset{notation}{=} M_g, \quad \forall t \in [a,b].$$

Moreover, considering

$$M = \max\{M_0, \max\{|f(t,u,v)| : t \in [a,b], u, v \in [-R,R]\}\}$$

it follows that

$$|f(s, x_m(s), x_m(\varphi(s)))| \le M, \quad \forall s \in [a,b], \ \forall m \in \mathbb{N}.$$

Consider the uniform partition of $[a,b]$

$$\Delta : a = t_0 < t_1 < ... < t_{n-1} < t_n = b$$

with the knots $t_i = a + \frac{i(b-a)}{n}$, $i = \overline{0,n}$. On these knots, the relation (3.75) becomes:

$$x_m(t_i) = g(t_i) + \int_a^b G(t_i, s) \cdot f(s, x_{m-1}(s), x_{m-1}(\varphi(s)))\, ds, \quad i = \overline{0,n}. \tag{3.78}$$

We define the functions $F_{m,i} : [a,b] \to \mathbb{R}$, $m \in \mathbb{N}$, $i = \overline{0,n}$

$$F_{m,i}(s) = G(t_i, s) \cdot f(s, x_m(s), x_m(\varphi(s))), \quad s \in [a,b]$$

and obtain:

Theorem 76 *(see [86]) Under the conditions (i)-(iv), the functions $F_{m,i}$, $m \in \mathbb{N}$, $i = \overline{0,n}$, are uniformly Lipschitz with the constant*

$$L = \frac{3M}{2(b-a)} + \gamma + (\alpha + \beta\eta)\left(\frac{3M}{2} + \frac{8|c| + 8|d|}{b-a} + 3|w| + 3|r|\right). \tag{3.79}$$

The algorithm of successive interpolations

In the computation of the terms of the sequence of successive approximations (3.78) on the knots of the grid, we combine the trapezoidal quadrature rule with the natural cubic spline interpolation procedure in a similar manner as in the previous section and the following numerical method is obtained:

$$x_m(t_0) = c, \quad x_m(t_n) = d, \quad \forall m \in \mathbb{N} \tag{3.80}$$

$$x_0(t) = g(t), \quad t \in [a,b], \quad x_0(t_i) = g(t_i), \quad i = \overline{0,n} \tag{3.81}$$

$$x_m(t_i) = g(t_i) + \int_a^b G(t_i, s) \cdot f(s, x_{m-1}(s), x_{m-1}(\varphi(s)))\, ds = \tag{3.82}$$

$$= g(t_i) + \int_a^b F_{m-1,i}(s)\, ds = g(t_i) + \frac{(b-a)}{2n} \cdot [\sum_{j=1}^{i} [H(t_i, t_{j-1}) \cdot$$

$$\cdot f(t_{j-1}, x_{m-1}(t_{j-1}), x_{m-1}(\varphi(t_{j-1}))) + H(t_i, t_j) \cdot f(t_j, x_{m-1}(t_j), x_{m-1}(\varphi(t_j)))] +$$

$$+ \sum_{j=i+1}^{n} [H(t_i, t_{j-1}) \cdot f(t_{j-1}, x_{m-1}(t_{j-1}), x_{m-1}(\varphi(t_{j-1}))) +$$

$$+ H(t_i, t_j) \cdot f(t_j, x_{m-1}(t_j), x_{m-1}(\varphi(t_j)))]] + R_{m,i}, \quad i = \overline{1, n-1}, \ m \in \mathbb{N}^*$$

with the remainder estimate

$$|R_{m,i}| \le \frac{(b-a)^2 L}{4n}, \quad \forall i = \overline{0,n}, \ \forall m \in \mathbb{N}^*. \tag{3.83}$$

These lead to the following algorithm:

$$x_m(t_0) = c, \quad x_m(t_n) = d, \quad \forall m \in \mathbb{N}$$

$$x_0(t) = g(t), \quad t \in [a,b], \quad x_0(t_i) = g(t_i), \quad i = \overline{0,n}$$

$$x_1(t_i) = g(t_i) + \frac{(b-a)}{2n} \cdot [\sum_{j=1}^{i} [H(t_i, t_{j-1}) \cdot f(t_{j-1}, x_0(t_{j-1}), x_0(\varphi(t_{j-1}))) + \tag{3.84}$$

$$+H(t_i,t_j)\cdot f(t_j,x_0(t_j),x_0(\varphi(t_j)))]+\sum_{j=i+1}^{n}[K(t_i,t_{j-1})\cdot f(t_{j-1},x_0(t_{j-1}),x_0(\varphi(t_{j-1})))+$$

$$+K(t_i,t_j)\cdot f(t_j,x_0(t_j),x_0(\varphi(t_j)))]]+R_{1,i}=\overline{x_1(t_i)}+R_{1,i},\quad i=\overline{1,n-1}$$

$$x_1(t_0)=\overline{x_1(t_0)}=c,\quad x_1(t_n)=\overline{x_1(t_n)}=d.$$

By induction for $m\geq 2$ it obtains:

$$x_m(t_i)=g(t_i)+\frac{(b-a)}{2n}\cdot[\sum_{j=1}^{i}[H(t_i,t_{j-1})\cdot f\left(t_{j-1},\overline{x_{m-1}(t_{j-1})}+\overline{R_{m-1,j-1}},x_{m-1}(\varphi(t_{j-1}))\right)+ \tag{3.85}$$

$$+H(t_i,t_j)\cdot f\left(t_j,\overline{x_{m-1}(t_j)}+\overline{R_{m-1,j}},x_{m-1}(\varphi(t_j))\right)]+\sum_{j=i+1}^{n}[K(t_i,t_{j-1})\cdot$$

$$\cdot f\left(t_{j-1},\overline{x_{m-1}(t_{j-1})}+\overline{R_{m-1,j-1}},x_{m-1}(\varphi(t_{j-1}))\right)+$$

$$+K(t_i,t_j)\cdot f\left(t_j,\overline{x_{m-1}(t_j)}+\overline{R_{m-1,j}},x_{m-1}(\varphi(t_j))\right)]]+$$

$$+R_{m,i}=g(t_i)+\frac{(b-a)}{2n}\cdot[\sum_{j=1}^{i}[H(t_i,t_{j-1})\cdot f\left(t_{j-1},\overline{x_{m-1}(t_{j-1})},s_{m-1}(\varphi(t_{j-1}))\right)+$$

$$+H(t_i,t_j)\cdot f\left(t_j,\overline{x_{m-1}(t_j)},s_{m-1}(\varphi(t_j))\right)]+$$

$$+\sum_{j=i+1}^{n}[K(t_i,t_{j-1})\cdot f\left(t_{j-1},\overline{x_{m-1}(t_{j-1})},s_{m-1}(\varphi(t_{j-1}))\right)+$$

$$+K(t_i,t_j)\cdot f\left(t_j,\overline{x_{m-1}(t_j)},s_{m-1}(\varphi(t_j))\right)]]+\overline{R_{m,i}}=\overline{x_m(t_i)}+\overline{R_{m,i}},\quad i=\overline{1,n-1}$$

$$x_m(t_0)=\overline{x_m(t_0)}=c,\quad x_m(t_n)=\overline{x_m(t_n)}=d,\quad m\in\mathbb{N}^*$$

where $s_{m-1}:[a,b]\to\mathbb{R}$, is the natural cubic spline interpolating the values $\overline{x_{m-1}(t_i)}$, $i=\overline{0,n}$ and having the restrictions to the intervals $[t_{i-1},t_i]$, $i=\overline{1,n}$:

$$s_{m-1}^{(i)}(t)=\left[\frac{(t-t_{i-1})^2}{2}-\frac{(t-t_{i-1})^3}{6h}-\frac{h(t-t_{i-1})}{3}\right]\cdot M_{m-1}^{(i-1)}+ \tag{3.86}$$

$$+\left[\frac{(t-t_{i-1})^3}{6h}-\frac{h(t-t_{i-1})}{6}\right]\cdot M_{m-1}^{(i)}+\frac{t-t_{i-1}}{h}\cdot\overline{x_{m-1}(t_i)}+\frac{t_i-t}{h}\cdot\overline{x_{m-1}(t_{i-1})}$$

where $M_{m-1}^{(0)}=M_{m-1}^{(n)}=0$ and $M_{m-1}^{(i)}$, $i=\overline{1,n-1}$ are recurrently given by:

$$a_i=2,\quad b_i=c_i=\frac{1}{2},\quad d_i=\frac{3}{h^2}\cdot[\overline{x_{m-1}(t_{i+1})}+2\overline{x_{m-1}(t_i)}-\overline{x_{m-1}(t_{i-1})}],\quad i=\overline{1,n-1}$$

and

$$\alpha_1=\frac{c_1}{a_1},\quad \omega_i=a_i-\alpha_{i-1}\cdot b_i,\quad \alpha_i=\frac{c_i}{\omega_i},\quad i=\overline{2,n-2}$$

$$\omega_{n-1}=a_{n-1}-\alpha_{n-2}\cdot b_{n-1}$$

$$z_1=\frac{d_1}{2},\quad z_i=\frac{d_i-b_i\cdot z_{i-1}}{\omega_i},\quad i=\overline{2,n-1}.$$

Using the backward recurrence,

$$M_{m-1}^{(n-1)} = z_{n-1}, \quad M_{m-1}^{(i)} = z_i - \alpha_i \cdot M_{m-1}^{(i+1)}, \quad i = \overline{n-2,1}.$$

The effective computed approximations are $\overline{x_m(t_i)}$, $i = \overline{0,n}$ and the algorithm has the following practical stopping criterion:

For given $\varepsilon' > 0$ and given $n \in \mathbb{N}^*$ (previously chosen) it determines the first natural number $m \in \mathbb{N}^*$ for which,

$$\left| \overline{x_m(t_i)} - \overline{x_{m-1}(t_i)} \right| < \varepsilon' \qquad \text{for all } i = \overline{1,n-1}$$

and we stop to this m, retaining the approximations $\overline{x_m(t_i)}$, $i = \overline{0,n}$, of the solution.

The convergence analysis

Theorem 77 *(see [86]) Under the conditions (i)-(iv), if* $\frac{(b-a)(\alpha+\beta)}{64} < 1$, *the solution of the boundary value problem (3.69) is approximated on the knots* $t_i = a + \frac{i(b-a)}{n}$, $i = \overline{0,n}$, *by the terms of the sequence* $\left(\overline{x_m(t_i)}\right)_{m \in \mathbb{N}^*}$, *given in (3.84)-(3.86), and the apriori error estimate is:*

$$\left| x^*(t_i) - \overline{x_m(t_i)} \right| \le \frac{\frac{(b-a)^m(\alpha+\beta)^m}{(192)^m}}{1 - \frac{(b-a)(\alpha+\beta)}{192}} \cdot \frac{(b-a)M_0}{192} + \frac{(b-a)^2 L}{4n\left[1 - \frac{(b-a)(\alpha+\beta)}{64}\right]} + \tag{3.87}$$

$$+\frac{7\beta(b-a)}{256\left[1 - \frac{(b-a)(\alpha+\beta)}{64}\right]} \cdot \omega(V_{m-1}, h), \quad \forall i = \overline{1,n-1}, \ \forall m \in \mathbb{N}^*.$$

Here, the auxiliary function $V_m : [a,b] \to \mathbb{R}$ is given by its restrictions to the subintervals $[t_{i-1}, t_i]$, $i = \overline{1,n}$, as follows:

$$V_m(t) = x_m(t) + \left[\overline{x_m(t_i)} - x_m(t_i)\right] \cdot \frac{t - t_{i-1}}{h} + \left[\overline{x_m(t_{i-1})} - x_m(t_{i-1})\right] \cdot \frac{t_i - t}{h}.$$

The proof of this error estimation result is analogous to the proof of Theorem 70.

Remark 78 *Under the conditions of Theorem 77, we can obtain continuous approximation of the solution interpolating the computed values* $\overline{x_m(t_i)}$, $i = \overline{0,n}$ *and using the same procedure as in (3.86). So, we obtain the continuous approximation* $s_m : [a,b] \to \mathbb{R}$ *given by its restrictions to the intervals* $[t_{i-1}, t_i]$, $i = \overline{1,n}$*:*

$$s_m^{(i)}(t) = \left[\frac{(t-t_{i-1})^2}{2} - \frac{(t-t_{i-1})^3}{6h} - \frac{h(t-t_{i-1})}{3}\right] \cdot M_m^{(i-1)} + \tag{3.88}$$

$$+\left[\frac{(t-t_{i-1})^3}{6h} - \frac{h(t-t_{i-1})}{6}\right] \cdot M_m^{(i)} + \frac{t-t_{i-1}}{h} \cdot \overline{x_m(t_i)} + \frac{t_i - t}{h} \cdot \overline{x_m(t_{i-1})}$$

where $M_m^{(0)} = M_m^{(n)} = 0$ *and* $M_m^{(i)}$, $i = \overline{1,n-1}$ *are recurrently given by:*

$$a_i = 2, \quad b_i = c_i = \frac{1}{2}, \quad d_i = \frac{3}{h^2} \cdot \left[\overline{x_m(t_{i+1})} + 2\overline{x_m(t_i)} - \overline{x_m(t_{i-1})}\right], \quad i = \overline{1,n-1}$$

and

$$\alpha_1 = \frac{c_1}{a_1}, \quad \omega_i = a_i - \alpha_{i-1} \cdot b_i, \quad \alpha_i = \frac{c_i}{\omega_i}, \quad i = \overline{2,n-2}$$

$$\omega_{n-1} = a_{n-1} - \alpha_{n-2} \cdot b_{n-1}$$

$$z_1 = \frac{d_1}{2}, \quad z_i = \frac{d_i - b_i \cdot z_{i-1}}{\omega_i}, \quad i = \overline{2, n-1}.$$

The moments $M_m^{(i)}$, $i = \overline{1, n-1}$, *are obtained using the backward recurrence:*

$$M_m^{(n-1)} = z_{n-1}, \quad M_m^{(i)} = z_i - \alpha_i \cdot M_m^{(i+1)}, \quad i = \overline{n-2, 1}.$$

Corollary (see [86]): The error estimate in the continuous approximation (3.88) is:

$$|x^*(t) - s_m(t)| \leq \frac{\frac{(b-a)^m(\alpha+\beta)^m}{(192)^m}}{1 - \frac{(b-a)(\alpha+\beta)}{192}} \cdot \frac{(b-a)\,M_0}{192} + \frac{(b-a)^2 L}{4n\left[1 - \frac{(b-a)(\alpha+\beta)}{64}\right]} +$$

$$+ \frac{7\beta\,(b-a)}{256\left[1 - \frac{(b-a)(\alpha+\beta)}{64}\right]} \cdot \omega(V_{m-1}, h) + \frac{7}{4} \cdot \omega(V_m, h), \quad \forall t \in [a,b], \quad \forall m \in \mathbb{N}^*,$$

where $\omega(V_m, h) \overset{notation}{=} \max\{\omega(V_k, h) : k = \overline{1, m}\}$.

In order to investigate the numerical stability we have considered the same two-point boundary value problem

$$\begin{cases} x^{IV}(t) = f(t, x(t), x(\varphi(t))), & t \in [a,b] \\ x(a) = c', & x(b) = d' \\ x'(a) = w', & x'(b) = r' \end{cases} \tag{3.89}$$

but with modified boundary conditions such that for given small $\varepsilon, \epsilon > 0$ we have

$$\left|c - c'\right| < \varepsilon, \quad \left|d - d'\right| < \varepsilon$$

$$\left|w - w'\right| < \epsilon, \quad \left|r - r'\right| < \epsilon$$

and let

$$h(t) = \frac{(b-t)^2\left[2(t-a) + (b-a)\right]}{(b-a)^3} \cdot c' + \frac{(t-a)^2\left[2(b-t) + (b-a)\right]}{(b-a)^3} \cdot d' +$$

$$+ \frac{(b-t)^2(t-a)}{(b-a)^2} \cdot w' - \frac{(t-a)^2(b-t)}{(b-a)^2} \cdot r', \quad t \in [a,b].$$

The boundary value problem (3.89) is equivalent with the integral equation

$$x(t) = h(t) + \int_a^b G(t,s) \cdot f(s, x(s), x(\varphi(s)))\,ds, \quad t \in [a,b]. \tag{3.90}$$

Applying the above presented method to the integral equation (3.90) the terms of the sequence of successive approximations on the knots $t_i = a + \frac{i(b-a)}{n}$, $i = \overline{0, n}$, are

$$y_0(t_i) = h(t_i), \quad i = \overline{0, n}$$

$$y_m(t_i) = h(t_i) + \int_a^b G(t_i, s) \cdot f(s, y_{m-1}(s), y_{m-1}(\varphi(s)))\,ds, \quad i = \overline{0, n}$$

and the effective computed values are $\overline{y_0(t_i)} = y_0(t_i) = h(t_i)$, $i = \overline{0, n}$, $\overline{y_m(t_i)}$, $i = \overline{1, n-1}$, $y_m(t_0) = \overline{y_m(t_0)} = c'$, $y_m(t_n) = \overline{y_m(t_n)} = d'$, $m \in \mathbb{N}^*$. These values are computed in the same way as in (3.84)-(3.86) and $y_m(t_i) = \overline{y_m(t_i)} + \overline{R'_{m,i}}$, $i = \overline{1, n-1}$. We see that

$$|x_0(t) - y_0(t)| = |g(t) - h(t)| < \varepsilon + \frac{b-a}{4} \cdot \epsilon, \quad \forall t \in [a,b].$$

Definition 79 *(see [86]) We say that the method of successive interpolations applied to the two-point boundary value problem (3.69) is numerically stable with respect to the boundary values if there exist* $p \in \mathbb{N}^*$, $K_1, K_2, K_3, K_4 \in \mathbb{R}_+^*$ *and a sequence of continuous functions* $\mu_m : [0, b-a] \to [0, \infty)$, $m \in \mathbb{N}^*$ *with the property* $\lim_{h \to 0} \mu_m(h) = 0$, $\forall m \in \mathbb{N}^*$ *such that*

$$\left| \overline{x_m(t_i)} - \overline{y_m(t_i)} \right| \leq K_1 \varepsilon + K_2 \epsilon + K_3 \cdot h^p + K_4 \cdot \mu_m(h),$$

for all $i = \overline{0,n}$, $m \in \mathbb{N}^*$.

It is proved in [86] that under the conditions of Theorem 77, the method of successive interpolations applied to the two-point boundary value problem (3.69) is numerically stable with respect to the boundary values.

Numerical experiments

In order to test the theoretical results and to illustrate the accuracy of the method, we present three numerical examples.

Example 1: The two-point boundary value problem

$$\begin{cases} x^{IV}(t) = -4e^{-t} + \frac{1}{2} \cdot x(t) + e^{-t/2} \cdot x\left(\frac{t}{2}\right), & t \in [0,1] \\ x(0) = 0, \qquad x(1) = e^{-1} \\ x'(0) = 1, \qquad x'(1) = 0 \end{cases}$$

has the solution $x^*(t) = t \cdot e^{-t}$ and applying the method presented in Section 3 for $n = 10$, $\varepsilon' = 10^{-15}$ we get the number of iterations $m = 7$. The numerical results are in Table 1, where in the second column there are the errors $e_i = \left| x^*(t_i) - \overline{x_m(t_i)} \right|$, $i = \overline{0,10}$. We see that the accuracy is $O\left(10^{-7} \div 10^{-8}\right)$. In order to illustrate the numerical stability we consider $\varepsilon = \epsilon = 0.1$ and the corresponding differences $d_i = \left| \overline{x_m(t_i)} - \overline{y_m(t_i)} \right|$, $i = \overline{0,10}$ are in the fifth column of Table 1. In order to test the convergence of the method we consider $n = 100$ and the numerical results (with accuracy $O\left(10^{-11}\right)$) are presented in the third column on the same knots as in the case for $n = 10$. For $n = 1000$ the results reveal the accuracy $O\left(10^{-15}\right)$ in the fourth column. For $n = 100$ and $n = 1000$ the number of iterations is $m = 8$.

t_i	e_i, $n = 10$	e_i, $n = 100$	e_i, $n = 1000$	d_i
0.0	0	0	0	1.000000e-001
0.1	2.064584e-007	2.183817e-011	2.192690e-015	1.168038e+000
0.2	3.179154e-007	3.307021e-011	3.330669e-015	1.087064e+000
0.3	3.537479e-007	3.583955e-011	3.524958e-015	1.022881e+000
0.4	3.223338e-007	3.230949e-011	3.275158e-015	9.735645e-001
0.5	2.461068e-007	2.464273e-011	2.498002e-015	9.373462e-001
0.6	1.501541e-007	1.500305e-011	1.498801e-015	9.125954e-001
0.7	5.617307e-008	5.554446e-012	5.551115e-016	8.978094e-001
0.8	1.478956e-008	1.538436e-012	1.110223e-016	8.916070e-001
0.9	4.085599e-008	4.111433e-012	3.885781e-016	8.927224e-001
1.0	0	0	0	1.000000e-001

Table 1

Example 2: For the boundary value problem

$$\begin{cases} x^{IV}(t) = \frac{2}{3} \cdot x(t) + \frac{1}{3} e^{-t/2} \cdot x\left(\frac{t}{2}\right), & t \in [0,1] \\ x(0) = 1, \qquad x(1) = e \\ x'(0) = 1, \qquad x'(1) = e \end{cases}$$

the exact solution is $x^*(t) = e^t$ and with $n = 10$, $\varepsilon' = 10^{-15}$ we get the number of iterations $m = 7$. The numerical results are in Table 2, where in the second column there are the errors $e_i = \left| x^*(t_i) - \overline{x_m(t_i)} \right|$, $i = \overline{0, 10}$ and, in order to test the numerical stability, in the fifth column there are the differences $d_i = \left| \overline{x_m(t_i)} - \overline{y_m(t_i)} \right|$, $i = \overline{0, 10}$ corresponding to

$$|c - c'| = |d - d'| = |w - w'| = |r - r'| = 0.1.$$

The accuracy is $O\left(10^{-7} \div 10^{-8}\right)$. In order to test the convergence, the results for $n = 100$, and for $n = 1000$ are in the third and in the fourth column, with the same number of iterations $m = 7$ and the accuracy $O\left(10^{-11}\right)$ and $O\left(10^{-15}\right)$, respectively.

t_i	e_i, $n = 10$	e_i, $n = 100$	e_i, $n = 1000$	d_i
0.0	0	0	0	1.000000e-001
0.1	3.308139e-008	3.131051e-012	2.220446e-016	7.865138e-001
0.2	2.392622e-008	2.266187e-012	4.440892e-016	6.647727e-001
0.3	1.424324e-008	1.357359e-012	2.220446e-016	5.329052e-001
0.4	6.630944e-008	6.502354e-012	4.440892e-016	3.891109e-001
0.5	1.203055e-007	1.193312e-011	1.332268e-015	2.316549e-001
0.6	1.648088e-007	1.641043e-011	1.332268e-015	5.885539e-002
0.7	1.874581e-007	1.869571e-011	1.776357e-015	1.309290e-001
0.8	1.757160e-007	1.754907e-011	1.776357e-015	3.393122e-001
0.9	1.173055e-007	1.173106e-011	1.332268e-015	5.678950e-001
1.0	0	0	0	1.000000e-001

Table 2

Example 3: Consider the boundary value problem

$$\begin{cases} x^{IV}(t) = \frac{22}{(t+1)^5} + \frac{1}{(t+1)^2} \cdot \left([x(t)]^2 + |x(t)|^3 \right) \cdot x\left(\frac{t}{2}\right), \quad t \in [0, 1] \\ x(0) = 1, \qquad x(1) = \frac{1}{2} \\ x'(0) = -1, \qquad x'(1) = -\frac{1}{4} \end{cases}$$

for which the exact solution is $x^*(t) = \frac{1}{t+1}$. For $n = 10$, $\varepsilon' = 10^{-15}$ the number of iterations is $m = 8$ and the accuracy is $O\left(10^{-6}\right)$. The numerical results are in Table 3. In order to test the convergence, we consider $n = 100$ and $n = 1000$ obtaining the same number of iterations $m = 8$. The accuracy is $O\left(10^{-10}\right)$ and $O\left(10^{-14}\right)$, respectively.

t_i	e_i, $n = 10$	e_i, $n = 100$	e_i, $n = 1000$	d_i
0.0	0	0	0	1.000000e-001
0.1	4.966906e-006	5.219815e-010	5.218048e-014	2.663317e-001
0.2	7.669382e-006	8.046303e-010	8.038015e-014	1.942196e-001
0.3	8.653231e-006	9.061637e-010	9.059420e-014	1.320789e-001
0.4	8.346081e-006	8.730353e-010	8.726353e-014	7.837641e-002
0.5	7.136916e-006	7.461346e-010	7.460699e-014	3.160992e-002
0.6	5.392152e-006	5.635770e-010	5.617729e-014	9.706440e-003
0.7	3.463657e-006	3.620274e-010	3.630429e-014	4.705045e-002
0.8	1.695703e-006	1.773527e-010	1.776357e-014	8.189693e-002
0.9	4.285610e-007	4.496403e-011	4.551914e-015	1.157210e-001
1.0	0	0	0	1.000000e-001

Table 3

Remark 80 *Now, we consider for the above presented examples the same stepsize* $h = 0.2$ *as in [224] and [225] taking* $n = 5$ *and* $\varepsilon' = 10^{-15}$. *The number of iterations is* $m = 7$ *for all three examples. For this stepsize, the accuracy is illustrated by the errors* $e_i = \left| x^*(t_i) - \overline{x_7(t_i)} \right|$, $i = \overline{0,5}$ *which are presented in the second, in the third and in the fourth column of the following table. The accuracy of the method presented in [224] and [225] is* $O\left(10^{-4} \div 10^{-5}\right)$. *As can be observed in the following table, for Example 3 the accuracy is* $O\left(10^{-4} \div 10^{-5}\right)$ *and for Examples 1 and 2 it becomes* $O\left(10^{-6} \div 10^{-7}\right)$.

t_i	e_i for Example 1	e_i for Example 3	e_i for Example 2
0.0	0	0	0
0.2	4.547945e-006	1.083479e-004	4.748093e-007
0.4	4.355331e-006	1.182057e-004	9.622064e-007
0.6	2.085666e-006	7.651988e-005	2.654772e-006
0.8	2.855686e-007	2.401714e-005	2.831214e-006
1.0	0	0	0

Remark 81 *Since the values of the derivative of the solution of (3.69) on the end-points are known, instead of the natural cubic spline we can use the complete cubic spline with the end-conditions* $s'(a) = w$, $s'(b) = r$. *The sharp error bound for complete cubic splines on equidistant grid can be found in [173] and could be used in order to obtain the error estimate of the method of successive interpolations (that uses this complete cubic splne).*

3.7 Functional integral equations

In the last section of this chapter we present the results obtained in [56], [72], [74], [82], and [87], concerning the application of the method of successive interpolations to functional integral equations.

Fredholm and Hammerstein integral equations arise in physics (solid state physics, plasma physics, quantum mechanics), astrophysics (the radiative transfer being modelled by the well-known Chandrasekhar integral equation), fluid dynamics (the study of water waves on liquids of infinite depth uses the Nekrasov's integral equation), cell kinetics, chemical kinetics, the theory of gases, mathematical economics, hereditary phenomena in biology. Existence results for Fredholm functional integral equations are obtained using the measure of noncompactness in [147], [175], and in [174] (where Darbo conditions are involved), or by the use of the Krasnoselskii's fixed point theorem (see [112] and [113]).

For the numerical solution of Fredholm and Hammerstein integral equations the existing methods are generally based on: Nyström type methods (see [97]), iterative methods (see [13], [14], [15]) and projection methods which include the well-known collocations method and Galerkin methods (see [13], [14], [15], [92], [93], [94], [176], [239]). Other methods use spline functions and wavelets (see [185]). For functional Fredholm and functional Hammerstein integral equations the numerical approximation of the solution is studied in few papers. The numerical methods for functional Fredholm integral equations are based on collocation techniques (see [19]), homotopy perturbation methods (see [4]), Lagrange and Chebyshev polynomials (see [207] and [208]), successive approximations (see [104]), the variational iteration method (see [40] and [105]) and the spline functions method (see [185]). The functional integral equation studied in [4], [19], [40], [207] and [208] is of special type:

$$y(x) + p(x) \cdot y(h(x)) + \lambda \int_a^b K(x,t) \cdot y(t)\,dt = g(x), \quad x \in [a,b]. \tag{3.91}$$

We present here the application of the method of successive interpolations to the following Hammerstein functional integral equation of the second kind

$$x(t) = g(t) + \int_a^b H(t,s) \cdot f(s, x(s), x(\varphi(s)))\,ds, \quad t \in [a,b] \tag{3.92}$$

where $a, b \in \mathbb{R}$, $a < b$, $H : [a,b] \times [a,b] \to \mathbb{R}$, $\varphi, g : [a,b] \to \mathbb{R}$, $a \le \varphi(t) \le b$, $\forall t \in [a,b]$, under the following conditions:

(i) $\varphi, g \in C[a,b]$, $a \le \varphi(t) \le b$, $\forall t \in [a,b]$, $f \in C([a,b] \times \mathbb{R} \times \mathbb{R})$, $H \in C([a,b] \times [a,b])$

(ii) there exist $\alpha, \beta > 0$ such that

$$\left|f(s,u,v) - f(s,u',v')\right| \le \alpha\left|u - u'\right| + \beta\left|v - v'\right|$$

for all $s \in [a,b]$, $(u,v), (u',v') \in \mathbb{R} \times \mathbb{R}$,

(iii) $b - a < \frac{1}{K(\alpha+\beta)}$ where $K \ge 0$ is such that $|H(t,s)| \le K$ for all $(t,s) \in [a,b] \times [a,b]$

(iv) there exist $\gamma, \delta, \mu, \rho, \lambda > 0$ such that

$$\left|H(t,s) - H(t',s')\right| \le \delta\left|t - t'\right| + \lambda\left|s - s'\right|$$

$$\left|f(s,u,v) - f(s',u,v)\right| \le \gamma\left|s - s'\right|$$

$$\left|\varphi(t) - \varphi(t')\right| \le \mu\left|t - t'\right|$$

$$\left|g(t) - g(t')\right| \le \rho\left|t - t'\right|$$

for all $t, s, t', s' \in [a,b]$, $(u,v) \in \mathbb{R} \times \mathbb{R}$.

Let $f_0 : [a,b] \to \mathbb{R}$, $f_0(s) = f(s, g(s), g(\varphi(s)))$. Since f, g, φ are continuous we infer that f_0 is continuous on the compact set $[a,b]$ and therefore exists $M_0 \ge 0$ such that $|f_0(s)| \le M_0$ for all $s \in [a,b]$.

Under the conditions (i)-(iii), applying the classical Picard-Banach's fixed point technique, the existence and uniqueness of the solution of (3.92) is obtained. Let $x^* \in C[a,b]$ be the solution of (3.92) and the sequence of successive approximations

$$x_0(t) = g(t), \quad \forall t \in [a,b],$$

$$x_m(t) = g(t) + \int_a^b H(t,s) \cdot f(s, x_{m-1}(s), x_{m-1}(\varphi(s)))\, ds, \quad \forall t \in [a,b], m \in \mathbb{N}^* \tag{3.93}$$

which uniformly converges to x^*. The following error estimations hold:

$$|x^*(t) - x_m(t)| \leq \frac{(b-a)^m (K(\alpha+\beta))^m}{1 - K(b-a)(\alpha+\beta)} \cdot KM_0(b-a)$$

and

$$|x^*(t) - x_m(t)| \leq \frac{K(b-a)(\alpha+\beta)}{1-K(b-a)(\alpha+\beta)} \cdot \max_{t\in[a,b]} |x_m(t) - x_{m-1}(t)|,$$

for all $t \in [a,b]$, $m \in \mathbb{N}^*$. Similarly, as in the previous section, the uniform boundedness of the terms of the sequence of successive approximations is obtained in [74]:

$$|x_m(t)| \leq \frac{KM_0(b-a)}{1-K(b-a)(\alpha+\beta)} + M_g = R$$

for all $t \in [a,b]$ and $m \in \mathbb{N}^*$, where $M_g \geq 0$ is such that $|g(t)| \leq M_g$, for all $t \in [a,b]$. Moreover, considering

$$M = \max(M_0, \max\{|f(t,u,v)| : t \in [a,b], u,v \in [-R.R]\})$$

we get

$$|F_m(t)| = |f(t, x_m(t), x_m(\varphi(t)))| \leq M$$

for all $t \in [a,b]$ and $m \in \mathbb{N}$.

Consider the uniform partition of $[a,b]$:

$$\Delta : a = t_0 < t_1 < ... < t_{n-1} < t_n = b$$

with $t_i = a + \frac{i(b-a)}{n}$, $i = \overline{0,n}$. Let $h = \frac{b-a}{n}$. On these knots, the relation (3.93) becomes:

$$x_m(t_i) = g(t_i) + \int_a^b H(t_i,s) \cdot f(s, x_{m-1}(s), x_{m-1}(\varphi(s)))\, ds, \quad i = \overline{0,n}.$$

We define the functions $F_{m,i} : [a,b] \to \mathbb{R}$,

$$F_{m,i}(s) = H(t_i,s) \cdot f(s, x_m(s), x_m(\varphi(s))), \quad s \in [a,b],\ i = \overline{0,n},\ m \in \mathbb{N}$$

and analogous as in the previous section it obtains the following uniform Lipschitz property (see [74]):

$$\left|F_{m,i}(s) - F_{m,i}(s')\right| \leq L \cdot \left|s - s'\right|$$

for all $i = \overline{0,n}$, $m \in \mathbb{N}$, where

$$L = \lambda M + K(\gamma + [\rho + \delta M(b-a)] \cdot (\alpha + \beta\mu)).$$

Applying the trapezoidal quadrature rule and the natural cubic spline interpolation procedure at each iterative step, the following algorithm is obtained:

$$x_0(t_i) = g(t_i), \quad i = \overline{0,n}$$

$$x_1(t_i) = g(t_i) + \frac{(b-a)}{2n} \cdot \sum_{j=1}^{n} [H(t_i, t_{j-1}) \cdot f(t_{j-1}, g(t_{j-1}), g(\varphi(t_{j-1}))) +$$

$$+H(t_i,t_j)\cdot f(t_j,g(t_j),g(\varphi(t_j)))]+R_{1,i}=\overline{x_1(t_i)}+R_{1,i},\quad i=\overline{0,n} \tag{3.94}$$

By induction for $m\geq 2$, it obtains:

$$x_m(t_i)=g(t_i)+\frac{(b-a)}{2n}\cdot\sum_{j=1}^{n}[H(t_i,t_{j-1})\cdot f\left(t_{j-1},\overline{x_{m-1}(t_{j-1})}+\overline{R_{m-1,j-1}},x_{m-1}(\varphi(t_{j-1}))\right)+$$

$$+H(t_i,t_j)\cdot f\left(t_j,\overline{x_{m-1}(t_j)}+\overline{R_{m-1,j}},x_{m-1}(\varphi(t_j))\right)]+R_{m,i}=g(t_i)+$$

$$+\frac{(b-a)}{2n}\cdot\sum_{j=1}^{n}[H(t_i,t_{j-1})\cdot f\left(t_{j-1},\overline{x_{m-1}(t_{j-1})},s_{m-1}(\varphi(t_{j-1}))\right)+$$

$$+H(t_i,t_j)\cdot f\left(t_j,\overline{x_{m-1}(t_j)},s_{m-1}(\varphi(t_j))\right)]+\overline{R_{m,i}}=\overline{x_m(t_i)}+\overline{R_{m,i}},\quad \forall i=\overline{0,n} \tag{3.95}$$

where $s_{m-1}:[0,a]\to\mathbb{R}$, is the natural cubic spline interpolating the values $\overline{x_{m-1}(t_i)}$, $i=\overline{0,n}$ and having the restrictions to the intervals $[t_{i-1},t_i]$, $i=\overline{1,n}$:

$$s_{m-1}^{(i)}(t)=\left[\frac{(t-t_{i-1})^2}{2}-\frac{(t-t_{i-1})^3}{6h_i}-\frac{h_i(t-t_{i-1})}{3}\right]\cdot M_{m-1}^{(i-1)}+$$

$$+\left[\frac{(t-t_{i-1})^3}{6h_i}-\frac{h_i(t-t_{i-1})}{6}\right]\cdot M_{m-1}^{(i)}+\frac{t-t_{i-1}}{h_i}\cdot\overline{x_{m-1}(t_i)}+\frac{t_i-t}{h_i}\cdot\overline{x_{m-1}(t_{i-1})} \tag{3.96}$$

where $M_{m-1}^{(0)}=M_{m-1}^{(n)}=0$ and $M_{m-1}^{(i)}$, $i=\overline{1,n-1}$ are recurrently given by:

$$a_i=2,\quad b_i=c_i=\frac{1}{2},\quad d_i=\frac{3}{h^2}\cdot[\overline{x_{m-1}(t_{i+1})}+2\overline{x_{m-1}(t_i)}-\overline{x_{m-1}(t_{i-1})}],\quad i=\overline{1,n-1}$$

and

$$\alpha_1=\frac{c_1}{a_1},\quad \omega_i=a_i-\alpha_{i-1}\cdot b_i,\quad \alpha_i=\frac{c_i}{\omega_i},\quad i=\overline{2,n-2}$$

$$\omega_{n-1}=a_{n-1}-\alpha_{n-2}\cdot b_{n-1}$$

$$z_1=\frac{d_1}{2},\quad z_i=\frac{d_i-b_i\cdot z_{i-1}}{\omega_i},\quad i=\overline{2,n-1}.$$

Using the backward recurrence it follows that,

$$M_{m-1}^{(n-1)}=z_{n-1},\quad M_{m-1}^{(i)}=z_i-\alpha_i\cdot M_{m-1}^{(i+1)},\quad i=\overline{n-2,1}.$$

This algorithm has a practical stopping criterion presented below:

For given $\varepsilon'>0$ and given $n\in\mathbb{N}^*$ (previously chosen) it determines the first natural number $m\in\mathbb{N}^*$ for which,

$$\left|\overline{x_m(t_i)}-\overline{x_{m-1}(t_i)}\right|<\varepsilon'\qquad\text{for all } i=\overline{0,n}$$

and we stop to this m, retaining the approximations $\overline{x_m(t_i)}$, $i=\overline{0,n}$, of the solution.

In an analogous manner as in Theorem 70, the following result is proved:

Theorem 82 *(see [74]) Under the conditions (i)-(iv), the sequence* $\left(\overline{x_m(t_i)}\right)_{m\in\mathbb{N}^*}$ *approximates the solution* $x^*(t_i)$ *on the knots* $t_i=a+\frac{i(b-a)}{n}$, $i=\overline{0,n}$ *and the 'apriori' error estimate is:*

$$|x^*(t_i)-x_m(t_i)|\leq\frac{(b-a)^m(K(\alpha+\beta))^m}{1-K(b-a)(\alpha+\beta)}\cdot KM_0(b-a)+$$

$$+\frac{L\left(b-a\right)^{2}}{4n[1-K\left(b-a\right)\left(\alpha+\beta\right)]}+\frac{7\beta\left(b-a\right)\omega\left(V_{m-1},h\right)}{4[1-K\left(b-a\right)\left(\alpha+\beta\right)]},\quad\forall m\in\mathbb{N}^{*},i=\overline{0,n}\qquad(3.97)$$

where $V_m:[a,b]\to\mathbb{R}$ *is defined by its restrictions to the subintervals* $[t_{i-1},t_i]$, $i=\overline{1,n}$, *as follows:*

$$V_m\left(t\right)=x_m\left(t\right)+\left[\overline{x_m\left(t_i\right)}-x_m\left(t_i\right)\right]\cdot\frac{t-t_{i-1}}{h}+\left[\overline{x_m\left(t_{i-1}\right)}-x_m\left(t_{i-1}\right)\right]\cdot\frac{t_i-t}{h}.$$

Remark 83 *(see [74]) Under the conditions of Theorem 82, we can obtain continuous approximation of the solution interpolating the computed values* $\overline{x_m\left(t_i\right)}$, $i=\overline{0,n}$, *using the same procedure as in (3.96). So, we obtain the continuous approximation of the solution,* $s_m:[a,b]\to\mathbb{R}$ *given by its restrictions to the intervals* $[t_{i-1},t_i]$, $i=\overline{1,n}$*:*

$$s_m^{(i)}\left(t\right)=\left[\frac{(t-t_{i-1})^2}{2}-\frac{(t-t_{i-1})^3}{6h_i}-\frac{h_i(t-t_{i-1})}{3}\right]\cdot M_m^{(i-1)}+$$

$$+\left[\frac{(t-t_{i-1})^3}{6h_i}-\frac{h_i(t-t_{i-1})}{6}\right]\cdot M_m^{(i)}+\frac{t-t_{i-1}}{h_i}\cdot\overline{x_m\left(t_i\right)}+\frac{t_i-t}{h_i}\cdot\overline{x_m\left(t_{i-1}\right)}\qquad(3.98)$$

where $M_m^{(0)}=M_m^{(n)}=0$ *and* $M_m^{(i)}$, $i=\overline{1,n-1}$ *are recurrently given by:*

$$a_i=2,\quad b_i=c_i=\frac{1}{2},\quad d_i=\frac{3}{h^2}\cdot\left[\overline{x_m(t_{i+1})}+2\overline{x_m(t_i)}-\overline{x_m(t_{i-1})}\right],\quad i=\overline{1,n-1}$$

and

$$\alpha_1=\frac{c_1}{a_1},\quad\omega_i=a_i-\alpha_{i-1}\cdot b_i,\quad\alpha_i=\frac{c_i}{\omega_i},\quad i=\overline{2,n-2}$$

$$\omega_{n-1}=a_{n-1}-\alpha_{n-2}\cdot b_{n-1}$$

$$z_1=\frac{d_1}{2},\quad z_i=\frac{d_i-b_i\cdot z_{i-1}}{\omega_i},\quad i=\overline{2,n-1}.$$

The moments $M_m^{(i)}$, $i=\overline{1,n-1}$, *are obtained using the backward recurrence:*

$$M_m^{(n-1)}=z_{n-1},\quad M_m^{(i)}=z_i-\alpha_i\cdot M_m^{(i+1)},\quad i=\overline{n-2,1}.$$

Corollary (see [74]): The error estimate in the continuous approximation (3.98) is:

$$\left|x^*\left(t\right)-s_m\left(t\right)\right|\leq\frac{(b-a)^m\left(K\left(\alpha+\beta\right)\right)^m}{1-K\left(b-a\right)\left(\alpha+\beta\right)}\cdot M_0\left(b-a\right)+\frac{L\left(b-a\right)^2}{4n[1-K\left(b-a\right)\left(\alpha+\beta\right)]}+$$

$$+\frac{\beta\left(b-a\right)+1-K\left(b-a\right)\left(\alpha+\beta\right)}{4[1-K\left(b-a\right)\left(\alpha+\beta\right)]}\cdot7\omega\left(V_m,h\right),\quad\forall t\in[a,b],\quad\forall m\in\mathbb{N}^*.$$

Now, in order to prove the numerical stability of the method we consider a small perturbation in the first iterative step $x_0=g$. Therefore, we investigate the Hammerstein integral equation:

$$x\left(t\right)=h\left(t\right)+\int_a^b H\left(t,s\right)\cdot f\left(s,x\left(s\right),x\left(\varphi\left(s\right)\right)\right)ds,\quad t\in[a,b]\qquad(3.99)$$

such that $h\in C[a,b]$ and $\left|g\left(t\right)-h\left(t\right)\right|<\varepsilon$ for small $\varepsilon>0$. Let $\rho',M_0',M_g'\geq0$ be such that

$$\left|h\left(t\right)\right|\leq M_g',\quad\forall t\in[a,b],\quad\left|h\left(t\right)-h\left(t'\right)\right|\leq\rho'\left|t-t'\right|,\quad\forall t,t'\in[a,b]$$

and

$$M_0' = \max\{|f(s, h(s), h(\varphi(s)))| : s \in [a,b]\}.$$

Applying the above presented numerical method to the integral equation (3.99) we obtain the sequence of successive approximations on the knots $t_i = \frac{i\cdot(b-a)}{n}$, $i = \overline{0,n}$:

$$y_0(t) = h(t), \quad \forall t \in [a,b]$$

$$y_0(t_i) = h(t_i), \qquad i = \overline{0,n},$$

$$y_m(t_i) = h(t_i) + \int_a^b H(t_i, s) \cdot f(s, y_{m-1}(s), y_{m-1}(\varphi(s)))\, ds, \quad i = \overline{0,n}, \; m \in \mathbb{N}^*.$$

Similarly,

$$|y_m(t)| \leq \frac{KM_0'(b-a)}{1 - K(b-a)(\alpha+\beta)} + M_g' = R', \quad \forall t \in [a,b], \; \forall m \in \mathbb{N}^*$$

and the constants M and L become

$$M' = \max\left(M_0', \max\{|f(t,u,v)| : t \in [a,b], u, v \in [-R'.R']\}\right),$$

$$L' = \lambda M' + K\left(\gamma + \left[\rho + \delta M'(b-a)\right] \cdot (\alpha + \beta\mu)\right).$$

The effective computed values are

$$y_0(t_i) = h(t_i), \qquad i = \overline{0,n},$$

and $\overline{y_m(t_i)}, i = \overline{0,n}$, $m \in \mathbb{N}^*$. These values are computed in the same way as in (3.94)-(3.96) and $y_m(t_i) = \overline{y_m(t_i)} + \overline{R_{m,i}'}$, $\forall i = \overline{0,n}$, $m \in \mathbb{N}^*$. We see that

$$|x_0(t) - y_0(t)| < \varepsilon, \quad \forall t \in [a,b].$$

Definition 84 *(see [74]) We say that the method of successive interpolations applied to the integral equation (3.92) is numerically stable with respect to the first iteration if there exist $p \in \mathbb{N}^*$, a sequence of continuous functions $\mu_m : [0, b-a] \to [0,\infty)$, $m \in \mathbb{N}^*$ with the property $\lim_{h\to 0} \mu_m(h) = 0$, $\forall m \in \mathbb{N}^*$ and the constants $K_1, K_2, K_3 > 0$ which not depend by h, such that*

$$\left|\overline{x_m(t_i)} - \overline{y_m(t_i)}\right| \leq K_1\varepsilon + K_2 \cdot h^p + K_3 \cdot \mu_m(h),$$

for all $i = \overline{0,n}$, $m \in \mathbb{N}^*$.

It is proved in [74], that under the conditions of Theorem 82, the method of successive interpolations applied to the integral equation (3.92) is numerically stable with respect to the first iteration.

Remark 85 *In [72] the approximation of the solution of the Urysohn-Fredholm type functional integral equation*

$$x(t) = g(t) + \int_a^b f(t, s, x(s), x(\varphi(s)))\, ds, \quad t \in [a,b]$$

is obtained using the method of successive interpolations in an analogous manner. Similarly are approached the pantograph type Volterra functional integral equation

$$x(t) = g(t) + \int_0^t f(t, s, x(s), x(\lambda s))\, ds, \quad t \in [0, a],\ 0 < \lambda < 1$$

in [56], and the Hammerstein-Volterra functional integral equation

$$x(t) = g(t) + \int_a^t H(t, s) \cdot f(s, x(s), x(\varphi(s)))\, ds, \quad t \in [a, b]$$

in [87].

Numerical experiments

Example 1: The Hammerstein-Volterra functional integral equation

$$x(t) = 1 - t + \int_0^t (t - s) \left[(x(s))^2 + x\left(\frac{s}{2}\right) \cdot |x(s)|^3 \right] ds, \quad t \in [0, 0.5]$$

has the exact solution $x^*(t) = \frac{1}{t+1}$ and for $\varepsilon' = 10^{-15}$, $n = 10$ we get $m = 9$ iterations and the accuracy is $O\left(10^{-3} \div 10^{-4}\right)$. The effective errors $e_i = \left| x^*(t_i) - \overline{x_9(t_i)} \right|$, $i = \overline{0, 10}$ are in the second column of Table 1. The numerical stability is tested for $\varepsilon = 0.1$ in the fifth column of Table 1. In order to test the convergence of the method we consider $n = 100$ and $n = 1000$ and the errors on the same knots as in the case $n = 10$, are in the third and in the fourth column.

t_i	e_i, $n = 10$	e_i, $n = 100$	e_i, $n = 1000$	d_i
0.00	0	0	0	1.000000e-001
0.05	1.190476e-004	1.209551e-006	1.214240e-008	1.006496e-001
0.10	2.308973e-004	2.370975e-006	2.380507e-008	1.026875e-001
0.15	3.378024e-004	3.511897e-006	3.526197e-008	1.059885e-001
0.20	4.486825e-004	4.653244e-006	4.672317e-008	1.104498e-001
0.25	5.590919e-004	5.811353e-006	5.835264e-008	1.159946e-001
0.30	6.738945e-004	6.999200e-006	7.028066e-008	1.225564e-001
0.35	7.910692e-004	8.227275e-006	8.261254e-008	1.300828e-001
0.40	9.143533e-004	9.504196e-006	9.543480e-008	1.385275e-001
0.45	1.041937e-003	1.083717e-005	1.088198e-007	1.478527e-001
0.50	1.176480e-003	1.223230e-005	1.228289e-007	1.580236e-001

Table 1

Example 2: The Fredholm functional integral equation

$$x(t) = -\frac{t}{t+1} + 2t \ln\left(\frac{2t+3}{2t+2}\right) + \int_0^1 \frac{1}{1 + t \cdot \left| x\left(\frac{s}{2}\right) \right|} ds, \quad t \in [0, 1]$$

has the exact solution $x^*(t) = \frac{1}{t+1}$. The accuracy of the algorithm is tested for $\varepsilon' = 10^{-15}$, $n = 10$, being $O\left(10^{-5}\right)$ and the number of iterations is $m = 21$. For the numerical stability we consider $\varepsilon = 0.1$ and the convergence is tested for $n = 100$ and $n = 1000$. The results are in Table 2.

t_i	e_i, $n = 10$	e_i, $n = 100$	e_i, $n = 1000$	d_i
0.0	0	0	0	1.000000e-001
0.1	2.111584e-005	1.654728e-007	1.608081e-009	9.259713e-002
0.2	3.386942e-005	2.619313e-007	2.541308e-009	8.710396e-002
0.3	4.140565e-005	3.159932e-007	3.060681e-009	8.299503e-002
0.4	4.562333e-005	3.435699e-007	3.322062e-009	7.990768e-002
0.5	4.770240e-005	3.544431e-007	3.421156e-009	7.758577e-002
0.6	4.839363e-005	3.547626e-007	3.418052e-009	7.584481e-002
0.7	4.818400e-005	3.484646e-007	3.351160e-009	7.454960e-002
0.8	4.739435e-005	3.381034e-007	3.245353e-009	7.359959e-002
0.9	4.623863e-005	3.253527e-007	3.116898e-009	7.291899e-002
1.0	4.486072e-005	3.113150e-007	2.976482e-009	7.245004e-002

Table 2

Example 3: For the Hammerstein-Fredholm functional integral equation

$$x(t) = \frac{9}{16} - \frac{t}{12} + \int_0^1 (t-s) \cdot \left(\left[x\left(\frac{s}{2}\right) \right]^2 + 1 \right) ds, \quad t \in [0,1]$$

the exact solution is $x^*(t) = t$ and applying the method of successive interpolations with $n = 10$ and $\varepsilon' = 10^{-15}$ it obtains $m = 16$ (the number of iterations to be made). The numerical stability is tested for $\varepsilon = 0.1$ and the results are in Table 3.

t_i	e_i, $n = 10$	e_i, $n = 100$	e_i, $n = 1000$	d_i
0.0	4.878738e-004	4.881628e-006	4.881657e-008	7.480581e-002
0.1	4.671025e-004	4.674521e-006	4.674556e-008	7.840184e-002
0.2	4.463313e-004	4.467414e-006	4.467455e-008	8.199788e-002
0.3	4.255600e-004	4.260307e-006	4.260355e-008	8.559392e-002
0.4	4.047887e-004	4.053201e-006	4.053254e-008	8.918995e-002
0.5	3.840175e-004	3.846094e-006	3.846153e-008	9.278599e-002
0.6	3.632462e-004	3.638987e-006	3.639053e-008	9.638203e-002
0.7	3.424750e-004	3.431881e-006	3.431952e-008	9.997806e-002
0.8	3.217037e-004	3.224774e-006	3.224851e-008	1.035741e-001
0.9	3.009324e-004	3.017667e-006	3.017751e-008	1.071701e-001
1.0	2.801612e-004	2.810560e-006	2.810650e-008	1.107662e-001

Table 3

Chapter 4

Fuzzy numbers and fuzzy integral equations

In this chapter we present the results obtained in [23], [67], [68], [75], and [77], concerning the algebraic structure on the set of continuous fuzzy numbers and numerical methods for fuzzy integral equations.

4.1 Fuzzy numbers

4.1.1 Fuzzy numbers and fuzzy-number-valued functions

The arithmetic operations

A fuzzy number is a fuzzy subset of the real line generalizing the classical concept real number.

Definition 86 *(see [114] and [28])A fuzzy subset of the real line* $u:\mathbb{R}\to[0,1]$ *is a fuzzy number if it satisfies the following properties:*
(i) u *is normal, i.e.* $\exists x_0\in\mathbb{R}$ *such that* $u(x_0)=1$;
(ii) u *is fuzzy convex, i.e.* $u(tx+(1-t)y)\geq\min(u(x),u(y))$, $\forall t\in[0,1]$, $\forall x,y\in\mathbb{R}$;
(iii) u *is upper semicontinuous on* $\mathbb{R}$ *(i.e.* $\forall x_0\in\mathbb{R}$, $\forall\varepsilon>0,\exists\delta>0$ *such that* $|x-x_0|<\delta$ *implies that* $u(x)-u(x_0)<\varepsilon$*);*
(iv) u *is compactly supported, i.e.* $cl\{x\in\mathbb{R},u(x)>0\}$ *is compact, where* clA *denotes the closure of the set* A.

The set of fuzzy numbers is denoted by $\mathbb{R}_{\mathcal{F}}$. Since each real number u can be identified with the function $\overline{u}:\mathbb{R}\to[0,1]$, $\overline{u}(x)=\begin{cases}1, & x=u\\ 0, & x\neq u\end{cases}$, we infer that each real number is a fuzzy number, and so $\mathbb{R}\subset\mathbb{R}_{\mathcal{F}}$.

For a fuzzy number $u\in\mathbb{R}_{\mathcal{F}}$ and for $r\in(0,1]$ are defined the subsets of $\mathbb{R}$,

$$u_r=\{x\in\mathbb{R}:u(x)\geq r\}$$

called the r-level sets and the set

$$u_0=cl\{x\in\mathbb{R},u(x)>0\}$$

called the support of u. The 1-level set is called the core of u. If $u_1^-=u_1^+$ then the corresponding fuzzy number is called unimodal.

It is proved that each r-level set $u_r=[u_r^-,u_r^+]$ is a closed interval for any $r\in[0,1]$, and $0\leq r_1\leq r_2\leq 1$ implies that $u_{r_2}\subset u_{r_1}$ (see [28]).

From calculus point of view, more convenient is another representation of a fuzzy number, the LU-representation (u_r^-, u_r^+) (lower-upper representation) introduced in [137]. The connection between the above definition and the LU-representation can be viewed in the following two theorems.

Theorem 87 *(see [137] and [28])Let u be a fuzzy number with its r-level sets $u_r = [u_r^-, u_r^+] = \{x \in \mathbb{R} : u(x) \geq r\}$. Then the functions $u^-, u^+ : [0,1] \to \mathbb{R}$, defining the end-points of the r-level sets, satisfy the following conditions:*
(i) the function u^- given by $u^-(r) = u_r^-$, $r \in [0,1]$, is bounded, non-decreasing, left-continuous in $(0,1]$ and it is right-continuous at 0;
(ii) the function u^+ given by $u^+(r) = u_r^+$, $r \in [0,1]$, is bounded, non-increasing, left-continuous in $(0,1]$ and it is right-continuous at 0;
(iii) $u_1^- \leq u_1^+$.

Theorem 88 *(see [137] and [28])Let us consider the functions $u^-, u^+ : [0,1] \to \mathbb{R}$, that satisfy the following conditions:*
(i) the function u^- given by $u^-(r) = u_r^-$, $r \in [0,1]$, is bounded, non-decreasing, left-continuous in $(0,1]$ and it is right-continuous at 0;
(ii) the function u^+ given by $u^+(r) = u_r^+$, $r \in [0,1]$, is bounded, non-increasing, left-continuous in $(0,1]$ and it is right-continuous at 0;
(iii) $u_1^- \leq u_1^+$.
Then there is a fuzzy number $u \in \mathbb{R}_{\mathcal{F}}$ that has u_r^-, u_r^+ as end-points of its r-level sets, u_r.

Considering the functions u^-, u^+ as functions by the *cut* parameter r, the LU-representation is also called the parametric representation (see [140] and [222]). Important subsets of the set $\mathbb{R}_{\mathcal{F}}$ are: $L - R$ fuzzy numbers, trapezoidal fuzzy numbers, triangular fuzzy numbers, Gaussian fuzzy numbers, exponential fuzzy numbers (see [28]), centered fuzzy numbers (see [164]).

The arithmetic operations with fuzzy numbers are defined by using the Zadeh's extension principle, which is a rule to extend any operation between real numbers to the fuzzy framework. The extension principles works in the following way (see [28], [142]):

If we assume that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, then we can extend it to $F : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ such that $w = F(u, v)$ has its level-sets:

$$w_r = \{f(x, y) : x \in u_r, y \in v_r\},$$

for any $u, v \in \mathbb{R}_{\mathcal{F}}$, that is $w_r = [w_r^-, w_r^+]$ with

$$w_r^- = \inf\{f(x, y) : x \in u_r, y \in v_r\}$$

and

$$w_r^+ = \sup\{f(x, y) : x \in u_r, y \in v_r\}.$$

For the usual arithmetic operations sum and product, these level sets become

$$[(u + v)_r^-, (u + v)_r^+] = [u_r^- + v_r^-, u_r^+ + v_r^+]$$

and

$$(u \cdot v)_r^- = \min\{u_r^- \cdot v_r^-, u_r^- \cdot v_r^+, u_r^+ \cdot v_r^-, u_r^+ \cdot v_r^+\}$$
$$(u \cdot v)_r^+ = \max\{u_r^- \cdot v_r^-, u_r^- \cdot v_r^+, u_r^+ \cdot v_r^-, u_r^+ \cdot v_r^+\}$$

for any $r \in [0,1]$. The scalar multiplication is defined by

$$(\lambda \cdot u)_r = \lambda \cdot [u_r^-, u_r^+] = \begin{cases} [\lambda u_r^-, \lambda u_r^+], & \text{if } \lambda \geq 0 \\ [\lambda u_r^+, \lambda u_r^-], & \text{if } \lambda < 0 \end{cases}, \quad \forall r \in [0,1].$$

The sum and the product defined on $\mathbb{R}_{\mathcal{F}}$ are associative and commutative, $0 = \chi_{\{0\}}$ is the zero element, $1 = \chi_{\{1\}}$ is the unit element. None of $u \in \mathbb{R}_{\mathcal{F}} \setminus \mathbb{R}$ has an opposite in $(\mathbb{R}_{\mathcal{F}}, +)$, none of $u \in \mathbb{R}_{\mathcal{F}} \setminus \mathbb{R}$ has an inverse in $(\mathbb{R}_{\mathcal{F}}, \cdot)$, and generally, the distributivity does not holds. The properties of the scalar multiplication are the following:

$$\begin{gathered} \lambda \cdot (u+v) = \lambda \cdot u + \lambda \cdot v, \quad \forall u, v \in \mathbb{R}_{\mathcal{F}},\ \forall \lambda \in \mathbb{R} \\ \lambda \cdot (\mu \cdot u) = \lambda\mu \cdot u, \quad \forall u \in \mathbb{R}_{\mathcal{F}},\ \forall \lambda, \mu \in \mathbb{R} \\ \text{if } \lambda \cdot \mu \geq 0, \text{ then } (\lambda+\mu) \cdot u = \lambda \cdot u + \mu \cdot u, \quad \forall u \in \mathbb{R}_{\mathcal{F}} \end{gathered}$$

and the property $(\lambda+\mu) \cdot u = \lambda \cdot u + \mu \cdot u$ does not hold for general $\lambda, \mu \in \mathbb{R}$ (see [119]).

Since for the standard difference induced by the Zadeh's extension principle we have $u - u \neq 0$, several other differences were proposed. We present below some of them.

Definition 89 *(see [28])The Hukuhara difference (H-difference, $\ominus_H$) is defined by*

$$u \ominus_H v = w \Longleftrightarrow \exists w \in \mathbb{R}_{\mathcal{F}} \text{ such that } u = v + w.$$

If $u \ominus_H v$ exists, then its r-level sets are

$$[u \ominus_H v]_r = [u_r^- - v_r^-, u_r^+ - v_r^+].$$

Of course, $u \ominus_H u = 0$, but the Hukuhara difference rarely exists. To cover this gap, the generalized Hukuhara difference was proposed.

Definition 90 *(see [28])Given two fuzzy numbers $u, v \in \mathbb{R}_{\mathcal{F}}$, the generalized Hukuhara difference (shortly, gH-difference) is the fuzzy number w, if it exists, such that*

$$u \ominus_{gH} v = w \Longleftrightarrow \begin{cases} u = v + w \\ or,\ \ v = u - w. \end{cases}$$

It is proved that in terms of r-level sets we have

$$[u \ominus_{gH} v]_r = [\min\{u_r^- - v_r^-, u_r^+ - v_r^+\}, \max\{u_r^- - v_r^-, u_r^+ - v_r^+\}], \quad \forall r \in [0,1].$$

According to [28], the generalized Hukuhara exists in many more situations than the usual Hukuhara difference, but it does not always exists. Therefore, another generalized difference was proposed.

Definition 91 *(see [29])The generalized difference (shortly, g-difference) of two fuzzy numbers $u, v \in \mathbb{R}_{\mathcal{F}}$ is given by its level sets as*

$$[u \ominus_g v]_\alpha = [\inf_{\beta \geq \alpha} \min\{u_\beta^- - v_\beta^-, u_\beta^+ - v_\beta^+\}, \sup_{\beta \geq \alpha} \max\{u_\beta^- - v_\beta^-, u_\beta^+ - v_\beta^+\}].$$

Theorem 92 *(see [28])For any fuzzy numbers $u, v \in \mathbb{R}_{\mathcal{F}}$ the g-difference $u \ominus_g v$ exists and it is a fuzzy number.*

In other words, the g-difference always exists.

The topological structure

The most used metric on the set of fuzzy numbers is the Hausdorff type distance:

Definition 93 *(see [114] and [28])Let $D_\infty : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+ \cup \{0\}$,*

$$D_\infty(u,v) = \sup_{r \in [0,1]} \max\{|u_r^- - v_r^-|, |u_r^+ - v_r^+|\}.$$

Theorem 94 *(see [114])*$(\mathbb{R}_{\mathcal{F}}, D_{\infty})$ *is a complete metric space and the following properties hold true:*
(i) $D_{\infty}(u+w, v+w) = D_{\infty}(u,v), \quad \forall u,v,w \in \mathbb{R}_{\mathcal{F}}$
(ii) $D_{\infty}(\lambda \cdot u, \lambda \cdot v) = |\lambda| D_{\infty}(u,v), \quad \forall u,v \in \mathbb{R}_{\mathcal{F}}, \forall \lambda \in \mathbb{R}$
(iii) $D_{\infty}(u+v, w+e) \leq D_{\infty}(u,w) + D_{\infty}(v,e), \quad \forall u,v,w,e \in \mathbb{R}_{\mathcal{F}}$
(iv) the space $(\mathbb{R}_{\mathcal{F}}, D_{\infty})$ *is not separable*
(v) the closed unit ball of $(\mathbb{R}_{\mathcal{F}}, D_{\infty})$

$$\overline{B}(0,1) = \{u \in \mathbb{R}_{\mathcal{F}} : D_{\infty}(u,0) \leq 1\}$$

is not compact.

The norm of a fuzzy number $u \in \mathbb{R}_{\mathcal{F}}$ is defined by $\|u\| = D_{\infty}(u,0)$ and it obtains the following result.

Theorem 95 *(see [28] and [136])The norm on* $\mathbb{R}_{\mathcal{F}}$ *has the following properties:*
(i) $\|u\| = 0$ *if and only if* $u = 0$
(ii) $\|\lambda \cdot u\| = |\lambda| \cdot \|u\|, \quad \forall u \in \mathbb{R}_{\mathcal{F}}, \forall \lambda \in \mathbb{R}$
(iii) $\|u+v\| \leq \|u\| + \|v\|, \quad \forall u,v \in \mathbb{R}_{\mathcal{F}}$
(iv) $|\|u\| - \|v\|| \leq D_{\infty}(u,v), \quad \forall u,v \in \mathbb{R}_{\mathcal{F}}$
(v) $D_{\infty}(\lambda \cdot u, \mu \cdot u) = |\lambda - \mu| \cdot \|u\|, \quad \forall u \in \mathbb{R}_{\mathcal{F}}$ *and for any real numbers* λ *and* μ *having the same sign;*
(vi) $D_{\infty}(u,v) = \|u \ominus_{gH} v\|, \quad \forall u,v \in \mathbb{R}_{\mathcal{F}}$.

We see that the norm of fuzzy numbers has properties similar as the properties of a norm in the crisp sense, but $(\mathbb{R}_{\mathcal{F}}, \|\cdot\|)$ is not a normed space because $\mathbb{R}_{\mathcal{F}}$ is not a linear space.

Remark 96 *(see [28])According to the above presented metric and norm on* $\mathbb{R}_{\mathcal{F}}$ *we can consider the analysis concepts as limit, convergence, and continuity for functions* $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ *taking for the topology of* $\mathbb{R}_{\mathcal{F}}$*, the metric topology induced by* D_{∞}*. For instance,* $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ *is continuous in* $x_0 \in [a,b]$ *if* $\forall \varepsilon > 0, \exists \delta > 0$ *such that for any* $x \in [a,b]$ *with* $|x - x_0| < \delta$ *it follows that* $D_{\infty}(f(x), f(x_0)) < \varepsilon$*. By the definition of the metric* D_{∞} *it follows that*

$$\sup_{r \in [0,1]} \max\{\left|f_r^-(x) - f_r^-(x_0)\right|, \left|f_r^+(x) - f_r^+(x_0)\right|\} < \varepsilon$$

which means that the families of functions $\{f_r^- : r \in [0,1]\}$ *and* $\{f_r^+ : r \in [0,1]\}$ *are equicontinuous. This equicontinuity implies the continuity of any of the functions* $f_r^- : [a,b] \to \mathbb{R}$, $f_r^+ : [a,b] \to \mathbb{R}$, $r \in [0,1]$*. These functions are called the* $r-$*level functions of the function* f*.*

A function $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ is bounded iff there exists $M \geq 0$ such that $D_{\infty}(f(t), 0) \leq M$, $\forall t \in [a,b]$.

Remark 97 *(see [238] and [130])If* $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ *is continuous then it is bounded and its supremum* $\sup_{t \in [a,b]} f(t)$ *must exist and is determined by* $u \in \mathbb{R}_{\mathcal{F}}$ *with* $u_-^r = \sup_{t \in [a,b]} f_-^r(t)$ *and* $u_+^r = \sup_{t \in [a,b]} f_+^r(t)$*. A similar conclusion for the infimum is also true. Consequently, on the set*

$$C([a,b], \mathbb{R}_{\mathcal{F}}) = \{f : [a,b] \to \mathbb{R}_{\mathcal{F}} \mid f \text{ continuous on } [a,b]\}$$

it is defined the metric $D^*(f,g) = \sup_{t\in[a,b]} D(f(t),g(t))$, $\forall f,g \in C([a,b],\mathbb{R}_{\mathcal{F}})$. *We see that* $(C([a,b],\mathbb{R}_{\mathcal{F}}), D^*)$ *is complete metric space and similar properties as above can be obtained for the metric* D^*

Considering the set $\overline{C}[0,1] = \{F : [0,1] \to \mathbb{R},\ F$ bounded on $[0,1]$, left continuous on $(0,1]$ and right continuous at $0\}$ and the following norm

$$\|F\| = \sup\{|F(x)| : x \in [0,1]\},$$

$(\overline{C}[0,1], \|\cdot\|)$ is Banach space and the following embedding result could be obtained.

Theorem 98 *(see [171])Let* $j : \mathbb{R}_{\mathcal{F}} \to \overline{C}[0,1] \times \overline{C}[0,1]$ *given by* $j(u) = (u^-, u^+)$, *where* $u^-, u^+ : [0,1] \to \mathbb{R}$, *are* $u^-(r) = u_r^-$, $u^+(r) = u_r^+$, $r \in [0,1]$. *Then* $j(\mathbb{R}_{\mathcal{F}})$ *is a closed convex cone having the vertex at* 0 *in the Banach space* $\overline{C}[0,1] \times \overline{C}[0,1]$. *In this space the norm is* $\|(f,g)\| = \max\{\|f\|, \|g\|\}$. *Moreover,*

$$j(a \cdot u + b \cdot v) = a \cdot j(u) + b \cdot j(v), \quad \forall u,v \in \mathbb{R}_{\mathcal{F}},\ a,b \geq 0$$

and

$$D_\infty(u,v) = \|j(u) - j(v)\|.$$

The function j is injective and its image is

$$j(\mathbb{R}_{\mathcal{F}}) = \{(u^-,u^+) \in \overline{C}[0,1] \times \overline{C}[0,1] \mid u^-\text{-nondecreasing},\ u^+\text{-nonincreasing},\ u_1^- \leq u_1^+\}.$$

The fuzzy integrability of fuzzy-number-valued functions was introduced under two aspects: fuzzy-Riemann integrability and fuzzy-Henstock integrability, presented below.

Definition 99 *(see [28] and [136])A function* $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ *is called Riemann integrable on* $[a,b]$ *if there exists* $I \in \mathbb{R}_{\mathcal{F}}$ *with the property:* $\forall \varepsilon > 0, \exists \delta > 0$ *such that for any division of* $[a.b]$, $\Delta : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ *of norm* $\nu(\Delta) < \delta$ *and for any points* $\xi_i \in [x_i, x_{i+1}]$, $i = \overline{0, n-1}$, *we have*

$$D_\infty\left(\sum_{i=0}^{n-1}(x_{i+1} - x_i) \cdot f(\xi_i), I\right) < \varepsilon.$$

Then we denote $I = (FR)\int_a^b f(x)\,dx$ *and it is called fuzzy Riemann integral.*

Definition 100 *(see [237] and [30])Let* $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ *be a fuzzy-number valued function. For* $\Delta_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ *a partition of the interval* $[a,b]$, *we consider the points* $\xi_i \in [x_{i-1}, x_i]$, $i = 1,...,n$, *and the function* $\delta : [a,b] \to \mathbb{R}_+$. *The partition* $P = \{([x_{i-1}, x_i]; \xi_i); i = 1,...,n\}$ *denoted by* $P = (\Delta_n, \xi)$ *is called* δ*-fine iff* $[x_{i-1}, x_i] \subseteq (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$. *For* $I \in \mathbb{R}_{\mathcal{F}}$, *the function* f *is fuzzy Henstock integrable on* $[a,b]$ *if for any* $\epsilon > 0$ *there is a function* $\delta : [a,b] \to \mathbb{R}_+$ *such that for any partition* δ*-fine* P, $D\left(\sum_{i=1}^n (x_i - x_{i-1}) \cdot f(\xi_i), I\right) < \epsilon$. *The fuzzy number* I *is called the fuzzy Henstock integral of* f *and will be denoted by* $(FH)\int_a^b f(t)\,dt$.

When the function $\delta : [a,b] \to \mathbb{R}_+$ is constant, then we obtain the Riemann integrability for fuzzy-number-valued functions. Consequently, the fuzzy-Riemann integrability is a particular case of the fuzzy-Henstock integrability, and therefore the properties of the integral (FH) will be valid for the integral (FR), too. Other results concerning fuzzy Riemann integrability can be found in [121]-[123] and [138].

According to [28], a continuous fuzzy-number valued function is fuzzy-Riemann integrable and fuzzy-Henstock integrable, too, and

$$(FH)\int_a^b f(x)\,dx = (FR)\int_a^b f(x)\,dx.$$

Moreover,

$$\left[(FH)\int_a^b f(t)\,dt\right]^r = [(H)\int_a^b f_-^r(t)\,dt, (H)\int_a^b f_+^r(t)\,dt],$$

and

$$\left[(FR)\int_a^b f(t)\,dt\right]^r = [\int_a^b f_-^r(t)\,dt, \int_a^b f_+^r(t)\,dt], \quad \forall r \in [0,1],$$

and the Henstock and Riemann fuzzy integrals possess the linearity property and the additivity for intervals

Theorem 101 *(see [30])If f and g are fuzzy-Henstock integrable functions and if the function given by $D(f(t), g(t))$ is Lebesgue integrable, then*

$$D\left((FH)\int_a^b f(t)\,dt, (FH)\int_a^b g(t)\,dt\right) \le \int_a^b D(f(t), g(t))\,dt.$$

Definition 102 *(see [30])For $L \ge 0$, a function $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ is L-Lipschitz if*

$$D(f(x), f(y)) \le L|x-y|$$

for any $x, y \in [a,b]$.

Numerical computation of fuzzy-Riemann integrals can be realized using quadrature rules. The trapezoidal quadrature rule for L-Lipschitz functions can be found in [30] and [68]. In this context, let $\Delta : a = t_0 < t_1 < ... < t_n = b$ be an uniform partition of the interval $[a,b]$ with $t_i = a + \frac{i\cdot(b-a)}{n}$, $i = \overline{0,n}$. We can mention the following result:

Theorem 103 *(see [68]) If $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ is L-Lipschitz, then the following trapezoidal inequality holds:*

$$D\left((FR)\int_a^b f(t)dt, \quad \sum_{i=0}^{n-1} \frac{(t_{i+1}-t_i)}{2}\cdot[f(t_i)+f(t_{i+1})]\right) \le \frac{L(b-a)^2}{4n}.$$

4.1.2 Algebraic structures on the set of fuzzy numbers

We present now the categorial framework realized in [67] for the algebraic structure on the set of fuzzy numbers.

Definition 104 *(see [67])Consider a commutative monoid* $(M, *)$*, an automorphism* $\varphi : M \to M$ *with the property* $\varphi \circ \varphi = 1_M$ *(that is an involutive automorphism) and a submonoid* H *of* M *with the following properties :*
(i) $\varphi(H) = H$;
(ii) $x * \varphi(x) \in H, \forall x \in M$;
(iii) $[(M \setminus H) * H] \cap H = \varnothing$,
where for $A, B \subset M$ *we have* $A * B = \{x * y : x \in A, \quad y \in B\}$.
The equivalence relation generated on M *by* H *is the relation defined by :*
for $x, y \in M$ *we have* $x \sim_H y \Longleftrightarrow x * \varphi(y) \in H$.

It is proved in [67] that the equivalence relation $\sim_H$ is a congruence on $(M, *)$. Denoting by M_H the quotient set $M /{\sim_H} = \{x * H : x \in M\}$ and by $\circledast$ the operation on M_H defined by

$$P_H(x) \circledast P_H(y) = (x * H) \circledast (y * H) = (x * y) * H, \quad \forall x, y \in M,$$

we can see that $(M_H, \circledast)$ is an abelian group where the inverse of $x*H$ is $\varphi(x)*H$. Moreover, the automorphism φ generates the involutive automorphism $\Phi : M_H \to M_H, \quad \Phi(x*H) = \varphi(x) * H$ with $\Phi(H) = H$.

In [67] was considered the following category $AutMon$, where
$ObAutMon = \{(M, \varphi) : M$ commutative monoid, $\varphi : M \to M, \ \varphi \circ \varphi = 1_M\}$ and for $(M, \varphi), (N, \lambda) \in ObAutMon$ we have

$$Hom((M, \varphi), (N, \lambda)) = \{\chi : M \to N \mid \chi \text{ unital morphism}, \ \chi \circ \varphi = \lambda \circ \chi\}.$$

which is a veritable category. In this category, the objects (M, φ) and (M_H, Φ) have the following universality property :

Theorem 105 *(see [67])Let* $(M, *)$ *and* $H \subset M$ *be as in Definition 104. For each* $(N, \gamma) \in ObAutMon$ *and* $\chi \in Hom((M, \varphi), (N, \gamma))$ *with* $\ker P_H \subset \ker \chi$*, exists a unique homomorphism in AutMon,* $\eta : M_H \to N$*, such that* $\eta \circ P_H = \chi$.

Moreover, the following result is obtained as a consequence :

Corollary 106 *(see [67])Let* $(M, \varphi), (N, \gamma) \in ObAutMon$ *and* $\chi \in Hom((M, \varphi), (N, \gamma))$ *such that* $\chi(H) = K$*, where* $H \le M$ *and* $K \le N$ *are submonoids of* M *and* N*, respectively, which generate the objects* $(M_H, \Phi), (N_K, \Gamma)$*, and for which the conditions (i), (ii), (iii) are fulfilled. Then there is a unique homomorphism* $\eta \in Hom((M_H, \Phi), (N_K, \Gamma))$ *such that for* $\delta = P_K \circ \chi$ *we have* $\eta \circ P_H = \delta$*. If* χ *is isomorphism, then* η *is an isomorphism between the abelian groups* $(M_H, \circledast)$ *and* $(N_K, \odot)$.

In [67] is considered the following subset $\mathcal{F}$ of the set of fuzzy numbers $\mathbb{R}_{\mathcal{F}}$.

Definition 107 *(see [67])A real fuzzy number lies in the set* $\mathcal{F}$ *if it is a function* $f : \mathbb{R} \to [0, 1]$ *which satisfies the properties:*
a) there exists $a, b, c \in \mathbb{R}$, $a < c < b$*, such that* $f(x) = 0, \forall x \in (-\infty, a) \cup (b, \infty)$, $f(a) = f(b) = 0$ *and* $f(c) = 1$;
b) there exists $f_1 : [a, c] \to [0, 1]$ *and* $f_2 : [c, b] \to [0, 1]$ *such that* $f(x) = f_1(x), \forall x \in [a, c]$ *and* $f(x) = f_2(x), \forall x \in [c, b]$;
c) f_1 *and* f_2 *are onto (surjective) such that* f_1 *is strictly increasing and* f_2 *is strictly decreasing.*

Remark 108 *The functions f_1 and f_2 from the above definition are almost everywhere continuous and bijective. Moreover, $f_1^{-1}, f_2^{-1} \in \widetilde{C}[0,1]$, where $\widetilde{C}[0,1]$ is the set of almost everywhere continuous functions defined on $[0,1]$, and therefore a fuzzy number $f \in \mathcal{F}$ can be defined by the pair $(f_1^{-1}, f_2^{-1}) \in \widetilde{C}[0,1] \times \widetilde{C}[0,1]$. For this reason, since it is too easy to use the pairs from $\widetilde{C}[0,1] \times \widetilde{C}[0,1]$, the fuzzy numbers from $\mathcal{F}$ will be represented by the elements of the set*

$$FV = \{(A_-, A_+) \in \widetilde{C}[0,1] \times \widetilde{C}[0,1] : A_-(1) = A_+(1),$$

$$A_- \nearrow \text{ and } A_+ \searrow\} \cup \{\overline{r}, r \in \mathbb{R}\}$$

where the symbols $\nearrow$ and $\searrow$ mean the property of strictly increasing and strictly decreasing, respectively. Here, $\overline{r} : \mathbb{R} \to [0,1]$, $\overline{r}(r) = 1$ and $\overline{r}(x) = 0$, $\forall x \neq r$. We will use the notation $A = (A_-, A_+) \in FV$ for an element of $\mathcal{F}$

Definition 109 *(see [67])A fuzzy number $A \in FV$, $A = (A_-, A_+)$ is :*
(i) with symmetric support, if $A_-(1) - A_-(0) = A_+(0) - A_+(1)$;
(ii) symmetric, if the following equality holds

$$A_-(1) - A_-(\varepsilon) = A_+(\varepsilon) - A_+(1); \ \forall \varepsilon \in [0,1];$$

(iii) zero-symmetric, if it is symmetric and $A(1) = 0$, that is

$$A_-(\varepsilon) = -A_+(\varepsilon), \forall \varepsilon \in [0,1].$$

We will use the notations

$$S_0 = \{A \in FV : A \text{ is zero-symmetric }\}$$

$$S = \{A \in FV : A \text{ is symmetric}\}.$$

In the following, we define the operations between elements of FV. Let $A, B \in FV$ and $\alpha \in \mathbb{R}$:
(i) $A + B = ((A+B)_-, (A+B)_+) = (A_- + B_-, A_+ + B_+)$, where $A_- + B_-$ and $A_+ + B_+$ represent the addition from $\widetilde{C}[0,1]$ and $\overline{r} + A = ((\overline{r}+A)_-, (\overline{r}+A)_+)$, where $(\overline{r}+A)_-(\varepsilon) = r + A_-(\varepsilon)$, $(\overline{r}+A)_+(\varepsilon) = r + A_+(\varepsilon)$, $\forall \varepsilon \in [0,1]$.
(ii) $\alpha \cdot A = (\alpha \cdot A_-, \alpha \cdot A_+)$, if $\alpha \geq 0$ and $\alpha \cdot A = (\alpha \cdot A_+, \alpha \cdot A_-)$, if $\alpha < 0$, where $\alpha \cdot A_-$ and $\alpha \cdot A_+$ is the product of the elements from $\widetilde{C}[0,1]$ with real numbers.

Using the order from $\widetilde{C}[0,1]$ we can define a partial order on FV :

$$A \leq B \Longleftrightarrow A_- \leq B_- \quad \text{and} \quad A_+ \leq B_+,$$

which is compatible with the addition defined above. It is easy to prove that $(FV, +, \leq)$ is ordered monoid where the zero element is $\overline{0}$. We see that $(S_0, +) \leq (FV, +)$ as sub-semigroup and $\overline{0} \in S_0$. We can define in $(FV, +)$ the inverse element of $A \in FV$, by $-A = (-1) \cdot A$. obtaining the function $\varphi : FV \to FV$, $\varphi(A) = -A$.

Using Theorem 105 and Corollary 106, it obtains:

Theorem 110 *(see [67])The function φ is automorphism of $(FV, +)$, $\varphi \circ \varphi = 1_{FV}$ and has the following properties:*
(i) $\varphi(S_0) = S_0$
(ii) $A + \varphi(A) \in S_0, \quad \forall A \in FV$
(iii) $[(FV \setminus S_0) + S_0] \cap S_0 = \varnothing$.

From the above presented theorem we see that $(FV, \varphi) \in ObAutMon$ and the submonoid S_0 has the properties (i)-(iii) from Definition 104, which defines the equivalence relation on FV :

$$A \sim_{S_0} B \Longleftrightarrow A + (-B) \in S_0,$$

that is a congruence relation. We will denote this relation $\sim_{S_0}$ by $\sim_{\oplus}$ and we see that $FV_{S_0} = FV \ / \sim_{\oplus}$ is an abelian group. It is easy to prove that

$$A \sim_{\oplus} B \Longleftrightarrow A_-(\varepsilon) + A_+(\varepsilon) = B_-(\varepsilon) + B_+(\varepsilon), \quad \forall \varepsilon \in [0,1].$$

The property $A + \varphi(A) \in S_0, \quad \forall A \in FV$, means that

$$A + (-A) \sim_{S_0} \overline{0}, \forall A \in FV,$$

since $\overline{0} \in S_0$.

Remark 111 *Let $(FV, \varphi), (FV_{S_0}, \Phi) \in ObAutMon$. The automorphism*

$$\Phi : FV_{S_0} \to FV_{S_0},$$

is uniquely defined by the property of the canonical projection,

$$P_{S_0} \circ \varphi = \Phi \circ P_{S_0},$$

that is,

$$\Phi(A + S_0) = \varphi(A) + S_0 = -A + S_0, \quad \forall A \in FV.$$

According to [23] and [67], the properties of the scalar product are:
(i) $a \cdot (A + B) = a \cdot A + a \cdot B, \forall A, B \in FV, a \in \mathbb{R}$
(ii) $a \cdot (b \cdot A) = (a \cdot b) \cdot A, \forall A \in FV, a, b \in \mathbb{R}$
(iii) $(a + b) \cdot A = a \cdot A + b \cdot A, \forall A \in FV$, for $a, b \in \mathbb{R}$ with $a \cdot b \geq 0$
(iv) $(a + b) \cdot A \sim_{\oplus} a \cdot A + b \cdot A, \forall A \in FV$, for $a, b \in \mathbb{R}$ with $a \cdot b < 0$
(v) $1 \cdot A = A, \forall A \in FV$.

For the multiplicative operation, in order to obtain $A \cdot B \in FV$ for $A, B \in FV$ we need to consider a subset of FV, of the fuzzy numbers with positive support,

$$FV_+ = \{A = (A_-, A_+) \in FV : A_-(0) \geq 0\}$$

$$FV_+^* = \{A = (A_-, A_+) \in FV : A_-(0) > 0\}.$$

Definition 112 *(see [67])Let $A = (A_-, A_+), B = (B_-, B_+) \in FV_+$. We define $C = A \cdot B = ((A \cdot B)_-, (A \cdot B)_+) \in \widetilde{C}[0,1] \times \widetilde{C}[0,1]$ by*

$$(A \cdot B)_-(\varepsilon) = A_-(\varepsilon) \cdot B_-(\varepsilon)$$

and

$$(A \cdot B)_+(\varepsilon) = A_+(\varepsilon) \cdot B_+(\varepsilon), \quad \forall \varepsilon \in [0,1].$$

It is proved in [67] that $(FV_+^*, \cdot)$ is a commutative monoid.

For $A = (A_-, A_+) \in FV_+^*$ it defines $\overline{A} = (\overline{A}_-, \overline{A}_+) \in \widetilde{C}[0,1] \times \widetilde{C}[0,1]$ by

$$\overline{A}_-(\varepsilon) = \frac{1}{A_+(\varepsilon)}, \qquad \overline{A}_+(\varepsilon) = \frac{1}{A_-(\varepsilon)}, \quad \forall \varepsilon \in [0,1].$$

Since A_- is strictly increasing and A_+ is strictly decreasing, it is easy to prove that $\overline{A}_-$ is strictly increasing and $\overline{A}_+$ is strictly decreasing. Moreover, $\overline{A}_-(1) = \overline{A}_+(1)$ and

$$A_+(0) > A_-(0) > 0 \Longrightarrow \overline{A}_-(0) > 0.$$

Then $\overline{A} \in FV_+^*, \quad \forall A \in FV_+^*$. In this way we define the function $\theta : FV_+^* \longrightarrow FV_+^*$, $\theta(A) = \overline{A}$.
Consider a subset of FV_+^*, of the elements "projectable on one",

$$U_1 = \{(A_-, A_+) \in FV_+^* : A_-(1) = A_+(1) = 1,\ A_-(\varepsilon) \cdot A_+(\varepsilon) = 1,\ \forall \varepsilon \in [0,1]\}.$$

Similarly, as for Theorem 110, it obtains:

Theorem 113 *(see [67])The function θ is an involutive automorphism of $(FV_+^*, \cdot)$ and the subset U_1 is a submonoid of $(FV_+^*, \cdot)$, having the properties:*
(i) $\theta(U_1) = U_1$
(ii) $A \cdot \theta(A) \in U_1, \quad \forall A \in FV_+^*$
(iii) $[(FV_+^* \setminus U_1) \cdot U_1] \cap U_1 = \emptyset$.

Remark 114 *For any $A, B, C \in FV_+$ it is easy to obtain*

$$A \cdot (B + C) = A \cdot B + A \cdot C$$

$$(A + B) \cdot C = A \cdot C + B \cdot C$$

and therefore $(FV_+, +, \cdot)$ is semiring. Moreover, considering the set

$$FV_-^* = \{A = (A_-, A_+) \in FV : A_+(0) < 0\}$$

we can extend the multiplication on $FV_+ \cup FV_-^*$ in the following way:
(i) if $A \in FV_-^*$ and $B \in FV_+$, then $\varphi(A) \in FV_+^*$ and $A \cdot B = \varphi(\varphi(A) \cdot B) \in FV_-^*$,
(ii) if $A \in FV_+$ and $B \in FV_-^*$, then $\varphi(B) \in FV_+^*$ and $A \cdot B = \varphi(A \cdot \varphi(B)) \in FV_-^*$,
(iii) if $A, B \in FV_-^*$, then $\varphi(A), \varphi(B) \in FV_+^*$ and $A \cdot B = \varphi(A) \cdot \varphi(B) \in FV_+^*$,
and the above distributivity properties can be extended without difficulty.

We see that $(FV_+^*, \theta) \in ObAutMon$ and the submonoid U_1 has the properties (i)-(iii) from Definition 104, which defines the equivalence relation on FV_+^*

$$A \sim_{U_1} B \Longleftrightarrow A \cdot \theta(B) \in U_1,$$

which is a congruence relation. We will denote this relation $\sim_{U_1}$ by $\sim_\odot$ and we see that $(FV_+^*)_{U_1} = FV_+^* / \sim_\odot$ is an abelian group. It is easy to prove that

$$A \sim_\odot B \Longleftrightarrow A_-(\varepsilon) \cdot A_+(\varepsilon) = B_-(\varepsilon) \cdot B_+(\varepsilon), \quad \forall \varepsilon \in [0,1].$$

Remark 115 *The property $A \cdot \theta(A) \in U_1, \quad \forall A \in FV_+^*$, means that $A \cdot \overline{A} \sim_\odot \overline{1}, \quad \forall A \in FV_+^*$. Since $(FV_+^*, \theta), ((FV_+^*)_{U_1}, \Psi) \in ObAutMon$, the automorphism*

$$\Psi : (FV_+^*)_{U_1} \longrightarrow (FV_+^*)_{U_1},$$

is uniquely defined by the property of the canonical projection, $P_{U_1} \circ \theta = \Psi \circ P_{U_1}$, that is, $\Psi(A \cdot U_1) = \theta(A) \cdot U_1 = \overline{A} \cdot U_1, \quad \forall A \in FV_+^*$.

As a novelty, in [67] it is extended the isomorphism $\ln : (\mathbb{R}_+^*, \cdot) \longrightarrow (\mathbb{R}, +)$ for fuzzy numbers.

Definition 116 *(see [67])Let the commutative monoids $(FV_+^*, \cdot)$ and $(FV, +)$. Consider the function $Ln : FV_+^* \longrightarrow FV$, defined by $Ln(A) = (Ln(A)_-, Ln(A)_+)$, with*

$$Ln(A)_-(\varepsilon) = \ln[A_-(\varepsilon)], \quad \forall \varepsilon \in [0,1]$$

$$Ln(A)_+(\varepsilon) = \ln[A_+(\varepsilon)], \quad \forall \varepsilon \in [0,1].$$

This function is called the canonical fuzzy logarithm.

Theorem 117 *(see [67])The canonical fuzzy logarithm is an isomorphism between $(FV_+^*, \cdot)$ and $(FV, +)$ with $Ln(U_1) = S_0$.*

Sketch of proof: Firstly it is proved that $Ln(A) \in FV, \quad \forall A \in FV_+^*$ and

$$Ln(A \cdot B) = Ln(A) + Ln(B), \qquad A, B \in FV_+^*.$$

Considering the function $Exp : FV \longrightarrow FV_+^*$ defined by,

$$Exp(A) = (Exp(A)_-, Exp(A)_+)$$

with

$$Exp(A)_-(\varepsilon) = e^{A_-(\varepsilon)}, \qquad \forall \varepsilon \in [0,1]$$
$$Exp(A)_+(\varepsilon) = e^{A_+(\varepsilon)}, \qquad \forall \varepsilon \in [0,1],$$

after elementary calculus, it follows that

$$(Ln \circ Exp)(A) = A, \qquad \forall A \in FV$$

and

$$(Exp \circ Ln)(A) = A, \qquad \forall A \in FV_+^*$$

and so, Ln is isomorphism.
In order to prove that $Ln(U_1) = S_0$, let $A \in U_1$ and $B = (B_-, B_+) \in FV$, be given by

$$B_-(\varepsilon) = \ln(A_-(\varepsilon)), \quad B_+(\varepsilon) = \ln(A_+(\varepsilon)), \quad \forall \varepsilon \in [0,1]$$

that is $B = Ln(A)$. We have that $A_-(\varepsilon) \cdot A_+(\varepsilon) = 1, \quad \forall \varepsilon \in [0,1]$, $A(1) = 1$ and $A_-(0) > 0$. From here it follows that

$$B_-(1) = \ln(A_-(1)) = 0, \qquad B_+(1) = \ln(A_+(1)) = 0$$

and

$$B_-(\varepsilon) + B_+(\varepsilon) = \ln(A_-(\varepsilon)) + \ln(A_+(\varepsilon)) = \ln(A_-(\varepsilon) \cdot A_+(\varepsilon)) = 0, \quad \forall \varepsilon \in [0,1].$$

Then $B \in S_0$ and consequently, $Ln(U_1) \subset S_0$.
Analogous it is proved that $S_0 \subset Ln(U_1)$. Consequently, $Ln(U_1) = S_0$.

Corollary 118 *(see [67])The abelian groups $(FV_{S_0}, +)$ and $((FV_+^*)_{U_1}, \cdot)$ are isomorphic.*

Remark 119 *(see [67])We can define on FV the relation $\sim_1$ by*

$$A \sim_1 B \Longleftrightarrow A(1) = B(1).$$

It is easy to prove that this relation is an equivalence and since in each equivalence class we can choose the representing element as the corresponding element $\overline{r}$, we infer that *there exists a bijection between* $FV\ /\sim_1$ *and* $\mathbb{R}$. *We can see that* $\sim_\oplus \subsetneq \sim_1$. *Indeed, let* $A, B \in FV$,

$$A_-(\varepsilon) = \varepsilon + 1, \qquad A_+(\varepsilon) = 3 - \varepsilon, \qquad \forall \varepsilon \in [0,1]$$

$$B_-(\varepsilon) = \varepsilon + 1, \qquad B_+(\varepsilon) = 3 - \sqrt{\varepsilon}, \qquad \forall \varepsilon \in [0,1].$$

We see that $A(1) = B(1) = 2$, *and then* $A \sim_1 B$. *On the other hand,*

$$A_-(\varepsilon) + A_+(\varepsilon) \neq B_-(\varepsilon) + B_+(\varepsilon), \qquad \forall \varepsilon \in [0,1]$$

since $A_-(\varepsilon) + A_+(\varepsilon) = 4, \quad \forall \varepsilon \in [0,1]$ *and* $B_-(\frac{1}{4}) + B_+(\frac{1}{4}) = \frac{15}{4}$. *Then* A *and* B *there are not equivalent in the relation* $\sim_\oplus$. *Therefore* $\mathbb{R} \subset FV\ /\sim_\oplus$ *and* $FV\ /\sim_\oplus \neq \mathbb{R}$. *So, the above construction of the group* $(FV_{S_0}, +)$ *is not trivial.*

Remark 120 *Recently, the problem to specify the quotient set* $FV\ /\sim_\oplus$ *was solved in [144], showing that* $FV\ /\sim_\oplus$ *is isomorphic to* $(C[0,1], +)$.

4.1.3 One-sided fuzzy numbers

Here we present the results obtained in [77] concerning the one-sided fuzzy numbers. The attention is focused on right-sided fuzzy numbers which formalize the expressions such as " the number is at least x and no more than y ". Such expressions appear in many fields of real life. For instance, a right-sided fuzzy number can represent a real description of the number of infected individuals with an infectious disease having the rate of contact that varies seasonally (like grippe). The reason is that the real number of infected individuals is greater than the number of registered infectious individuals, but not exceed a reasonable proportion in the whole population. Moreover, the existence of right-sided fuzzy numbers is justified by many other fields in medical statistics where the real number of ill persons, regarding to a specific disease, is greater than those registered by the health care organizations and by authorities.

The one-sided fuzzy numbers are particular cases of fuzzy numbers. The first ideas for the use of one-sided fuzzy numbers appears in [196]. A distinct approach of one-sided fuzzy numbers is realized in [196], where the authors affirm that they define the notions of "left fuzzy set" and "right fuzzy set", but in Definition 2.3, page 512, these notions represent one-sided fuzzy numbers having only left side and right side, respectively. They prove that the sum of two left fuzzy sets is a left fuzzy set and the sum of two right fuzzy sets is a right fuzzy set, and obtain some properties of the mean value and of the variance of one-sided fuzzy sets.

Here we present some algebraic properties of the set of one-sided fuzzy numbers and prove that the sets of right-sided fuzzy numbers and left-sided fuzzy numbers are bounded in the metric topology of the set of all fuzzy numbers. We define the notion of fuzzy function preserving the side and prove that the fuzzy-Henstock integral of a continuous side-preserving function is a one-sided fuzzy number.

As an application of right-sided fuzzy numbers we study the delay integral equation

$$x(t) = g(t) + \int_{t-\tau}^{t} f(s, x(s))\, ds \qquad (4.1)$$

with $x(t)$ be a right-sided fuzzy number for each $t \in \mathbb{R}$. In crisp context, equation (4.1) is a mathematical model for the spread of infectious diseases with the rate of contact that varies seasonally (see [108] and [109]). In this model, the length of time for an individual to be infective is $\tau > 0$, the number of individuals who immigrated with the infection and still have the disease at time t is $g(t)$, the number of infected individuals at time t is $x(t)$, and the number of new infectious cases on unit time is $f(t, x(t))$. Here, we consider that g, x and f are time variable right-sided-fuzzy-number-valued functions.

We study the fuzzy initial value problem

$$\begin{cases} x(t) = g(t) + \int_{t-\tau}^{t} f(s, x(s))\, ds, \quad t \in [0, T] \\ x(t) = \varphi(t), \quad t \in [-\tau, 0] \end{cases} \tag{4.2}$$

where x, φ, g, f are right-sided-fuzzy-number-valued functions and $T = p \cdot \tau$ for some $p \in \mathbb{N}^*$. Our investigation includes the existence and uniqueness of the solution with positive support and the approximation of this solution.

According to [237] we denote the set of all fuzzy numbers by E^1 and we see that $\left(E^1, +, \widetilde{0}, \cdot\right)$ is a semimodule over the commutative semiring $(\mathbb{R}_+, +, 0, \cdot, 1)$ and $\left(E^1_+, +, \widetilde{0}, *, \widetilde{1}, \cdot\right)$ is a commutative semialgebra over the commutative semiring $(\mathbb{R}_+, +, 0, \cdot, 1)$, where the operations are given by

$$[u + v]^r = [u]^r + [v]^r = [u^r_- + v^r_-, u^r_+ + v^r_+], \forall r \in [0, 1],$$

$$[k \cdot u]^r = k \cdot [u]^r = \begin{cases} [ku^r_-, ku^r_+], & \text{if } k \geq 0 \\ [ku^r_+, ku^r_-], & \text{if } k < 0 \end{cases}$$

and for $u, v \in E^1_+$ we have $u * v \in E^1_+$ given by the $r-$level sets

$$[u * v]^r = [u^r_- \cdot v^r_-, u^r_+ \cdot v^r_+], \forall r \in [0, 1]$$

with

$$E^1_+ = \{u \in E^1 : suppu \subset [0, \infty)\}, \quad E^{1*}_+ = \{u \in E^1 : suppu \subset (0, \infty)\}.$$

Definition 121 *(see [106], Definition 4.2) For $u, v \in E^1$ we have $u \preccurlyeq_2 v$ if and only if $u^r_- \leq v^r_-$ and $u^r_+ \leq v^r_+$ for all $r \in (0, 1]$. A fuzzy number $u \in E^1$ is positive if $\widetilde{0} \preccurlyeq_2 u$.*

Remark 122 *$\left(E^1, \preccurlyeq_2\right)$ is partially ordered set and $u \preccurlyeq_2 v$, $u' \preccurlyeq_2 v'$ implies that $u + u' \preccurlyeq_2 v + v'$ for any $u, v, u', v' \in E^1$.*

Definition 123 *A function $f : E^1 \to E^1$ is said to be increasing if $u \preccurlyeq_2 v$ implies that $f(u) \preccurlyeq_2 f(v)$ for any $u, v \in E^1$ and positive if $\widetilde{0} \preccurlyeq_2 f(u)$ for any $u \in E^1$ with $\widetilde{0} \preccurlyeq_2 u$.*

The structure of one-sided fuzzy numbers

The the $r-$level sets of a right-sided fuzzy number are $[u^1_-, u^r_+]$ for all $r \in [0, 1]$, and the $r-$level sets of a left-sided fuzzy number are $[u^r_-, u^1_+]$ for all $r \in [0, 1]$. For right-sided fuzzy numbers we have $u^r_- = 0$ for all $r \in [0, 1)$ and for left-sided fuzzy numbers we have $u^r_+ = 0$ for all $r \in [0, 1)$. The set of all left-sided fuzzy numbers will be denoted by L^1 and the set of all right-sided fuzzy numbers will be denoted by R^1. We can use the notations $L^1_+ = \{u \in L^1 :\text{supp } u \subset [0, \infty)\}$, $L^{1*}_+ = \{u \in L^1 :\text{supp } u \subset (0, \infty)\}$, $R^1_+ = \{u \in R^1 :\text{supp } u \subset [0, \infty)\}$, $R^{1*}_+ = \{u \in R^1 :\text{supp } u \subset (0, \infty)\}$.

Remark 124 *(see [77])We have $u \in L^1$ if and only if $\forall r, r' \in (0, 1]$ with $r < r'$: $u^r_+ = u^{r'}_+$ and $u^r_- \leq u^{r'}_-$. Similarly, $u \in R^1$ if and only if $\forall r, r' \in (0, 1]$ with $r < r'$: $u^r_- = u^{r'}_-$ and $u^{r'}_+ \leq u^r_+$. So, $L^1 \subset E^1$, $R^1 \subset E^1$,*

$$L^1_+ = \{u \in L^1 : u^0_- \geq 0\}, \quad L^{1*}_+ = \{u \in L^1 : u^0_- > 0\}$$

and

$$R^1_+ = \{u \in R^1 : u^1_- \geq 0\}, \quad R^{1*}_+ = \{u \in R^1 : u^1_- > 0\}.$$

For unimodal one-sided fuzzy numbers we can write:

$$D(u,v) = \sup_{r\in[0,1]} \left|u^r_- - v^r_-\right| \text{ for any } u,v \in L^1$$

and

$$D(u,v) = \sup_{r\in[0,1]} \left|u^r_+ - v^r_+\right| \text{ for any } u,v \in R^1.$$

Definition 125 *(see [77])(i) A function $f : A \subset L^1 \to E^1$ preserves the side if $\forall u \in A$ and for any r, $r' \in (0,1]$ with $r < r'$ we have $f(u)^r_+ = f(u)^{r'}_+$ and $f(u)^r_- \leq f(u)^{r'}_-$.*
(ii) A function $f : A \subset R^1 \to E^1$ preserves the side if $\forall u \in A$ and for any r, $r' \in (0,1]$ with $r < r'$ we have $f(u)^r_- = f(u)^{r'}_-$ and $f(u)^{r'}_+ \leq f(u)^r_+$.

Example: The functions Ln and Exp are defined in [67] by $Exp : E^1 \to E^{1*}_+$,

$$Exp(u)^r_- = \exp\left(u^r_-\right), \quad Exp(u)^r_+ = \exp\left(u^r_+\right), \quad \forall r \in [0,1]$$

and $Ln : E^{1*}_+ \to E^1$,

$$Ln(u)^r_- = \ln\left(u^r_-\right), \quad Ln(u)^r_+ = \ln\left(u^r_+\right), \quad \forall r \in [0,1].$$

We can observe that the functions $Exp : L^1 \to E^{1*}_+$, $Exp : R^1 \to E^{1*}_+$ and $Ln : L^{1*}_+ \to E^1$, $Ln : R^{1*}_+ \to E^1$ preserve the side. Defining $Pow_2 : E^1_+ \to E^1_+$ and $Pow_k : E^1_+ \to E^1_+$ by

$$Pow_2(u) = u * u, \quad Pow_k(u) = Pow_{k-1}(u) * u, \forall k \in \mathbb{N}^*, \forall u \in E^1_+,$$

we see that $Pow_2 : L^1_+ \to E^1_+$, $Pow_2 : R^1_+ \to E^1_+$ and $Pow_k : L^1_+ \to E^1_+$, $Pow_k : R^1_+ \to E^1_+$, preserve the side.

As was observed above, in [67] have been defined the category $AutMon$. In this context we consider the following construction.

Definition 126 *(see [77])Let $(M,\varphi) \in ObAutMon$ and N, $N' \subset M$ two submonoids such that $\varphi(N) = N'$ and $\varphi(N') = N$. We say that the pair (N,N') realizes a dual decomposition of the monoid M if $M =< N \cup N' >$, where*

$$< N \cup N' >= \cap\{A : A \text{ is submonoid of } M \text{ and } N \cup N' \subset A\}$$

is the submonoid generated by $N \cup N'$. Denoting by e the neutral element of M, we say that the dual decomposition (N,N') is prime if $N \cap N' = \{e\}$ and it is proper if $N \cap N' \neq \{e\}$.

Examples of prime dual decompositions can be easily obtained considering some algebraic constructions in the classical sets of numbers: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$. The algebraic structure of L^1 and R^1 is presented in the following result.

Theorem 127 *(see [77])(i) In the monoid $\left((E^1,+),\varphi\right) \in ObAutMon$ with the automorphism $\varphi : E^1 \to E^1$, $\varphi(u) = -u$, $\forall u \in E^1$, the pair $\left(L^1, R^1\right)$ realizes a proper dual decomposition.*
(ii) $\left(L^1, +, \widetilde{0}, \cdot\right)$ and $\left(R^1, +, \widetilde{0}, \cdot\right)$ are semimodules over the commutative semiring $(\mathbb{R}_+, +, 0, \cdot, 1)$ and $\left(L^1_+, +, \widetilde{0}, *, \widetilde{1}, \cdot\right)$ and $\left(R^1_+, +, \widetilde{0}, *, \widetilde{1}, \cdot\right)$ are commutative semialgebras over the commutative semiring $(\mathbb{R}_+, +, 0, \cdot, 1)$.

Sketch of proof: (i) Obviously, $\left(L^1,+\right)$ and $\left(R^1,+\right)$ are submonoids in $\left(E^1,+\right)$. It is easy to prove that the opposite of a left-sided fuzzy number is a right-sided fuzzy number and conversely, the opposite of a right-sided fuzzy number is a left-sided fuzzy number. So, $\varphi\left(L^1\right)=R^1$, $\varphi\left(R^1\right)=L^1$, and $<L^1\cup R^1>\subset E^1$. In [77], after elementary computations it follows that $E^1\subset<L^1\cup R^1>$. It is easy to observe that $L^1\cap R^1=\mathbb{R}\neq\{\widetilde{0}\}$. Consequently, $E^1=<L^1\cup R^1>$ and the pair $\left(L^1,R^1\right)$ realizes a proper dual decomposition.
(ii) After elementary calculus we obtain:
$a\cdot u\in L^1$, $a\cdot v\in R^1$, for any $a\in\mathbb{R}_+$, $u\in L^1$, $v\in R^1$
$u+v\in L^1$, $u'+v'\in R^1$, for any $u,v\in L^1$, $u',v'\in R^1$
$u*v\in L^1_+$, $u'*v'\in R^1_+$, for any $u,v\in L^1_+$, $u',v'\in R^1_+$
and since $\widetilde{0},\widetilde{1}\in L^1_+\cap R^1_+$, the result (ii) follows.

We present now the topological structure of the sets L^1 and R^1.

Theorem 128 *(see [77])L^1 and R^1 are closed subsets in the metric space $\left(E^1,D\right)$, (here we have denoted $D=D_\infty$).*

In the proof of this result it is considered a sequence $(u_n)_{n\in\mathbb{N}}\subset R^1$ and $u\in E^1$ such that $\lim\limits_{n\to\infty}D\left(u_n,u\right)=0$. It is proved that $u\in R^1$ and then R^1 is a closed subset. Similarly it is proved that L^1 is closed.

Corollary 129 *(see [77])$\left(L^1,D\right)$ and $\left(R^1,D\right)$ are complete metric spaces. $\left(C\left([a,b],L^1\right),D^*\right)$ and $\left(C\left([a,b],R^1\right),D^*\right)$ are complete metric spaces for any interval $[a,b]\subset\mathbb{R}$, where*

$$C\left([a,b],L^1\right)=\{f:[a,b]\to L^1\ \mid f\text{ continuous on }[a,b]\}$$

and

$$C\left([a,b],R^1\right)=\{f:[a,b]\to R^1\ \mid f\text{ continuous on }[a,b]\}.$$

Theorem 130 *(see [77])(i) If $f:L^1\to E^1$ is a continuous function that preserves the side, then for any continuous $x:[a,b]\to L^1$ we have $(FH)\int_a^b f\left(x\left(s\right)\right)ds\in L^1$.*
(ii) If $f:R^1\to E^1$ is a continuous function that preserves the side, then for any continuous $x:[a,b]\to R^1$ we have $(FH)\int_a^b f\left(x\left(s\right)\right)ds\in R^1$.

Sketch of proof: (i) Since $x\left(s\right)\in L^1$ for all $s\in[a,b]$ and f preserves the side it follows that $x\left(s\right)^r_-\leq x\left(s\right)^{r'}_-$ and $x\left(s\right)^r_+=x\left(s\right)^{r'}_+$ implies that $f\left(x\left(s\right)^r_-\right)\leq f\left(x\left(s\right)^{r'}_-\right)$ and $f\left(x\left(s\right)^r_+\right)=f\left(x\left(s\right)^{r'}_+\right)$, $\forall r,r'\in(0,1]$ with $r<r'$. So, $f\circ x:[a,b]\to L^1$ is fuzzy-Riemann integrable and

$$\left[(FR)\int_a^b f\left(x\left(s\right)\right)ds\right]^r=\left[\int_a^b f\left(x\left(s\right)\right)^r_-ds,\int_a^b f\left(x\left(s\right)\right)^r_+ds\right].$$

Using the monotony of the integral we infer that

$$\int_a^b f\left(x\left(s\right)\right)^r_-ds\leq\int_a^b f\left(x\left(s\right)\right)^{r'}_-ds,\quad\int_a^b f\left(x\left(s\right)\right)^r_+ds=\int_a^b f\left(x\left(s\right)\right)^{r'}_+ds$$

for any $r, r' \in (0,1]$ with $r < r'$, and consequently, $(FR)\int_a^b f(x(s))\,ds \in L^1$.

(ii) It follows analogously.

Application of the right-sided fuzzy numbers to delay integral equations arising in epidemiology

Consider the fuzzy initial value problem (4.2) under the following conditions:

(i) $g : [0,T] \to R^1_+$, $\varphi : [-\tau.0] \to R^1_+$, $f : [-\tau, T] \times R^1 \to R^1$ are continuous functions. Since we expect positive solutions, suppose that f is positive (i.e. $\widetilde{0} \preccurlyeq_2 f(s,u)$ for all $s \in [-\tau, T]$ and $u \in R^1$, $\widetilde{0} \preccurlyeq_2 u$)

(ii) there exist $\gamma, \alpha \geq 0$ such that

$$D\left(f(s,u), f\left(s', v\right)\right) \leq \gamma\left|s - s'\right| + \alpha D(u,v), \quad \forall s, s' \in [-\tau, T],\ u, v \in R^1$$

(iii) f preserves the side, that is $f(s,u) \in R^1$ for all $s \in [-\tau, T]$, $u \in R^1$

(iv) (the compatibility condition)

$$\varphi(0) = g(0) + \int_{-\tau}^{0} f(s, \varphi(s))\,ds \tag{4.3}$$

(v) (the contraction condition): $\alpha\tau < 1$

(vi) there exist $\beta, \mu \geq 0$ such that

$$D\left(g(s), g\left(s'\right)\right) \leq \beta\left|s - s'\right|, \quad D\left(\varphi(t), \varphi\left(t'\right)\right) \leq \mu\left|t - t'\right|, \ \forall s, s' \in [0,T],\ \forall t, t' \in [-\tau, 0].$$

We define the set

$$C_\varphi\left([-\tau, T], R^1\right) = \{x \in C\left([-\tau, T], R^1\right) : x(t) = \varphi(t), \forall t \in [-\tau, 0]\}$$

and it is obvious that $C_\varphi\left([-\tau, T], R^1\right)$ is a bounded set in the complete metric space $C\left([-\tau, T], R^1\right)$ and therefore $\left(C_\varphi\left([-\tau, T], R^1\right), D^*\right)$ is a complete metric space too, with the metric $D^*(u,v) = \max\limits_{s \in [-\tau, T]} D(u(s), v(s))$. On the same set we can consider the Bielecki's type metric

$$D^*_\theta(u,v) = \max_{s \in [-\tau, T]} \{D(u(s), v(s)) \cdot e^{-\theta(s+\tau)}\}$$

with $\theta > 0$ conveniently chosen. By denoting

$$X = \left(C_\varphi\left([-\tau, T], R^1\right), D^*\right), \quad X_\theta = \left(C_\varphi\left([-\tau, T], R^1\right), D^*_\theta\right)$$

we see that X_θ is a complete metric space for any $\theta > 0$.

Define the operator $A : C_\varphi\left([-\tau, T], E^1\right) \to \{x \mid x : [-\tau, T] \to R^1\}$ by

$$A(x)(t) = \begin{cases} g(t) + (FR)\displaystyle\int_{t-\tau}^{t} f(s, x(s))\,ds, & t \in [0,T] \\ \varphi(t), & t \in [-\tau, 0] \end{cases} \tag{4.4}$$

and the sequence of successive approximations $(x_m)_{m \in \mathbb{N}} \subset C_\varphi\left([-\tau, T], R^1\right)$ given by

$$x_0(t) = \varphi(t), \forall t \in [-\tau, 0], \quad x_0(t) = \varphi(0), \forall t \in [0,T], \quad x_m = A(x_{m-1}), \forall m \in \mathbb{N}^*. \tag{4.5}$$

Lemma 131 *(see [77])Suppose that $g : [a,b] \to E^1$ is continuous and $f : [a,b] \times E^1 \to E^1$ has the following Lipschitz property:*

$$\text{there exist } \alpha, \gamma > 0 \text{ such that } D(f(t,u), f(s,v)) \le \gamma |t-s| + \alpha D(u,v),$$

for all $t, s \in [a,b]$, $u, v \in E^1$. Then the function $F : [a,b] \to E^1$ given by $F(s) = f(s, g(s))$, $\forall s \in [a,b]$ is uniformly continuous and the function $H : [a,b] \to E^1$ given by $H(t) = (FR)\int_a^t f(s, g(s))\,ds$, $\forall t \in [a,b]$, is Lipschitzian.

Sketch of proof: From the continuity of g we infer its uniform continuity. Then for any $\varepsilon > 0$ there exists $\delta > 0$ which can be $\delta \le \frac{\varepsilon}{2\gamma}$ such that $D(g(s), g(s')) \le \frac{\varepsilon}{2\alpha}$ for any $s, s' \in [a,b]$ with $|s - s'| < \delta$. Moreover,

$$D(F(s), F(s')) = D(f(s, g(s)), f(s', g(s'))) \le \gamma |s - s'| +$$

$$+\alpha D(g(s), g(s')) \le \gamma \cdot \frac{\varepsilon}{2\gamma} + \alpha \cdot \frac{\varepsilon}{2\alpha} = \varepsilon$$

that is the uniform continuity of F.
Let arbitrary $t, t' \in [a,b]$. Suppose that $t \le t'$. Since $f_0 : [a,b] \to E^1$ given by $f_0(s) = f\left(s, \widetilde{0}\right)$ is continuous, we infer that there is $M_0 \ge 0$ such that $D\left(\widetilde{0}, f_0(s)\right) \le M_0$ for all $s \in [a,b]$. Then,

$$D(H(t), H(t')) \le D\left((FR)\int_a^t f(s, g(s))\,ds, (FR)\int_a^t f(s, g(s))\,ds\right) +$$

$$+D\left(\widetilde{0}, (FR)\int_t^{t'} f(s, g(s))\,ds\right) \le \int_t^{t'} [D\left(\widetilde{0}, f_0(s)\right) + \alpha D\left(\widetilde{0}, g(s)\right)]ds \le (M_0 + \alpha M_g) \cdot |t - t'|,$$

where $M_g \ge 0$ is such that $D\left(\widetilde{0}, g(s)\right) \le M_g$ for all $s \in [a,b]$, according to the continuity of g. The case $t' \le t$ is analogous.

In the following theorem we obtain the existence, uniqueness, and positivity of the solution of (4.2).

Theorem 132 *(see [77]) Under the conditions (i)-(v),*

$$A\left(C_\varphi\left([-\tau, T], R^1\right)\right) \subset C_\varphi\left([-\tau, T], R^1\right),$$

i.e. the operator A is well-defined, and the initial value problem (4.2) has a unique positive solution $x^* \in X$, $\lim_{m\to\infty} D^*(x_m, x^*) = 0$, and the following error estimates hold:

$$D(x_m(t), x^*(t)) \le \frac{(\alpha\tau)^m}{1 - \alpha\tau} \cdot [M_0\tau + M_g + M_\varphi], \quad \forall t \in [0, T], \tag{4.6}$$

$$D(x_m(t), x^*(t)) \le \frac{\alpha\tau}{1 - \alpha\tau} \cdot D(x_m(t), x_{m-1}(t)), \quad \forall t \in [0, T] \tag{4.7}$$

for all $m \in \mathbb{N}^*$.

Sketch of proof: Firstly it is proved that $A\left(C_\varphi\left([-\tau,T],R^1\right)\right) \subset C_\varphi\left([-\tau,T],R^1\right)$ and then applying the fixed point technique it obtains

$$D\left(A\left(u\right)\left(t\right),A\left(v\right)\left(t\right)\right) \leq \alpha\tau D^*\left(u,v\right), \quad \forall u,v \in C_\varphi\left([-\tau,T],R^1\right), \ \forall t \in [0,T].$$

By the Banach's fixed point theorem it follows the existence and uniqueness of the solution of (4.2) and the estimate (4.7). We see that

$$D^*\left(x_1,x_0\right) \leq \sup_{t\in[0,T]} D\left(g\left(t\right),\widetilde{0}\right) + D\left(\varphi\left(0\right),\widetilde{0}\right) + M_0\tau \leq M_g + M_\varphi + M_0\tau$$

where $M_g, M_\varphi, M_0 \geq 0$ are such that

$$D\left(\varphi\left(t\right),\widetilde{0}\right) \leq M_\varphi, \quad \forall t \in [-\tau,0], \quad D\left(g\left(t\right),\widetilde{0}\right) \leq M_g, \quad \forall t \in [0,T]$$

and $D\left(f\left(s,x_0\left(s\right)\right),\widetilde{0}\right) \leq M_0$, $\forall s \in [-\tau,T]$. Now, the inequality (4.6) follows.
Since $\varphi\left(t\right) \in R^1_+, \forall t \in [-\tau,0]$ we have $x_0 \in C_\varphi\left([-\tau,T],R^1_+\right)$ and because $g\left(t\right) \in R^1_+, \forall t \in [0,T]$ and f is positive preserving the side, we infer that $x_1 \in C_\varphi\left([-\tau,T],R^1_+\right)$.
By induction for $m \geq 1$ it follows that $x_m \in C_\varphi\left([-\tau,T],R^1_+\right)$, $\forall m \in \mathbb{N}^*$. Moreover, $\lim_{m\to\infty} D^*\left(x_m,x^*\right) = 0$ implies that

$$\lim_{m\to\infty}\left|x_m\left(t\right)^r_- - \left(x^*\left(t\right)\right)^r_-\right| = 0 \text{ and } \lim_{m\to\infty}\left|x_m\left(t\right)^r_+ - \left(x^*\left(t\right)\right)^r_+\right| = 0$$

for all $r \in [0,1]$, $t \in [-\tau,T]$ and because

$$x_m\left(t\right)^r_- \geq 0, \quad x_m\left(t\right)^r_+ \geq 0 \text{ for all } m \in \mathbb{N}^*, \ r \in [0,1], \ t \in [-\tau,T],$$

we infer that $\left(x^*\left(t\right)\right)^r_- \geq 0$, $\left(x^*\left(t\right)\right)^r_+ \geq 0$, for all $t \in [-\tau,T]$, $r \in [0,1]$. Then, $x^* \in C_\varphi\left([-\tau,T],R^1_+\right)$ and the solution of (4.2) is positive.

If we use Bielecki's type metric, the contraction condition (v) is not necessary.

Theorem 133 *(see [77])Under the conditions (i)-(iv) the initial value problem (4.2) has unique positive solution $x^* \in X_\theta$ and $\lim_{m\to\infty} D^*_\theta\left(x_m,x^*\right) = 0$. Furthermore, the following inequality holds:*

$$D^*_\theta\left(x_m,x^*\right) \leq \frac{\left(\frac{\alpha}{\theta}\right)^m}{1-\frac{\alpha}{\theta}} \cdot D^*_\theta\left(x_1,x_0\right), \quad \forall m \in \mathbb{N}^* \tag{4.8}$$

where $\theta > 0$ is chosen such that $\frac{\alpha}{\theta} < 1$.

We define the sequence of functions $F_m : [-\tau,T] \to R^1$, $m \in \mathbb{N}$, given by

$$F_m\left(s\right) = f\left(s,x_m\left(s\right)\right), \quad \forall s \in [-\tau,T]$$

and in the following theorem we obtain the uniformly boundedness of the sequence of successive approximations.

Theorem 134 *(see [77])If all the conditions (i)-(vi) are satisfied, then the sequences $\left(x_m\right)_{m\in\mathbb{N}} \subset C_\varphi\left([-\tau,T],R^1_+\right)$ and $\left(F_m\right)_{m\in\mathbb{N}} \subset C\left([-\tau,T],R^1\right)$ are uniformly bounded and uniformly Lipschitz, and the solution of (4.2) is bounded.*

Sketch of proof: The uniformly boundedness of the sequence $(x_m)_{m\in\mathbb{N}}$ means: $\exists R \geq 0$ such that $D\left(x_m(t), \widetilde{0}\right) \leq R$ for all $t \in [-\tau, T]$ and $m \in \mathbb{N}$.
The uniformly Lipschitz property means: $\exists L_0 \geq 0$ such that $D(x_m(t), x_m(t')) \leq L_0 |t - t'|$ for all $t, t' \in [-\tau, T]$ and $m \in \mathbb{N}$. The sequence $(x_m)_{m\in\mathbb{N}}$ is given by

$$x_m(t) = \varphi(t), \quad \forall t \in [-\tau, 0], \ \forall m \in \mathbb{N},$$

$x_0(t) = \varphi(0), \forall t \in [0, T]$, and

$$x_m(t) = g(t) + (FR)\int_{t-\tau}^{t} f(s, x_{m-1}(s))\, ds, \quad \forall t \in [0, T], \ \forall m \in \mathbb{N}^*. \tag{4.9}$$

By induction it obtains

$$D(x_m(t), x_{m-1}(t)) \leq (\alpha\tau)^{m-1} \cdot D^*(x_1, x_0), \quad \forall t \in [0, T].$$

Then,

$$D(x_m(t), x_0(t)) \leq D(x_m(t), x_{m-1}(t)) + D(x_{m-1}(t), x_{m-2}(t)) + ...+$$

$$+D(x_1(t), x_0(t)) \leq \frac{M_g + M_\varphi + M_0\tau}{1 - \alpha\tau}, \quad \forall t \in [0, T], \ \forall m \in \mathbb{N}^*.$$

Consequently,

$$D\left(x_m(t), \widetilde{0}\right) \leq D(x_m(t), x_0(t)) + D\left(x_0(t), \widetilde{0}\right) \leq \frac{M_g + M_\varphi + M_0\tau}{1 - \alpha\tau} + M_\varphi \overset{notation}{=} R$$

for all $t \in [0, T]$ and $m \in \mathbb{N}^*$. So, the sequence $(x_m)_{m\in\mathbb{N}^*}$ is uniformly bounded and

$$D\left(x^*(t), \widetilde{0}\right) \leq D(x^*(t), x_m(t)) + D\left(x_m(t), \widetilde{0}\right) \leq \frac{2(M_g + M_\varphi + M_0\tau)}{1 - \alpha\tau} + M_\varphi, \quad \forall t \in [0, T]$$

that is the boundeness of the solution of (4.2). Now, we can obtain

$$D\left(f(t, x_m(t)), \widetilde{0}\right) \leq \max\left(\frac{\alpha(M_g + M_\varphi + M_0\tau)}{1 - \alpha\tau} + M_0, M_\varphi\right) \overset{notation}{=} M$$

for all $t \in [-\tau, T], m \in \mathbb{N}^*$.
For the uniformly Lipschitz property it obtains

$$D\left(x_m(t), x_m(t')\right) \leq \beta\left|t - t'\right| + \int_{t-\tau}^{t'-\tau} D\left(f(s, x_{m-1}(s)), \widetilde{0}\right) ds + \int_{t}^{t'} D\left(f(s, x_{m-1}(s)), \widetilde{0}\right) ds \leq$$

$$\leq (\beta + 2M) \cdot \left|t - t'\right|, \quad \forall t, t' \in [-\tau, T], m \in \mathbb{N}^*.$$

Let $L_0 = \max(\mu, \beta + 2M)$. Then,

$$D\left(x_m(t), x_m(t')\right) \leq L_0\left|t - t'\right|, \quad \forall t, t' \in [-\tau, T], \ \forall m \in \mathbb{N}.$$

Finally,

$$D\left(f(t, x_m(t)), f\left(t', x_m(t')\right)\right) \leq (\gamma + \alpha L_0) \cdot \left|t - t'\right| = L\left|t - t'\right|, \quad \forall t, t' \in [-\tau, T], \ \forall m \in \mathbb{N}.$$

Remark 135 *Under the conditions (i)-(vi) the solution of (4.2) is Lipschitzian,*

$$D\left(x^{*}\left(t\right),x^{*}\left(t'\right)\right)\leq\max\{\mu,\beta+2\left(M+\alpha\frac{\alpha(M_{g}+M_{\varphi}+M_{0}\tau)}{1-\alpha\tau}\right)\}\cdot\left|t-t'\right|$$

for any $t,t'\in[-\tau,T]$.

The solution of of the initial value problem (4.2) is approximated on $[0,T]$ by the terms of the sequence of successive approximations given in (4.9). In order to compute the integrals from (4.9) we consider the uniform partition of $[-\tau,T]$

$$\Delta:-\tau=t_{0}<t_{1}<...<t_{n}=0<t_{n+1}<...<t_{q}=T$$

with $q=(p+1)\,n$, $t_{i}=t_{i-1}+h=t_{i-1}+\frac{\tau}{n}=-\tau+\frac{i\cdot\tau}{n}$, $i=\overline{1,q}$ and apply the trapezoidal quadrature rule. It obtains the following algorithm that gives the effective computed values:

$$\overline{x}_{m}\left(t_{i}\right)=x_{m}\left(t_{i}\right)=\varphi\left(t_{i}\right),\quad i=\overline{0,n},\ m\in\mathbb{N},$$

$$\overline{x}_{0}\left(t_{i}\right)=x_{0}\left(t_{i}\right)=\varphi\left(0\right),\ i=\overline{n+1,q},$$

$$\overline{x}_{1}\left(t_{i}\right)=g\left(t_{i}\right)+\frac{\tau}{2n}\cdot\sum_{j=0}^{n-1}[f\left(t_{i+j-n},\overline{x}_{0}\left(t_{i+j-n}\right)\right)+f\left(t_{i+j-n+1},\overline{x}_{0}\left(t_{i+j-n+1}\right)\right)],\ i=\overline{n+1,q}. \tag{4.10}$$

By induction for $m\in\mathbb{N}^{*}$, $m\geq2$ and $i=\overline{n+1,q}$ we get,

$$\overline{x}_{m}\left(t_{i}\right)=g\left(t_{i}\right)+\frac{\tau}{2n}\cdot\sum_{j=0}^{n-1}[f\left(t_{i+j-n},\overline{x}_{m-1}\left(t_{i+j-n}\right)\right)+f\left(t_{i+j-n+1},\overline{x}_{m-1}\left(t_{i+j-n+1}\right)\right)]. \tag{4.11}$$

This algorithm has the following practical stopping criterion:

For given $\varepsilon'>0$ and given $n\in\mathbb{N}^{*}$ (previously chosen) it is determined the first natural number $m\in\mathbb{N}^{*}$ for which

$$D\left(\overline{x}_{m}\left(t_{i}\right),\overline{x}_{m-1}\left(t_{i}\right)\right)<\varepsilon',\quad\forall i=\overline{n+1,q}$$

and we stop to this m retaining the approximations $\overline{x}_{m}\left(t_{i}\right)$, $i=\overline{0,q}$ of the solution.

The convergence of this algorithm is proved in the following result:

Theorem 136 *(see [77])Suppose that the conditions (i)-(vi) are satisfied. Then, the solution of the initial value problem (4.2) is approximated on the knots* $t_{i}=-\tau+\frac{i\cdot\tau}{n}$, $i=\overline{0,q}$, *by the sequence* $\left(\overline{x}_{m}\left(t_{i}\right)\right)_{m\in\mathbb{N}}$, $i=\overline{0,q}$, *with the following apriori error estimate:*

$$D\left(x^{*}\left(t_{i}\right),\overline{x}_{m}\left(t_{i}\right)\right)\leq\frac{\left(\alpha\tau\right)^{m}}{1-\alpha\tau}\cdot\left[M_{0}\tau+M_{g}+M_{\varphi}\right]+\frac{L\tau^{2}}{4n\left(1-\alpha\tau\right)}$$

for all $i=\overline{n+1,q}$, $m\in\mathbb{N}^{*}$.

4.2 Nonlinear Hammerstein and Fredholm fuzzy integral equations

We present now the iterative numerical method developed in [80] for fuzzy Hammerstein integral equations.

The study of fuzzy integral equations begins with the investigations of Kaleva (see [158]) and Seikkala (see [218]) for the fuzzy Volterra integral equation that is equivalent to the initial value problem for first order fuzzy differential equations, where the Banach's fixed point theorem and the method of successive approximations are applied in the problem of the existence and uniqueness of the solution. Afterwards, the distinct study of the existence of an unique solution for fuzzy Fredholm integral equations is carried out in [187].

The main problems that arise for fuzzy integral equations are: the existence, the existence and uniqueness of the solution, and the construction of numerical methods to approximate it. The fixed point theorems like the Darbo's theorem and the Banach's fixed point principle were the tools used to prove on the one hand the existence and on the other hand the existence and uniqueness of the solution of fuzzy integral equations (see [20], [22], [141], [198], [199], [200], [221], and [223]). The boundedness of the solutions is studied in [191]. Some applications of the fuzzy integral equations to control models with fuzzy uncertainties are presented in [165] and [115].

The numerical methods for fuzzy integral equations involve various techniques. Iterative techniques are applied in [30], [68], [131], [133], [134], [197], and [129]. The analytic-numeric methods like Adomian decomposition, homotopy analysis and homotopy perturbation are used in [1], [18], [16], [162], [168], and [186]. Other techniques used in the construction of the numerical methods for fuzzy integral equations are: Nyström techniques (see [216]), predictor-corrector procedures (see [220]), Lagrange interpolation (see [12]), spline interpolation (see [154]), divided and finite differences (see [195]), Bernstein polynomials (see [128] and [194]), Legendre wavelets (see [139]), fuzzy Haar wavelets (see [248]), and Galerkin type techniques (see [168]).

In this section we consider the nonlinear fuzzy Hammerstein integral equation

$$x(t) = g(t) + (FR)\int_a^b H(t,s)\cdot f(s,x(s))\,ds, \qquad t\in[a,b] \tag{4.12}$$

under the following conditions:

(i) $g\in C([a,b],E^1)$, $f\in C([a,b]\times E^1,E^1)$, $H\in C([a,b]\times[a,b],\mathbb{R})$, $H(t,s)\geq 0$, $\forall t,s\in[a,b]$;

(ii) there exist $\alpha,\gamma\geq 0$ such that $D(f(s,u),f(s',v))\leq\gamma|s-s'|+\alpha D(u,v)$ for all $s,s'\in[a,b]$, $u,v\in E^1$;

(iii) $\alpha M_H(b-a)<1$, where $M_H\geq 0$ is such that $|H(t,s)|\leq M_H$, $\forall t,s\in[a,b]$, according to the continuity of H;

(iv) there exists $\beta\geq 0$ such that $D(g(t),g(t'))\leq\beta|t-t'|$ for all $t,t'\in[a,b]$;

(v) there exists $\mu\geq 0$ such that $|H(t,s)-H(t',s)|\leq\mu|t-t'|$ for all $t,t',s\in[a,b]$;

(vi) there exists $\delta\geq 0$ such that $|H(t,s)-H(t,s')|\leq\delta|s-s'|$ for all $t,s,s'\in[a,b]$.

Lemma 137 *(see [80])If $f\in C([a,b]\times[a,b],E^1)$, $g\in C([a,b],E^1)$, and $a\in C(a,b],\mathbb{R}_+)$ then the functions $a\cdot g:[a,b]\to E^1$ and $F:[a,b]\to E^1$ given by $(a\cdot g)(t)=a(t)\cdot g(t)$, $\forall t\in[a,b]$ and $F(t)=(FH)\int_a^b f(t,s)\,ds$, are continuous.*

Theorem 138 *(see [80])(a). Under the conditions (i)-(iii) the integral equation (4.12) has unique solution in $C([a,b],E^1)$, $x^*\in C([a,b],E^1)$ and the sequence of successive approximations $(x_m)_{m\in\mathbb{N}}\subset C([a,b],E^1)$,*

$$x_m(t) = g(t) + (FR)\int_a^b H(t,s)\cdot f(s,x_{m-1}(s))\,ds, \quad t\in[a,b],\ m\in\mathbb{N}^* \tag{4.13}$$

converges to x^* *in* $C\left([a,b],E^1\right)$ *for any choice of* $x_0 \in C\left([a,b],E^1\right)$. *In addition, the following error estimates hold:*

$$D\left(x^*\left(t\right),x_m\left(t\right)\right) \leq \frac{\left[\alpha M_H\left(b-a\right)\right]^m}{1-\alpha M_H\left(b-a\right)} \cdot D\left(x_1\left(t\right),x_0\left(t\right)\right), \ \forall t \in [a,b], \ m \in \mathbb{N}^* \tag{4.14}$$

and

$$D\left(x^*\left(t\right),x_m\left(t\right)\right) \leq \frac{\alpha M_H\left(b-a\right)}{1-\alpha M_H\left(b-a\right)} \cdot D\left(x_m\left(t\right),x_{m-1}\left(t\right)\right), \ \forall t \in [a,b], \ m \in \mathbb{N}^*. \tag{4.15}$$

Choosing $x_0 \in C\left([a,b],E^1\right)$, $x_0 = g$, *the inequality (4.14) becomes*

$$D\left(x^*\left(t\right),x_m\left(t\right)\right) \leq \frac{\left[\alpha M_H\left(b-a\right)\right]^m}{1-\alpha M_H\left(b-a\right)} \cdot M_0 M_H\left(b-a\right), \ \forall t \in [a,b], \ m \in \mathbb{N}^*, \tag{4.16}$$

where $M_0 \geq 0$ *is given in (4.17). Moreover, the sequence of successive approximations (4.13) is uniformly bounded, that is, there exists a constant* $R \geq 0$ *such that* $D\left(x_m\left(t\right),\widetilde{0}\right) \leq R$, *for all* $m \in \mathbb{N}$ *and* $t \in [a,b]$, *and the solution* x^* *is bounded, too.*
(b). Under the conditions (i)-(v) the sequence of successive approximations (4.13) is uniformly Lipschitz, that is, there exist a constant $L_0 \geq 0$ *such that* $D\left(x_m\left(t\right),x_m\left(t'\right)\right) \leq L_0\left|t-t_0\right|$, *for all* $m \in \mathbb{N}$ *and* $t,t' \in [a,b]$.

Sketch of proof: Firstly it is proved that $A\left(C\left([a,b],E^1\right)\right) \subset C\left([a,b],E^1\right)$ for the operator defined by

$$A\left(x\right)\left(t\right) = g\left(t\right) + \left(FR\right)\int_a^b H\left(t,s\right) \cdot f\left(s,x\left(s\right)\right) ds, \ \forall t \in [a,b], \ \forall x \in C\left([a,b],E^1\right)$$

using the above presented Lemma. Then, using the fixed point technique it obtains

$$D^*\left(A\left(u\right),A\left(v\right)\right) \leq \alpha M_H\left(b-a\right) D^*\left(u,v\right), \quad \forall u,v \in C\left([a,b],E^1\right)$$

and using the Banach's fixed point theorem, the existence and uniqueness of the solution, $x^* \in C\left([a,b],E^1\right)$, of (4.12) follows. Moreover, the sequence of successive approximations converges to this solution in $C\left([a,b],E^1\right)$ and the estimates (4.14) and (4.15) hold. Choosing $x_0 = g$ and denoting $F_0 : [a,b] \to E^1$, $F_0\left(s\right) = f\left(s,g\left(s\right)\right)$, $s \in [a,b]$, we infer that F_0 is continuous since $f \in C\left([a,b] \times E^1, E^1\right)$ and $g \in C\left([a,b],E^1\right)$. Using the previous Lemma, there are $M_g, M_0 \geq 0$ such that $D\left(x_0\left(s\right),\widetilde{0}\right) = D\left(g\left(s\right),\widetilde{0}\right) \leq M_g$ and

$$D\left(F_0\left(s\right),\widetilde{0}\right) = D\left(f\left(s,x_0\left(s\right)\right),\widetilde{0}\right) \leq M_0, \quad \text{for all } s \in [a,b]. \tag{4.17}$$

By induction it follows that,

$$D\left(x_m\left(t\right),x_{m-1}\left(t\right)\right) \leq \left[\alpha M_H\left(b-a\right)\right]^{m-1} \cdot D^*\left(x_1,x_0\right), \quad \forall t \in [a,b], \ m \in \mathbb{N}^*.$$

Consequently,

$$D\left(x_m\left(t\right),x_0\left(t\right)\right) \leq D\left(x_m\left(t\right),x_{m-1}\left(t\right)\right) + D\left(x_{m-1}\left(t\right),x_{m-2}\left(t\right)\right) + ...+$$

$$+D\left(x_1\left(t\right),x_0\left(t\right)\right) \leq \frac{1-\left[\alpha M_H\left(b-a\right)\right]^m}{1-\alpha M_H\left(b-a\right)} \cdot M_0 M_H\left(b-a\right), \quad \forall t \in [a,b], \ m \in \mathbb{N}^*$$

and

$$D\left(x_m(t),\widetilde{0}\right) \le D\left(x_m(t),x_0(t)\right)+D\left(x_0(t),\widetilde{0}\right) \le \frac{M_0 M_H (b-a)}{1-\alpha M_H (b-a)}+M_g \overset{notation}{=} R$$

for any $t \in [a,b]$ and for all $m \in \mathbb{N}$, that is the uniformly boundedness of the sequence $(x_m)_{m\in\mathbb{N}}$ in $C\left([a,b],E^1\right)$. For $m \in \mathbb{N}^*$, denoting $F_m : [a,b] \to E^1$, $F_m(s) = f(s, x_m(s))$, $s \in [a,b]$, it obtains,

$$D\left(F_m(s),\widetilde{0}\right) \le \frac{\alpha M_0 M_H (b-a)}{1-\alpha M_H (b-a)}+M_0 \overset{notation}{=} M, \quad \forall s \in [a,b],\ m \in \mathbb{N}^*.$$

So, the sequence of functions $(F_m)_{m\in\mathbb{N}}$ is uniformly bounded in $C\left([a,b],E^1\right)$, too.
Since

$$D\left(x^*(t),\widetilde{0}\right) \le D\left(x^*(t),x_m(t)\right)+D\left(x_m(t),\widetilde{0}\right) \le \frac{M_0 M_H (b-a)}{1-\alpha M_H (b-a)}+R, \quad \forall t \in [a,b]$$

we infer that the solution of (4.12) is bounded.
In inductive manner it follows that

$$D\left(x_m(t),x_m(t')\right) \le [\beta+\mu M(b-a)]\cdot\left|t-t'\right| = L_0\left|t-t'\right|, \quad \forall t,t' \in [a,b],\ m \in \mathbb{N}^*$$

and

$$D\left(F_m(t),F_m(t')\right) \le \gamma\left|t-t'\right|+\alpha D\left(x_m(t),x_m(t')\right) \le (\gamma+\alpha L_0)\cdot\left|t-t'\right| = L_1\cdot\left|t-t'\right|$$

for all $m \in \mathbb{N}^*$. So, the sequence of functions $(F_m)_{m\in\mathbb{N}}$ is uniformly Lipschitz with the constant $L_1 = \gamma+\alpha(\beta+\mu M(b-a))$.

Remark 139 *Under the conditions (i)-(vi), we can obtain,*

$$D\left(H(t,s)\cdot f(s,x_m(s)),H(t,s')\cdot f(s',x_m(s'))\right) \le L\left|s-s'\right|, \quad \forall s,s' \in [a,b],$$

for any fixed $t \in [a,b]$ *and* $m \in \mathbb{N}$, *where*

$$L = M_H L_1 + M\delta = M_H[\gamma+\alpha(\beta+\mu M(b-a))]+M\delta.$$

The algorithm

Now, we present the algorithm to approximate the solution of (4.12). In this purpose, we consider the uniform partition of $[a,b]$,

$$\Delta : a = t_0 < t_1 < ... < t_{n-1} < t_n = b$$

with $t_i = a+ih$, $i=\overline{0,n}$, $h=\frac{(b-a)}{n}$, and apply the trapezoidal quadrature rule in the computation of the terms of the sequence of successive approximations (4.13),

$$x_0(t_i) = g(t_i), \quad i=\overline{0,n}$$

$$x_m(t_i) = g(t_i)+(FR)\int_a^b H(t_i,s)\cdot F_{m-1}(s)\,ds, \quad i=\overline{0,n},\ m \in \mathbb{N}^*$$

obtaining

$$x_m(t_i) = g(t_i)+\frac{(b-a)}{2n}\cdot\sum_{j=1}^{n}[H(t_i,t_{j-1})\cdot f(t_{j-1},x_{m-1}(t_{j-1}))+$$

$$+H(t_i,t_j)\cdot f(t_j,x_{m-1}(t_j))]+R_{m,i},\quad i=\overline{0,n},\ m\in\mathbb{N}^*,\tag{4.18}$$

with

$$D\left(R_{m,i},\widetilde{0}\right)\leq\frac{L(b-a)^2}{4n},\quad\forall i=\overline{0,n},\ m\in\mathbb{N}^*,$$

We get the following iterative algorithm:

Step 0: There are introduced the data a,b,ε',n, and the functions f,g,H.

Step 1: For $i=\overline{0,n}$ we set

$$\overline{x_0}(t_i)=x_0(t_i)=g(t_i).$$

Step 2 (the first iterative step): For $m=1$ and for all $i=\overline{0,n}$, compute

$$\overline{x_1(t_i)}=g(t_i)+\frac{(b-a)}{2n}\cdot\sum_{j=1}^{n}[H(t_i,t_{j-1})\cdot f(t_{j-1},\overline{x_0}(t_{j-1}))+H(t_i,t_j)\cdot f(t_j,\overline{x_0}(t_j))].\tag{4.19}$$

Steps 3-4 (the generic iterative step): By induction for $m\in\mathbb{N}^*$, $m\geq 2$, we use the values computed at the previous step and obtain for $i=\overline{0,n}$, the values:

$$\overline{x_m(t_i)}=g(t_i)+\frac{(b-a)}{2n}\cdot\sum_{j=1}^{n}[H(t_i,t_{j-1})\cdot f(t_{j-1},\overline{x_{m-1}}(t_{j-1}))+H(t_i,t_j)\cdot f(t_j,\overline{x_{m-1}}(t_j))].\tag{4.20}$$

This algorithm has the following practical stopping criterion:

Steps 3-4 (a condition of "do-while" type): If

$$D(\overline{x}_m(t_i),\overline{x}_{m-1}(t_i))<\varepsilon',\quad\text{for all } i=\overline{0,n}$$

then we stop to this "m" and retain its value ($m=$...), and the values $\overline{x}_m(t_i)$, $i=\overline{0,n}$, computed at this last iterative step. This condition is active after *Step 2*.

Step 5: Print "m" and print $\overline{x}_m(t_i)$, $i=\overline{0,n}$. STOP.

The convergence analysis

In order to investigate the numerical stability of the computed values with respect to small perturbations in the first iteration we consider another first term $y_0\in C\left([a,b],E^1\right)$ such that $\exists\varepsilon>0$ for which $D(x_0(t),y_0(t))<\varepsilon$, $\forall t\in[a,b]$. Suppose that there exist $M_y,\beta'\geq 0$ with $D(y_0(t),y_0(t'))\leq\beta'|t-t'|$ for all $t,t'\in[a,b]$ and $D\left(y_0(t),\widetilde{0}\right)\leq M_y$, $\forall t\in[a,b]$. The obtained sequence of successive approximations is:

$$y_m(t)=v(t)+(FR)\int_a^b H(t,s)\cdot f(s,y_{m-1}(s))\,ds,\quad\forall t\in[a,b],\ \forall m\in\mathbb{N}^*\tag{4.21}$$

and applying the same algorithm, the computed values are: $\overline{y_0}(t_i)=y_0(t_i)$, $i=\overline{0,n}$ and

$$\overline{y_m}(t_i)=v(t_i)+\frac{(b-a)}{2n}\cdot$$

$$\cdot\sum_{j=1}^{n}[H(t_i,t_{j-1})\cdot f(t_{j-1},\overline{y_{m-1}}(t_{j-1}))+H(t_i,t_j)\cdot f(t_j,\overline{y_{m-1}}(t_j))],\quad\forall i=\overline{0,n},\ m\in\mathbb{N}^*.\tag{4.22}$$

Definition 140 *(see [80])We say that the algorithm of successive approximations applied to the integral equation (4.12) is numerically stable with respect to the choice of the first iteration iff there exist a natural number $k \in \mathbb{N}^*$ and two constants $K_1, K_2 > 0$ which are independent by $h = \frac{b-a}{n}$, such that*

$$D\left(\overline{x}_m\left(t_i\right), \overline{y_m}\left(t_i\right)\right) < K_1\varepsilon + K_2 h^k, \quad \forall i = \overline{0,n},\ m \in \mathbb{N}^*. \tag{4.23}$$

Theorem 141 *(see [80])Under the conditions (i)-(vi), the solution of the integral equation (4.12) is approximated on the knots $t_i = a + \frac{i\cdot(b-a)}{n}$, $i = \overline{0,n}$, by the sequence $\left(\overline{x}_m\left(t_i\right)\right)_{m\in\mathbb{N}}$, $i = \overline{0,n}$, with the following apriori error estimate:*

$$D\left(x^*\left(t_i\right), \overline{x}_m\left(t_i\right)\right) \le \frac{\left[\alpha M_H\left(b-a\right)\right]^m}{1-\alpha M_H\left(b-a\right)} \cdot M_0 M_H\left(b-a\right) + \frac{L\left(b-a\right)^2}{4n\left(1-\alpha M_H\left(b-a\right)\right)} \tag{4.24}$$

for all $i = \overline{0,n}$, $m \in \mathbb{N}^*$ and the algorithm (4.19)-(4.20) is convergent and numerically stable with respect to the choice of the first iteration.

Sketch of proof: Using the estimate (4.14) it obtains

$$D\left(x^*\left(t_i\right), x_m\left(t_i\right)\right) \le \frac{\left[\alpha M_H\left(b-a\right)\right]^m}{1-\alpha M_H\left(b-a\right)} \cdot M_0 M_H\left(b-a\right), \quad \forall i = \overline{0,n},\ m \in \mathbb{N}^*,$$

and by induction, for $m \in \mathbb{N}^*$, $m \ge 3$, it follows that

$$D\left(x_m\left(t_i\right), \overline{x}_m\left(t_i\right)\right) \le \left[1 + \alpha\left(b-a\right)M_H + \ldots + \left(\alpha\left(b-a\right)M_H\right)^{m-1}\right] \cdot \frac{L\left(b-a\right)^2}{4n} =$$

$$= \frac{1-\left(\alpha\left(b-a\right)M_H\right)^m}{1-\alpha\left(b-a\right)M_H} \cdot \frac{L\left(b-a\right)^2}{4n} \le \frac{L\left(b-a\right)^2}{4n\left[1-\alpha\left(b-a\right)M_H\right]}, \quad \forall i = \overline{0,n},\ m \in \mathbb{N}^* \tag{4.25}$$

and consequently, the estimate (4.24) holds. In inductive manner we get

$$D\left(x_m\left(t\right), y_m\left(t\right)\right) \le \alpha M_H \int_a^b D\left(x_{m-1}\left(s\right), y_{m-1}\left(s\right)\right) ds < \left[\alpha M_H\left(b-a\right)\right]^m \varepsilon < \varepsilon$$

for all $t \in [a,b]$ and $m \in \mathbb{N}^*$. Then,

$$D\left(\overline{x}_m\left(t_i\right), \overline{y}_m\left(t_i\right)\right) \le$$

$$\le D\left(x_m\left(t_i\right), y_m\left(t_i\right)\right) + \frac{L\left(b-a\right)^2}{4n\left(1-\alpha M_H\left(b-a\right)\right)} + \frac{L'\left(b-a\right)^2}{4n\left(1-\alpha M_H\left(b-a\right)\right)} <$$

$$< \varepsilon + \frac{\left(L+L'\right)\left(b-a\right)}{4\left[1-\alpha M_H\left(b-a\right)\right]} \cdot \frac{b-a}{n} = K_1\varepsilon + K_2 h, \quad \forall i = \overline{0,n},\ \forall m \in \mathbb{N}^* \tag{4.26}$$

where

$$K_1 = 1, \quad K_2 = \frac{\left(L+L'\right)\left(b-a\right)}{4\left[1-\alpha M_H\left(b-a\right)\right]},$$

$$L' = M_H\left(\gamma + \alpha\left(\beta + \mu\left(b-a\right)M'\right)\right) + M'\delta, \quad M' = \alpha R' + M_0'$$

$$R' = \frac{M_0' M_H\left(b-a\right)}{1-\alpha M_H\left(b-a\right)} + M_y$$

and $M_0' \ge 0$ is such that

$$D\left(f\left(s, y_0\left(s\right)\right), \widetilde{0}\right) \le M_0', \quad \text{for all } s \in [a,b].$$

Remark 142 *(see [80])Combining the "a posteriori" error estimate (4.15) and the error estimate (4.24) we obtain a practical stopping criterion of the iterative algorithm, which can be stated as follows: For given $\varepsilon' > 0$ and given $n \in \mathbb{N}^*$ (previously chosen) it is determined the first natural number $m \in \mathbb{N}^*$ for which*

$$D\left(\overline{x}_m\left(t_i\right), \overline{x}_{m-1}\left(t_i\right)\right) < \varepsilon', \quad \forall i = \overline{0, n}$$

and we stop to this m retaining the approximations $\overline{x}_m\left(t_i\right)$, $i = \overline{0, n}$ of the solution. We can provide a short proof of this criterion as follows:

$$D\left(x^*\left(t_i\right), \overline{x}_m\left(t_i\right)\right) \leq D\left(x^*\left(t_i\right), x_m\left(t_i\right)\right) + D\left(x_m\left(t_i\right), \overline{x}_m\left(t_i\right)\right) \leq$$

$$\leq \frac{\alpha M_H\left(b-a\right)}{1-\alpha M_H\left(b-a\right)} \cdot D\left(x_m\left(t_i\right), x_{m-1}\left(t_i\right)\right) + \frac{L\left(b-a\right)^2}{4n\left(1-\alpha M_H\left(b-a\right)\right)}$$

and

$$D\left(x_m\left(t_i\right), x_{m-1}\left(t_i\right)\right) \leq D\left(x_m\left(t_i\right), \overline{x}_m\left(t_i\right)\right) + D\left(\overline{x}_m\left(t_i\right), \overline{x}_{m-1}\left(t_i\right)\right) +$$

$$+D\left(\overline{x}_{m-1}\left(t_i\right), x_{m-1}\left(t_i\right)\right) \leq \frac{L\left(b-a\right)^2}{2n\left(1-\alpha M_H\left(b-a\right)\right)} + D\left(\overline{x}_m\left(t_i\right), \overline{x}_{m-1}\left(t_i\right)\right).$$

So,

$$D\left(x^*\left(t_i\right), \overline{x}_m\left(t_i\right)\right) \leq$$

$$\leq \frac{\alpha M_H\left(b-a\right)}{1-\alpha M_H\left(b-a\right)} \cdot D\left(\overline{x}_m\left(t_i\right), \overline{x}_{m-1}\left(t_i\right)\right) + \frac{1+\left(b-a\right)M_H}{\left(1-\alpha M_H\left(b-a\right)\right)^2} \cdot \frac{L\left(b-a\right)^2}{4n}$$

and therefore, in order to obtain that $D\left(x^*\left(t_i\right), \overline{x}_m\left(t_i\right)\right) < \varepsilon$ we require

$$\frac{\alpha M_H\left(b-a\right)}{1-\alpha M_H\left(b-a\right)} \cdot D\left(\overline{x}_m\left(t_i\right), \overline{x}_{m-1}\left(t_i\right)\right) < \frac{\varepsilon}{2} \quad \text{and} \quad \frac{1+\alpha M_H\left(b-a\right)}{\left(1-\alpha M_H\left(b-a\right)\right)^2} \cdot \frac{L\left(b-a\right)^2}{4n} < \frac{\varepsilon}{2}.$$

Then we choose the smallest natural number $n \in \mathbb{N}^*$ for which

$$\frac{1+\alpha M_H\left(b-a\right)}{\left(1-\alpha M_H\left(b-a\right)\right)^2} \cdot \frac{L\left(b-a\right)^2}{4n} < \frac{\varepsilon}{2}, \quad \text{that is} \quad n > \frac{L\left(b-a\right)^2\left(1+\alpha M_H\left(b-a\right)\right)}{2\varepsilon\left(1-\alpha M_H\left(b-a\right)\right)^2}.$$

Finally, we find the smallest natural number $m \in \mathbb{N}^*$ (this is the last iterative step to be made) for which

$$D\left(\overline{x}_m\left(t_i\right), \overline{x}_{m-1}\left(t_i\right)\right) < \frac{\varepsilon}{2} \cdot \frac{1-\alpha\tau M_H}{\alpha\tau M_H} = \varepsilon'$$

for all $i = \overline{0, n}$. With these, the inequality $D\left(\overline{x}_m\left(t_i\right), \overline{x}_{m-1}\left(t_i\right)\right) < \varepsilon'$, $\forall i = \overline{0, n}$, leads to $D\left(x^*\left(t_i\right), \overline{x}_m\left(t_i\right)\right) < \varepsilon$ for all $i = \overline{0, n}$, and the desired accuracy ε is obtained.

Remark 143 *In [68], the method of successive approximations is applied to the nonlinear fuzzy Fredholm integral equation*

$$x\left(t\right) = g\left(t\right) + \int_a^b F\left(t, s, x\left(s\right)\right) ds, \quad t \in [a, b].$$

Numerical experiments

Example 1: Consider the nonlinear integral equation

$$x\left(t\right) = g\left(t\right) + \int_0^1 H\left(t, s\right) \cdot e^{-s} \cdot \left(x\left(s\right)\right)^2 ds, \quad t \in [0, 1]$$

with

$$H(t,s)=\begin{cases}\frac{1}{6}\cdot s^2(1-t)^2(3t-s-2ts), & 0\le s\le t\le 1\\ \frac{1}{6}\cdot t^2(1-s)^2(3s-t-2ts), & 0\le t\le s\le 1\end{cases},$$

$$g(t)=(1-t)^2(3t+1)\cdot\widetilde{1}+t^2(2-t)\cdot\widetilde{e},\quad t\in[0,1],$$

where

$$\widetilde{1},\widetilde{e}\in E^1,\quad \widetilde{1}=[1-\frac{1-r}{10},1+\frac{1-r}{10}],\quad \widetilde{e}=[e-\frac{1-r}{10},e+\frac{1-r}{10}],\quad r\in[0,1]$$

and in the expression of the function $f:[0,1]\times E^1\to E^1$, $f(s,u)=e^{-s}\cdot u^2$ the power $u^2=u\cdot u$ is computed by $\left[u^2\right]^r=\left[\left(u_-^r\right)^2,\left(u_+^r\right)^2\right]$, $r\in[0,1]$, and the exponential e^{-s} is crisp. The exact solution is

$$x^*(t)=[(x^*(t))_-^r,(x^*(t))_+^r]=[e^{t-0.1(1-r)},e^{t+0.1(1-r)}],\quad t,r\in[0,1].$$

Applying the algorithm for $n=10$, $\varepsilon'=10^{-15}$ we get $m=7$ and the results $e_{i,-}^r=\left|x^*(t_i)_-^r-x_m(t_i)_-^r\right|$, $e_{i,+}^r=\left|x^*(t_i)_+^r-x_m(t_i)_+^r\right|$, $i=\overline{0,n}$, for $r\in\{\frac{1}{4},\frac{1}{2},\frac{3}{4}\}$ can be viewed in Table 1. For $r=1$, $\varepsilon'=10^{-20}$ and $n\in\{10,100,1000\}$ we test the speed of convergence and the results are in Table 2. We see that the accuracy is $O\left(10^{-7}\div10^{-8}\right)$, $O\left(10^{-11}\div10^{-12}\right)$, $O\left(10^{-15}\right)$, respectively, and the convergence of the algorithm is confirmed.

t_i	$e_{i,-}^{0.25}$	$e_{i,-}^{0.5}$	$e_{i,-}^{0.75}$	$e_{i,+}^{0.25}$	$e_{i,+}^{0.5}$	$e_{i,+}^{0.75}$
0	0.07774	0.05123	0.02531	7.2115e-002	4.8729e-002	2.4685e-002
0.1	0.07024	0.04617	0.02275	6.4026e-002	4.3401e-002	2.2052e-002
0.2	0.06207	0.04065	0.01996	5.5189e-002	3.7579e-002	1.9171e-002
0.3	0.05302	0.03456	0.01689	4.5408e-002	3.1135e-002	1.5979e-002
0.4	0.04294	0.02776	0.01345	3.4509e-002	2.3953e-002	1.2425e-002
0.5	0.03168	0.02016	0.00961	2.2336e-002	1.5935e-002	8.4582e-003
0.6	0.01909	0.01167	0.00532	8.7565e-003	6.9926e-003	4.0387e-003
0.7	0.00509	0.00221	0.00053	6.3453e-003	2.9482e-003	8.6895e-004
0.8	0.01045	0.00828	0.00478	2.3067e-002	1.3951e-002	6.2943e-003
0.9	0.02759	0.01986	0.01067	4.1492e-002	2.6069e-002	1.2262e-002
1	0.04641	0.03257	0.01711	6.1711e-002	3.9369e-002	1.8814e-002

Table 1. Numerical results for $n=10$, $r\in\{\frac{1}{4},\frac{1}{2},\frac{3}{4}\}$, in Example 1

t_i	e_i^1, $n=10$	e_i^1, $n=100$	e_i^1, $n=1000$
0	0	0	0
0.1	3.1395e-008	3.1430e-012	2.2204e-016
0.2	2.2822e-008	2.2853e-012	1.1102e-015
0.3	1.3369e-008	1.3385e-012	3.3307e-015
0.4	6.4810e-008	6.4888e-012	4.8850e-015
0.5	1.1912e-007	1.1925e-011	5.5511e-015
0.6	1.6389e-007	1.6408e-011	4.8850e-015
0.7	1.8675e-007	1.8697e-011	4.4409e-015
0.8	1.7531e-007	1.7551e-011	3.1086e-015
0.9	1.1718e-007	1.1732e-011	1.7764e-015
1	0	0	0

Table 2. The results for $n=10$, $n=100$, $n=1000$, $r=1$, in Ex. 1

Example 2: For the nonlinear integral equation

$$x(t) = g(t) + \int_0^1 \frac{ts}{3} \cdot [x(s)]^2 \, ds, \quad t \in [0,1]$$

the exact solution is $x^*(t) = \left[t - \frac{1}{6}(1-r), t + \frac{1}{6}(1-r)\right]$, $t, r \in [0,1]$. Here,

$$H(t,s) = \frac{ts}{3}, \quad g(t) = \left[\frac{11t}{12} - \frac{1}{6}(1-r), \frac{11t}{12} + \frac{1}{6}(1-r)\right], \quad t, s, r \in [0,1]$$

and the product of two fuzzy numbers (in the expression of the function $f(s,u) = u^2 = u \cdot u$) is defined as $\left[u^2\right]^r = \left[\left(u_-^r\right)^2, \left(u_+^r\right)^2\right]$. The algorithm is applied with $n = 10$, $\varepsilon' = 10^{-15}$, obtaining $m = 18$, and the results $e_{i,-}^r = \left|x^*(t_i)_-^r - x_m(t_i)_-^r\right|$, $i = \overline{0,n}$, for $r \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$ and $e_{i,+}^r = \left|x^*(t_i)_+^r - x_m(t_i)_+^r\right|$, $i = \overline{0,n}$, for $r \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$ are presented in Table 3. In order to test the convergence we consider $r = 1$, and $n = 10$, $n = 100$, $n = 1000$, respectively, obtaining the accuracy $O\left(10^{-3} \div 10^{-4}\right)$, $O\left(10^{-5} \div 10^{-6}\right)$, $O\left(10^{-8}\right)$, as can be viewed in Table 4.

t_i	$e_{i,-}^{0}$	$e_{i,-}^{0.25}$	$e_{i,-}^{0.5}$	$e_{i,-}^{0.75}$	$e_{i,+}^{0}$	$e_{i,+}^{0.25}$	$e_{i,+}^{0.5}$	$e_{i,+}^{0.75}$
0	0	0	0	0	0	0	0	0
0.1	0.00364	0.00284	0.00195	0.00097	0.00541	0.00392	0.00254	0.00127
0.2	0.00728	0.00568	0.00390	0.00194	0.01081	0.00783	0.00508	0.00254
0.3	0.01092	0.00852	0.00585	0.00292	0.01621	0.01175	0.00762	0.00381
0.4	0.01456	0.01135	0.00780	0.00388	0.02162	0.01567	0.01016	0.00508
0.5	0.01821	0.01419	0.00975	0.00486	0.02702	0.01958	0.01271	0.00635
0.6	0.02185	0.01704	0.01171	0.00583	0.03242	0.02351	0.01524	0.00762
0.7	0.02549	0.01988	0.01366	0.00681	0.03783	0.02742	0.01778	0.00889
0.8	0.02914	0.02272	0.01561	0.00778	0.04323	0.03134	0.02032	0.01016
0.9	0.03278	0.02560	0.01756	0.00875	0.04863	0.03525	0.02286	0.01143
1	0.03642	0.02840	0.01951	0.00972	0.05404	0.03917	0.02541	0.01269

Table 3. The results for $n = 10$ and $r \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$, in Example 2

t_i	e_i^1, $n = 10$	e_i^1, $n = 100$	e_i^1, $n = 1000$
0	0	0	0
0.1	1.0021e-04	1.000021e-06	9.99998520e-09
0.2	2.0042e-04	2.000042e-06	1.99999705e-08
0.3	3.0063e-04	3.000063e-06	2.99999557e-08
0.4	4.0084e-04	4.000084e-06	3.99999410e-08
0.5	5.0105e-04	5.000105e-06	4.99999263e-08
0.6	6.0126e-04	6.000126e-06	5.99999115e-08
0.7	7.0147e-04	7.000147e-06	6.99998967e-08
0.8	8.0168e-04	8.000168e-06	7.99998819e-08
0.9	9.0189e-04	9.000189e-06	8.99998673e-08
1	1.0021e-03	1.000021e-05	9.99998526e-08

Table 4. The results for $r = 1$ and $n = 10$, $n = 100$, $n = 1000$, in Ex. 2

4.3 Hammerstein-Volterra fuzzy integral equations with constant delay

We present here the results obtained in [75] concerning the fuzzy Volterra integral equation having a constant delay

$$x(t) = g(t) + (FR)\int_{t-\tau}^{t} H(t,s)\cdot f(s,x(s))\,ds, \qquad t\in[0,T] \tag{4.27}$$

with the initial condition

$$x(t) = \varphi(t), \qquad t\in[-\tau,0] \tag{4.28}$$

under the following conditions:

(i) $H \in C([0,T]\times[-\tau,T],\mathbb{R})$, $g\in C\left([0,T],E^1\right)$, $f\in C\left([-\tau,T]\times E^1,E^1\right)$, $\varphi\in C\left([-\tau,0],E^1\right)$ and $H(t,s)\geq 0$, $\forall (t,s)\in[0,T]\times[-\tau,T]$

(ii) there exist $\alpha,\gamma>0$ such that

$$D\left(f(s,u), f\left(s',v\right)\right)\leq \gamma\left|s-s'\right| + \alpha D(u,v), \forall s,s'\in[-\tau,T], \forall u,v\in E^1$$

(iii) there exist $\delta,\eta>0$ such that

$$\left|H(t,s)-H\left(t',s'\right)\right|\leq \delta\left|t-t'\right| + \eta\left|s-s'\right|,\ \forall (t,s),\left(t',s'\right)\in[0,T]\times[-\tau,T]$$

(iv) (the compatibility condition)

$$\varphi(0) = g(0) + \int_{-\tau}^{0} H(0,s)\cdot f(s,\varphi(s))\,ds \tag{4.29}$$

(v) there exist $\beta,\mu>0$ such that

$$D\left(g(s), g\left(s'\right)\right)\leq\beta\left|s-s'\right|, \quad D\left(\varphi(t),\varphi\left(t'\right)\right)\leq\mu\left|t-t'\right|,\ \forall s,s'\in[0,T],\ \forall t,t'\in[-\tau,0]$$

(vi) (the contraction condition): $\alpha\tau M_H<1$, where $M_H\geq 0$ is such that $0\leq H(t,s)\leq M_H$, $\forall (t,s)\in[0,T]\times[-\tau,T]$, where $\tau>0$, $T>0$ are such that $T=p\cdot\tau$ for given $p\in\mathbb{N}^*$ and g, φ, f are fuzzy-number-valued functions. The weight function $H:[0,T]\times[-\tau,T]\to\mathbb{R}$ is supposed to be continuous and positive.

Defining the set

$$C_\varphi\left([-\tau,T],E^1\right) = \left\{x\in C\left([-\tau,T],E^1\right) : x(t)=\varphi(t), \forall t\in[-\tau,0]\right\}$$

it is easy to see that $C_\varphi\left([-\tau,T],E^1\right)$ is a bounded set in the complete metric space $C\left([-\tau,T],E^1\right)$ and therefore $\left(C_\varphi\left([-\tau,T],E^1\right), D^*\right)$ is a complete metric space too, with the metric $D^*(u,v) = \max_{s\in[-\tau,T]} D(u(s),v(s))$.

Define the operator $A: C_\varphi\left([-\tau,T],E^1\right)\to\left\{x \mid x:[-\tau,T]\to E^1\right\}$ by

$$A(x)(t) = \begin{cases} g(t) + (FR)\displaystyle\int_{t-\tau}^{t} H(t,s)\cdot f(s,x(s))\,ds, & t\in[0,T] \\ \varphi(t), & t\in[-\tau,0]\end{cases}$$

and the sequence of successive approximations $(x_m)_{m\in\mathbb{N}}\subset C_\varphi\left([-\tau,T],E^1\right)$ given by

$$x_0(t) = \varphi(t), \forall t\in[-\tau,0], \quad x_0(t)=\varphi(0), \forall t\in[0,T],$$

$$x_m(t) = g(t) + (FR)\int_{t-\tau}^{t} H(t,s)\cdot f(s, x_{m-1}(s))\,ds, \quad \forall t\in[0,T],\ \forall m\in\mathbb{N}^*. \tag{4.30}$$

Consider the sequence of functions $F_m : [0,T]\times[-\tau,T]\to E^1$, $m\in\mathbb{N}$, given by

$$F_m(t,s) = H(t,s)\cdot f(s, x_m(s)), \quad \forall (t,s)\in[0,T]\times[-\tau,T].$$

Theorem 144 *(see [75])Under the conditions (i)-(iv) and (vi) the initial value problem (4.27) has unique solution $x^*\in X$, $\lim\limits_{m\to\infty} D^*(x_m, x^*) = 0$, and the following error estimates hold:*

$$D(x_m(t), x^*(t)) \le \frac{(\alpha\tau M_H)^m}{1-\alpha\tau M_H}\cdot[M_H M_0\tau + M_g + M_\varphi], \quad \forall t\in[0,T], \tag{4.31}$$

$$D(x_m(t), x^*(t)) \le \frac{\alpha\tau M_H}{1-\alpha\tau M_H}\cdot D(x_m(t), x_{m-1}(t)), \quad \forall t\in[0,T] \tag{4.32}$$

for all $m\in\mathbb{N}^*$. Moreover, if all the conditions (i)-(vi) are satisfied, then the sequences $(x_m)_{m\in\mathbb{N}}\subset C_\varphi\left([-\tau,T],E^1\right)$ and $(F_m)_{m\in\mathbb{N}}\subset C\left([0,T]\times[-\tau,T],E^1\right)$ are uniformly bounded and uniform Lipschitz.

The proof of this theorem is similar to the proof of Theorem 138, obtaining

$$D\left(x_m(t),\widetilde{0}\right) \le D(x_m(t), x_0(t)) + D\left(x_0(t),\widetilde{0}\right) \le \frac{M_g + M_\varphi + M_H M_0\tau}{1-\alpha M_H\tau} + M_\varphi \overset{notation}{=} R$$

for all $t\in[0,T]$ and $m\in\mathbb{N}^*$, where $M_g, M_0\ge 0$ are such that $D\left(g(t),\widetilde{0}\right)\le M_g$, $\forall t\in[0,T]$ and $D\left(f(s,x_0(s)),\widetilde{0}\right)\le M_0$, $\forall s\in[-\tau,T]$, and $M_\varphi\ge 0$ is such that $D\left(\varphi(t),\widetilde{0}\right)\le M_\varphi$ for all $t\in[-\tau,0]$. Moreover,

$$D\left(f(t,x_m(t)),\widetilde{0}\right) \le \max\left(\frac{\alpha(M_g+M_\varphi+M_H M_0\tau)}{1-\alpha M_H\tau} + M_0, M_\varphi\right) \overset{notation}{=} M$$

for all $t\in[-\tau,T], m\in\mathbb{N}^*$ and

$$D\left(F_m(t,s),\widetilde{0}\right) = D\left(H(t,s)\cdot f(s,x_m(s)),\widetilde{0}\right) \le |H(t,s)|\cdot D\left(f(s,x_m(s)),\widetilde{0}\right) \le M_H M$$

for all $(t,s)\in[0,T]\times[-\tau,T]$ and $m\in\mathbb{N}^*$.
The Lipschitz properties are

$$D\left(x_m(t), x_m(t')\right) \le L_0\left|t-t'\right|, \quad \forall t\in[-\tau,T],\ \forall m\in\mathbb{N},$$

where $L_0 = \max(\mu, \beta+\delta M\tau + 2M_H M)$. Finally,

$$D\left(f(t,x_m(t)), f\left(t', x_m(t')\right)\right) \le (\gamma+\alpha L_0)\cdot\left|t-t'\right|, \quad \forall t\in[-\tau,T],\ \forall m\in\mathbb{N}$$

and

$$D\left(F_m(t,s), F_m\left(t',s'\right)\right) \le \delta M\left|t-t'\right| + [\eta M + M_H(\gamma+\alpha L_0)]\cdot\left|s-s'\right|$$

for all $(t,s), (t',s')\in[0,T]\times[-\tau,T]$.

Remark 145 *Denoting $L = \eta M + M_H(\gamma+\alpha L_0)$ we have $D(F_m(t,s), F_m(t,s'))\le L\cdot|s-s'|$, $\forall s,s'\in[-\tau,T]$, $\forall t\in[0,T]$, $\forall m\in\mathbb{N}$. Under the conditions (i)-(vi) the solution of (4.27) is bounded and Lipschitzian.*

The algorithm

From Theorem 144 it follows that the solution of the initial value problem (4.27) is approximated on $[0,T]$ by the terms of the sequence of successive approximations given in (4.30). In order to compute the integrals from (4.30) we consider the uniform partition of $[-\tau,T]$

$$\Delta : -\tau = t_0 < t_1 < ... < t_n = 0 < t_{n+1} < ... < t_q = T$$

with $q = (p+1)\,n$, $t_i = t_{i-1} + \frac{\tau}{n} = -\tau + \frac{i\cdot\tau}{n}$, $i = \overline{1,q}$ and apply the trapezoidal quadrature rule. On these knots the terms of the sequence of successive approximations are:

$$x_m(t_i) = \varphi(t_i)\,, \quad i = \overline{0,n},\ m \in \mathbb{N}, \quad x_0(t_i) = \varphi(0)\,,\ i = \overline{n+1,q}$$

$$x_m(t_i) = g(t_i) + (FR)\int_{t_i-\tau}^{t_i} H(t_i,s)\cdot f(s, x_{m-1}(s))\,ds, \quad i = \overline{n+1,q},\ m \in \mathbb{N}^* \tag{4.33}$$

and applying the trapezoidal quadrature rule to the integrals (4.33) we obtain the following algorithm that gives the effective computed values:

STEP 0: There are introduced the data τ, T, n, p, q, ε' and the functions φ, g, H, f.

STEP 1: For any $i = \overline{0,n}$ and for all $m \in \mathbb{N}$, we set

$$\overline{x}_m(t_i) = x_m(t_i) = \varphi(t_i)\,.$$

STEP 2: For $m = 0$ and for all $i = \overline{n+1,q}$ consider

$$\overline{x}_0(t_i) = x_0(t_i) = \varphi(0)\,.$$

STEP 3 (the first iterative step): For $m = 1$ and for all $i = \overline{n+1,q}$ compute

$$\overline{x}_1(t_i) = g(t_i) + \frac{\tau}{2n}\cdot\sum_{j=0}^{n-1}[H(t_i,t_{i+j-n})\cdot f(t_{i+j-n},\overline{x}_0(t_{i+j-n})) +$$

$$+H(t_i,t_{i+j-n+1})\cdot f(t_{i+j-n+1},\overline{x}_0(t_{i+j-n+1}))]. \tag{4.34}$$

STEP 4-5 (the generic iterative step): By induction for $m \in \mathbb{N}^*$, $m \geq 2$ we use the values computed at the previous step and obtain for $i = \overline{n+1,q}$, the values:

$$\overline{x}_m(t_i) = g(t_i) + \frac{\tau}{2n}\cdot$$

$$\cdot\sum_{j=0}^{n-1}[H(t_i,t_{i+j-n})\cdot f(t_{i+j-n},\overline{x}_{m-1}(t_{i+j-n}))+H(t_i,t_{i+j-n+1})\cdot f(t_{i+j-n+1},\overline{x}_{m-1}(t_{i+j-n+1}))]. \tag{4.35}$$

This algorithm has the following practical stopping criterion::

STEP 4-5 (a condition of "do-while" type): If

$$D(\overline{x}_m(t_i),\overline{x}_{m-1}(t_i)) < \varepsilon', \quad \text{for all } i = \overline{n+1,q}$$

then we stop to this "m" and retain its value ($m =$...), and the values $\overline{x}_m(t_i)\,,\ i = \overline{0,q}$, computed at this last iterative step. This condition is active after STEP 3.

STEP 6: Print "m" and print $\overline{x}_m(t_i)\,,\ i = \overline{0,q}$. STOP.

The convergence analysis

In order to investigate the numerical stability of the computed values with respect to small perturbations in the initial condition, we consider the same integral equation as in

(4.27) with the initial condition given by another function $\overline{\varphi} : [-\tau, 0] \to E^1$ such that $\exists \varepsilon > 0$ for which $D(\varphi(t), \overline{\varphi}(t)) < \varepsilon$ for all $t \in [-\tau, 0]$. The corresponding initial value problem is

$$\begin{cases} x(t) = g(t) + (FR) \int_{t-\tau}^{t} H(t,s) \cdot f(s, x(s))\, ds, \qquad t \in [0,T] \\ x(t) = \overline{\varphi}(t), \qquad t \in [-\tau, 0] \end{cases}$$

and using modified notations, the obtained sequence of successive approximations is

$$y_m(t) = \overline{\varphi}(t), \quad \forall t \in [-\tau, 0],\ \forall m \in \mathbb{N}, \quad y_0(t) = \overline{\varphi}(0), \quad \forall t \in [0,T]$$

$$y_m(t) = g(t) + (FR) \int_{t-\tau}^{t} H(t,s) \cdot f(s, y_{m-1}(s))\, ds, \quad \forall t \in [0,T],\ \forall m \in \mathbb{N}^*.$$

Similarly as in (4.35), the effective computed values are:

$$\overline{y}_m(t_i) = \overline{\varphi}(t_i), \quad i = \overline{0,n},\ m \in \mathbb{N}, \quad \overline{y}_0(t_i) = y_0(t_i) = \overline{\varphi}(0),\ i = \overline{n+1,q}$$

$$\overline{y}_m(t_i) = g(t_i) + \frac{\tau}{2n} \cdot \sum_{j=0}^{n-1} [H(t_i, t_{i+j-n}) \cdot f(t_{i+j-n}, \overline{y}_{m-1}(t_{i+j-n})) +$$

$$+ H(t_i, t_{i+j-n+1}) \cdot f(t_{i+j-n+1}, \overline{y}_{m-1}(t_{i+j-n+1}))], \quad i = \overline{n+1,q},\ \forall m \in \mathbb{N}^*$$

and because

$$y_0(t) = \begin{cases} \overline{\varphi}(t), & t \in [-\tau, 0] \\ \overline{\varphi}(0), & t \in [0,T] \end{cases}, \quad x_0(t) = \begin{cases} \varphi(t), & t \in [-\tau, 0] \\ \varphi(0), & t \in [0,T] \end{cases},$$

from $D(\varphi(t), \overline{\varphi}(t)) < \varepsilon$, we infer that

$$D(x_0(t), y_0(t)) < \varepsilon, \quad \text{for all } t \in [-\tau, T].$$

Definition 146 *(see [75])We say that the above presented method and its algorithm are numerically stable with respect to the initial condition iff there exist a natural number $k \in \mathbb{N}^*$ and two constants K_1, $K_2 > 0$ which are independent by $h = \frac{\tau}{n}$, such that*

$$D(\overline{x}_m(t_i), \overline{y}_m(t_i)) < K_1 \varepsilon + K_2 h^k, \quad \forall i = \overline{n+1,q},\ \forall m \in \mathbb{N}^*.$$

Theorem 147 *(see [75])Suppose that the conditions (i)-(vi) are satisfied. Then, the solution of the initial value problem (4.27) is approximated on the knots $t_i = -\tau + \frac{i \cdot \tau}{n}$, $i = \overline{0,q}$, by the sequence $(\overline{x}_m(t_i))_{m \in \mathbb{N}}$, $i = \overline{0,q}$, with the following apriori error estimate:*

$$D(x^*(t_i), \overline{x}_m(t_i)) \leq \frac{(\alpha \tau M_H)^m}{1 - \alpha \tau M_H} \cdot [M_H M_0 \tau + M_g + M_\varphi] + \frac{L \tau^2}{4n(1 - \alpha \tau M_H)} \tag{4.36}$$

for all $i = \overline{n+1,q}$, $m \in \mathbb{N}^*$ and the method and its algorithm are numerically stable with respect to the initial condition.

The proof is analogous to those of Theorem 141.

Example: The initial value problem

$$x(t)=\begin{cases} e^{t+1}-\tau-t\tau+\frac{\tau^2}{2}+\int\limits_{t-\tau}^{t}\ln x(s)\,ds, & t\in[0,1] \\ \varphi(t), & t\in[-\tau,0] \end{cases}$$

has $\varphi:[-0.5,0]\to E^1$ defined by its level sets

$$[\varphi(t)_-^r,\varphi(t)_+^r]=[e^{t+1}-0.3\left(1-r^2\right),e^{t+1}+0.3\left(1-r^2\right)],\quad r\in[0,1],\ t\in[-0.5,0].$$

and the exact solution $x^*:[-0.5.1]\to E^1$ is given by

$$[x^*(t)_-^r,x^*(t)_+^r]=[e^{t+1}-0.3\left(1-r^2\right),e^{t+1}+0.3\left(1-r^2\right)],\quad r\in[0,1],\ t\in[-0.5,1].$$

Here, $\tau=0.5$, $T=1$ and applying the algorithm for $n=10$, $p=2$, $q=30$, and $\varepsilon'=10^{-15}$, we obtain the number of iterations $m=8$. The fuzzy logarithm is defined for $u\in E^1$ with $u_-^0>0$ being constructed by using the monotonicity of ln, and has the level sets $[(\ln u)_-^r,(\ln u)_+^r]=[\ln\left(u_-^r\right),\ln\left(u_+^r\right)]$ for all $r\in[0,1]$. The obtained numerical results are presented in Tables 1 and 2. In Table 1 can be observed the errors

$$\begin{aligned}\left|e_i^1\right| &= \left|x^*(2t_i)^1-\overline{x}_m(2t_i)^1\right|, \quad i=\overline{n,q-n}\\ e_i &= x^*(2t_i)_-^r-\overline{x}_m(2t_i)_-^r, \quad i=\overline{n,q-n}\end{aligned}$$

for $n=10$ and $r=0$, 0.5, and 0.75. In Table 2 the errors

$$e_i=x^*(2t_i)_+^r-\overline{x}_m(2t_i)_+^r,\ i=\overline{n,q-n}$$

for $n=10$ and $r=0$, 0.5, and 0.75 are included.

t_i	$\left\|e_i^1\right\|$	e_i, for $r=0$	e_i, for $r=0.5$	e_i, for $r=0.75$
0	0	0	0	0
0.1	$O\left(10^{-17}\right)$	0.07165518455	0,0527092953	0.03003775522
0.2	2.6645352591E-15	0.06705683813	0,0493583521	0.02814947721
0.3	3.3306690738E-14	0.06281642385	0,0462585290	0.02639642846
0.4	2.1582735598E-13	0.05891364733	0,0433974014	0.02477304957
0.5	9.1038288019E-13	0.05532780143	0,0407618412	0.02327316723
0.6	2.8874680424E-12	0.04987670285	0,0367999655	0.02104750266
0.7	7.4136252692E-12	0.04433245288	0,0327681801	0.01878122136
0.8	1.6193268947E-11	0.03940886445	0,0291766631	0.01675511820
0.9	3.1027624913E-11	0.03503098968	0,0259746568	0.01494304394
1	5.3232973584E-11	0.03113367960	0,0231175053	0.01332165971

Table 1. The accuracy on the level sets from the left

t_i	e_i, for $r = 0$	e_i, for $r = 0.5$	e_i, for $r = 0.75$
0	0	0	0
0.1	-0.0623333993	-0.0474896811	-0.02826844940
0.2	-0.0585908475	-0.0446171884	-0.02654214205
0.3	-0.0550711571	-0.0419208883	-0.02492586437
0.4	-0.0517721066	-0.0393981964	-0.02341731427
0.5	-0.0486893885	-0.0370449774	-0.02201331330
0.6	-0.0443299781	-0.0336929087	-0.01999393990
0.7	-0.0398829499	-0.0302742666	-0.01793514708
0.8	-0.0358521404	-0.0271821284	-0.01607817095
0.9	-0.0322025222	-0.0243877689	-0.01440424006
1	-0.0289009238	-0.0218642473	-0.01289596666

Table 2. The accuracy on the level sets from the right

Chapter 5

New directions of research

5.1 Extending the QOA for parametric curves and surfaces

The QOA was investigated up now for plane curves given by explicit equation. A new direction in the developing these ideas and to extend the concept of QOA is related to parametric curves both in plane and in the Euclidean space with arbitrary finite dimension. A possibility to extend the QOA for parametric curves is presented in the following:

Consider the points $P_0, P_1, ..., P_{n-1}, P_n \in \mathbb{R}^q$, $q \geq 2$ and a partition of the interval $[a, b]$

$$\Delta : t_0 < t_1 < ... < t_{n-1} < t_n = b.$$

The polygonal line L in $\mathbb{R}^q$ joining these points can be defined by parametric equations, $L = (L_1, ..., L_q) : [a, b] \to \mathbb{R}^q$,

$$L(t) = \frac{t_i - t}{t_i - t_{i-1}} \cdot P_{i-1} + \frac{t - t_{i-1}}{t_i - t_{i-1}} \cdot P_i, \quad t \in [t_{i-1}, t_i], \quad i = \overline{1, n}.$$

Let $f = (f_1, ..., f_q) : [a, b] \to \mathbb{R}^q$ be a continuous function with the interpolation property, $f(t_i) = P_i$, $i = \overline{0, n}$, having the parametric representation:

$$\begin{cases} x_1 = f_1(t) \\ \quad .... \\ x_q = f_q(t) \end{cases}, t \in [a, b].$$

Denoting

$$C(([a, b], \mathbb{R}^q), \Delta, P) = \{f \in C([a, b], \mathbb{R}^q) : f(t_i) = P_i, \ \forall i = \overline{0, n}\}$$

we define the vectorial quadratic oscillation on average as a function $\rho = (\rho_1, ..., \rho_q) : C(([a, b], \mathbb{R}^q), \Delta, P) \to \mathbb{R}^q_+$, with the components

$$\rho_j(f) = \sqrt{\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} [f_j(t) - L_j(t)]^2 dt}, \quad j = \overline{1, q}.$$

The total oscillation of the vectorial quadratic oscillation, $\rho(f) = (\rho_1(f), ..., \rho_q(f))$, is the positive number (the Euclidean norm of $\rho(f)$),

$$T(\rho(f)) = \sqrt{[\rho_1(f)]^2 + ... + [\rho_q(f)]^2}.$$

It is easy to see that $\rho(f) = 0 = (0, ..., 0) \Longleftrightarrow f = L$ and the homogeneity property holds for $\left(\rho_1(f), ..., \rho_q(f)\right)$, too. Since the Hermite cubic spline interpolating in $\mathbb{R}^q$ the points $P_0, P_1, ..., P_{n-1}, P_n$ can be represented by parametric equations, the minimization of the vectorial quadratic oscillation ρ is realized component-wise using the same manner as in the previous section for each component ρ_j, $j = \overline{1, q}$. An interesting case is for $q = 2$, according to the connection between the quadratic oscillation in average introduced for explicit functions (curves) in Definition 43 and the vectorial quadratic oscillation defined above for parametric curves in plane.

Remark 148 *Consider the partition*

$$\Delta : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$$

of the interval $[a, b]$. *In the case* $q = 2$, *a curve in plane given by the explicit equation* $y = f(x)$, $x \in [a, b]$ *could have the parametric equations*

$$\begin{cases} x = f_1(t) = t \\ y = f_2(t) = f(t), \quad t \in [x_{i-1}, x_i],\ i = \overline{1, n} \end{cases}$$

denoting $x = t$. *Then, the same partition can be written as*

$$\Delta : t_0 < t_1 < ... < t_{n-1} < t_n = b$$

and the polygonal line joining the points $P_0(x_0, y_0), ..., P_n(x_n, y_n) \in \mathbb{R}^2$, $L = (L_1, L_2) : [a, b] \to \mathbb{R}^2$ *has the parametric equations*

$$\begin{cases} x = L_1(t) = \frac{t_i - t}{t_i - t_{i-1}} \cdot x_{i-1} + \frac{t - t_{i-1}}{t_i - t_{i-1}} \cdot x_i, \quad t \in [t_{i-1}, t_i] \\ y = L_2(t) = \frac{t_i - t}{t_i - t_{i-1}} \cdot y_{i-1} + \frac{t - t_{i-1}}{t_i - t_{i-1}} \cdot y_i, \quad t \in [t_{i-1}, t_i] \end{cases}, \ i = \overline{1, n}.$$

Observing that

$$\frac{t_i - t}{t_i - t_{i-1}} \cdot x_{i-1} + \frac{t - t_{i-1}}{t_i - t_{i-1}} \cdot x_i = \frac{t(x_i - x_{i-1}) + t_i x_{i-1} - t_{i-1} x_i}{t_i - t_{i-1}} =$$

$$= \frac{t(t_i - t_{i-1}) + t_i t_{i-1} - t_{i-1} t_i}{t_i - t_{i-1}} = t$$

we infer that the vectorial quadratic oscillation of $f = (f_1, f_2)$ *is* $\rho(f) = (\rho_1(f), \rho_2(f))$ *with* $\rho_2(f) = 0$, *namely* $(\rho_1(f), 0)$. *Since* $\rho_1(f)$ *coincides (as value) with the common quadratic oscillation in average of the function* $y = f(x)$, *namely the number* $\rho(f)$ *as in Definition 43, we can identify the vector with null second projection* $(\rho_1(f), 0)$ *to the number* $\rho(f)$. *So, in this case, the vectorial and scalar quadratic oscillations in average could be considered as equivalent. Moreover, in this case,* $T(\rho(f)) = \sqrt{[\rho_1(f)]^2 + 0} = \rho_1(f)$, *which is an argument for the above presented way of introducing the vectorial quadratic oscillation of parametric curves.*

The ideas presented above will be developed for periodic parametric curves in plane (such as cardinal, chordal, centripetal splines, and others, in a way different by [132]) and for parametric curves in the space $\mathbb{R}^q$, $q \geq 3$.

A real challenge will be the intention to extend the concept of quadratic oscillation in average from curves to parametrized surfaces, and for the future this is a direction where our attention will be focused. Such approach could have applications in the industrial design of aerodynamic profiles.

5.2 Extending the method of successive interpolations

5.2.1 The method of successive interpolations for functional differential equations with variable delay and advance

Until now, the method of successive interpolations was developed for functional differential equations of the type

$$x^{(p)}(t) = f(t, x(t), x(\varphi(t))), \quad t \in [a, b]$$

with functional argument satisfying $a \le \varphi(t) \le b$, $\forall t \in [a, b]$, or with vanishing deviating argument. A future research will focus our attention for the following type of initial value problems:

$$\begin{cases} x^{(p)}(t) = f(t, x(t), x(\varphi(t))), \quad t \in [a, b] \\ x(t) = \mu(t), ..., x^{(p-1)}(t) = \mu^{(p-1)}(t), \quad t \in [c, a] \end{cases}$$

with known function $\mu \in C^{p-1}[c, a]$ and variable functional argument $c = \min_{t \in [a,b]} \varphi(t) \le \varphi(t) \le b$, $t \in [a, b]$, including the case of affine argument. Such problems was usually approached by Runge-Kutta procedures or by collocation techniques (see [93]) and we will extend the method of successive interpolations.

In another direction of future research will be extended the method of successive interpolations for boundary value problems of the type

$$\begin{cases} x''(t) = f(t, x(t), x(\varphi(t))), \quad t \in [a, b] \\ x(t) = \mu(t), \quad t \in [c, a], \quad x(b) = \alpha \end{cases}$$

as can be found in [27], [101], and [107], or of the type

$$\begin{cases} x''(t) = f(t, x(t), x(g(t))), \quad t \in [a, b] \\ x(t) = \varphi(t), \quad t \in [c, a], \quad x(t) = \psi(t), \quad t \in [b, d] \end{cases}$$

with given functions $\varphi, \mu \in C[c, a]$, $\psi \in C[b, d]$ (see [204]). Such problems were early approached by finite differences, shooting techniques, spline functions methods, Galerkin and collocation methods.

The method of successive interpolations applied to the functional differential equation of fourth order (the generalized beam equation) with two-point boundary conditions as in (3.69) could be extended to problems with retardation and anticipation:

$$\begin{cases} x^{IV}(t) = f(t, x(t), x(\varphi(t))), \quad t \in [a, b] \\ x(t) = \varphi(t), \quad t \in [c, a], \quad x(t) = \psi(t), \quad t \in [b, d] \\ x'(a) = w, \qquad x'(b) = r \end{cases}$$

which were investigated with other iterative methods in [225] and [224], or to sixth order functional differential equations with two-point boundary conditions:

$$\begin{cases} x^{VI}(t) = f(t, x(t), x(\varphi(t))), \quad t \in [a, b] \\ x(a) = \alpha_0, \quad x''(a) = \alpha_2, \quad x^{iv}(a) = \alpha_4 \\ x(b) = \beta_0, \quad x''(b) = \beta_2, \quad x^{iv}(b) = \beta_4 \end{cases} \tag{5.1}$$

or

$$\begin{cases} x^{VI}(t) = f(t, x(t), x(\varphi(t))), \quad t \in [a, b] \\ x(a) = \alpha_0, \quad x'(a) = \alpha_1, \quad x''(a) = \alpha_2 \\ x(b) = \beta_0, \quad x'(b) = \beta_1, \quad x''(b) = \beta_2 \end{cases} \tag{5.2}$$

written in equivalent integral form and using a corresponding Green function. The ODE variant of (5.1) (without deviating argument) was investigated by Adomian decomposition in [10], by homotopy perturbation in [192], and by the use of septic splines in [161]. The ODE variant of (5.2) was approached by the collocation method in [143].

5.2.2 The method of successive interpolations for the neutral type pantograph differential equation

The method of successive interpolations will be extended to neutral type pantograph differential equations of first order:

$$\begin{cases} x'(t) = f(t, x(t), x(pt), x'(pt)), \quad t \in [0,a],\ 0 < p < 1 \\ x(0) = x_0 \end{cases}$$

by including in the algorithm the use, in the third and in the fourth argument, of the natural cubic spline and of its first order derivative. Such initial value problem was approached in [233] by using θ-methods.

The fourth order beam equation presented in Section 3.6 can be generalized to a neutral type one obtaining the following boundary value problem:

$$\begin{cases} x^{IV}(t) = f(t, x(t), x(\varphi(t)), x'(t), x'(\varphi(t))), \quad t \in [a,b] \\ x(a) = \alpha, \quad x(b) = \beta \\ x'(a) = w, \quad x'(b) = r \end{cases}$$

which can be approached by combining the Perov's fixed point theorem with the method of successive interpolations. More precisely, this problem is equivalent with the following integro-differential equation

$$x(t) = g(t) + \int_a^b G(t,s) \cdot f\left(s, x(s), x(\varphi(s)), x'(s), x'(\varphi(s))\right) ds, \quad t \in [a,b] \tag{5.3}$$

where

$$g(t) = \frac{(b-t)^2\left[2(t-a) + (b-a)\right]}{(b-a)^3} \cdot c + \frac{(t-a)^2\left[2(b-t) + (b-a)\right]}{(b-a)^3} \cdot d +$$

$$+ \frac{(b-t)^2(t-a)}{(b-a)^2} \cdot w - \frac{(t-a)^2(b-t)}{(b-a)^2} \cdot r, \quad t \in [a,b]$$

and.

$$G(t,s) = \begin{cases} H(t,s) = \frac{1}{6}\left(\frac{s-a}{b-a}\right)^2\left(1 - \frac{t-a}{b-a}\right)^2 \cdot \left[\left(\frac{t-a}{b-a} - \frac{s-a}{b-a}\right) + 2\left(1 - \frac{s-a}{b-a}\right)\left(\frac{t-a}{b-a}\right)\right], & s \le t \\ K(t,s) = \frac{1}{6}\left(\frac{t-a}{b-a}\right)^2\left(1 - \frac{s-a}{b-a}\right)^2 \cdot \left[\left(\frac{s-a}{b-a} - \frac{t-a}{b-a}\right) + 2\left(1 - \frac{t-a}{b-a}\right)\left(\frac{s-a}{b-a}\right)\right], & s \ge t. \end{cases}$$

Differentiating by t in equation (5.3) and denoting $x' = y$, it obtains the equivalent system of integral equations

$$\begin{cases} x(t) = g(t) + \int_a^b G(t,s) \cdot f(s, x(s), x(\varphi(s)), y(s), y(\varphi(s)))\, ds \\ y(t) = g'(t) + \int_a^b \frac{\partial G(t,s)}{\partial t} \cdot f(s, x(s), x(\varphi(s)), y(s), y(\varphi(s)))\, ds \end{cases}, \quad t \in [a,b].$$

For the existence and uniqueness of the solution of this system the Perov's fixed point theorem can be applied and in the approximation of the solution, the method of successive interpolations using natural cubic splines, will be extended.

5.2.3 The method of successive interpolations for fuzzy functional integral equations

Fuzzy Volterra functional integral equations with delayed argument are approached in [197] by applying the Banach's fixed point theorem and proving the existence and uniqueness of the solution and the convergence of the sequence of successive approximations to this solution. In [21] the existence of the solution of fuzzy Volterra integral equations with deviating argument is proved using the Darbo's fixed point theorem.

By combining the method of successive approximations with a proper fuzzy interpolation procedure (see for instance [90]) we can extend the method of successive interpolations to the following types of fuzzy functional integral equations:

$$x(t) = g(t) + \int_a^b F(t, s, x(s), x(\varphi(s)))\, ds, \quad t \in [a, b]$$

and

$$x(t) = g(t) + \int_a^t H(t, s) \cdot f(s, x(s), x(\varphi(s)))\, ds, \quad t \in [a, b].$$

5.3 Open problems for fuzzy numbers

5.3.1 Open problems on operations for fuzzy numbers with improved algebraic properties

The algebraic operations with fuzzy numbers are generated by using the Zadeh's extension principle, but the existence of an opposite (with addition), or of an inverse (with multiplication) for noncrisp fuzzy numbers is not ensured (see [28], [116], [117], [119], [140], [142], [144], [160],[170], [177]-[181], [222]) generating difficulties in operating with fuzzy numbers. Moreover, the distributivity of the scalar product and of the product of noncrisp fuzzy numbers not fulfils everywhere (see for instance, [23], [28], [177], and [181]). In order to avoid these, some equivalence relations were proposed in [23], [117], [177], [179], and [181], and only on the obtained quotient groups the inversability holds.

Concerning the distributivity of the scalar product, the following property holds

$$(\alpha + \beta) \cdot u = \alpha \cdot u + \beta \cdot u, \quad \forall u \in \mathbb{R}_{\mathcal{F}} \text{ if and only if } \alpha \cdot \beta \geq 0$$

only with this restriction, which can be avoided using the additive equivalence obtaining an equivalence, not an identity (see [23], [177], and [181]). So, the additive structure of the set of fuzzy numbers is not a linear space, its proper structure being the cancellative quasilinear space described in [183]. The distributivity of the product of noncrisp fuzzy numbers generally holds only for fuzzy numbers with positive support. Only the latticeal product proposed in [170] has distributivity everywhere, but this product operates from arithmetic point of view only for the middle of the core.

The middle-parametric representation of fuzzy numbers

In the purpose to improve and to overcome some of these above presented difficulties arising in fuzzy arithmetic, inspired by the approach presented in [88] and [170], we generalize it and propose a new representation of a fuzzy number, the *middle parametric representation*, and new type of operations using this representation.

Consider a fuzzy number $u \in \mathbb{R}_{\mathcal{F}}$ written in LU (or equivalently, parametric)-representation having the r-level sets $u_r = [u_r^-, u_r^+] = \{x \in \mathbb{R} : u(x) \geq r\}$. The functions $u^-, u^+ : [0, 1] \rightarrow$

$\mathbb{R}$, defining the end-points of the r-level sets, satisfy the following conditions:
(i) the function u^- given by $u^-(r) = u_r^-$, $r \in [0,1]$, is bounded, non-decreasing, left-continuous in $(0,1]$ and it is right-continuous at 0;
(ii) the function u^+ given by $u^+(r) = u_r^+$, $r \in [0,1]$, is bounded, non-increasing, left-continuous in $(0,1]$ and it is right-continuous at 0;
(iii) $u_1^- \leq u_1^+$.

We define the arithmetic mean function of the level functions u^-, u^+, by $m_u(t) = \frac{u^-(t)+u^+(t)}{2}$, $t \in [0,1]$ and observing that

$$u^+(t) - m_u(t) = m_u(t) - u^-(t) = \frac{u^+(t) - u^-(t)}{2}, \quad \forall t \in [0,1]$$

we define the dispersion function $\delta_u(t)$ by $\delta_u(t) = \frac{u^+(t)-u^-(t)}{2}$, $t \in [0,1]$. So, the pairs $(m_u(t), \delta_u(t))$, $t \in [0,1]$, completely determine the fuzzy number u and therefore the pair (m_u, δ_u) can be considered instead of the pair (u^-, u^+), being the so called *middle parametric* representation of the fuzzy number u. The level functions u^-, u^+ can be easily recuperated putting $u^+ = m_u + \delta_u$ and $u^- = m_u - \delta_u$.

In [170] is considered the middle core $u_0 = \frac{1}{2} \cdot (u^-(1) + u^+(1))$, named the location index number of u, and the functions

$$u_* = u_0 - u^-$$
$$u^* = u^+ - u_0$$

called the left and right fuzziness index functions of u. So, a fuzzy number is represented by the triple $(u_0, u_*, u^*) \in \mathbb{R} \times L \times L$, where

$$L = \{h : [0,1] \to [0,\infty) \mid h \text{ is non-increasing and left-continuous}\}.$$

The operations between $u = (u_0, u_*, u^*)$ and $v = (v_0, v_*, v^*)$ are defined by $u \circ v = (u_0 \circ v_0, u_* \vee v_*, u^* \vee v^*)$ were the operation $\circ$ is instead of $+, \cdot, -, /$. It is proved that all these four operations are defined everywhere and the commutativity, associativity and distributivity holds for all fuzzy numbers. But as can be viewed, only $u_0 \circ v_0$ is an arithmetic type operation being defined only for $t = 1$. The operations $u_* \vee v_*$ and $u^* \vee v^*$ are of latticeal type.

Using the middle parametric representation, $u = (m_u, \delta_u) \in M \times \Omega$, where

$$M = \{m : [0,1] \to \mathbb{R} \mid m \text{ is bounded, left-continuous in } (0,1] \text{ and right continuous at } 0\}$$

$$\Omega = \{\delta : [0,1] \to [0,\infty) \mid \delta \text{ is non-increasing and left-continuous in } (0,1] \text{ and right continuous at } 0\}$$

we propose new arithmetic operations. After operating we can comeback to the LU-representation in order to see the geometric image of the result of operation. We can see that $\overline{m_u} = \int_0^1 m_u(t)\,dt = \int_0^1 \frac{u^-(t)+u^+(t)}{2} dt$ is just the f-weighted possibilistic mean value of the fuzzy number u from [135] corresponding to the weight function $f = 1$, and therefore could be considered as the mean value of the fuzzy number.

Firstly, we define the addition of two fuzzy numbers and the scalar multiplication, on $\mathbb{R}_\mathcal{F}$. A fuzzy number $u \in \mathbb{R}_\mathcal{F}$ is called symmetric if in its middle parametric representation (MPR) $u = (m_u, \delta_u)$, the middle function m_u is constant, i.e. $m_u(t) = u_0$, $\forall t \in [0,1]$, and u is called nonsymmetric iff it is not symmetric. In this sense, the symmetric fuzzy numbers from [116] and [177] are here zero-symmetric corresponding to the particular case $u_0 = 0$. On the set of all symmetric fuzzy numbers $S\mathbb{R}_{\mathcal{F}}$ (which includes the set of crisp fuzzy numbers) the arithmetic operations could be those from [170], being now, $u \circ v = (u_0 \circ v_0, \delta_u \vee \delta_v)$. We define here another type of addition and scalar multiplication.

The set of nonsymmetric fuzzy numbers is $NS\mathbb{R}_{\mathcal{F}} = \mathbb{R}_{\mathcal{F}} \setminus S\mathbb{R}_{\mathcal{F}}$, and R is the set of all crisp fuzzy numbers, $R = \{\overline{r} : r \in \mathbb{R}\}$, where $\overline{r} : \mathbb{R} \to [0,1]$,

$$\overline{r}\,(x) = \begin{cases} 1, & \text{if } x = r \\ 0, & \text{if } x \neq r \end{cases}.$$

Of course, $R \subset S\mathbb{R}_{\mathcal{F}}$ and $\overline{r} = (r, 0)$ in this (MPR) representation for all $\overline{r} \in R$. We see that for $u \in R$, $\delta_u = 0$, and for $u \in \mathbb{R}_{\mathcal{F}} \setminus S\mathbb{R}_{\mathcal{F}}$, we have $\delta_u > 0$. For $u = (m_u, \delta_u) \in NS\mathbb{R}_{\mathcal{F}}$, let ϖ_u be the following positive number

$$\varpi_u = \sup_{t_1, t_2 \in [0,1]} |m_u\,(t_1) - m_u\,(t_2)|$$

and we see that $\delta_u = 0$, $\varpi_u = 0$ for $u \in R$ and $\delta_u > 0$, $\varpi_u = 0$ for $u \in S\mathbb{R}_{\mathcal{F}} \setminus R$. When $u \in \mathbb{R}_{\mathcal{F}} \setminus S\mathbb{R}_{\mathcal{F}}$, we have $\delta_u > 0$ and $\varpi_u > 0$. The addition on $\mathbb{R}_{\mathcal{F}}$ is defined as follows:

For $u, v \in \mathbb{R}_{\mathcal{F}}$, $u = (m_u, \delta_u)\,, v = (m_v, \delta_v)$ we have $u + v = (m_{u+v}, \delta_{u+v})$ with $m_{u+v}\,(t) = m_u\,(t) + m_v\,(t)\,, \forall t \in [0,1]$ and

$$\delta_{u+v}\,(t) = \delta_u\,(t) + \delta_v\,(t)\,, \quad \forall t \in [0,1]. \tag{5.4}$$

We see that the case $\delta_u \cdot \delta_v = 0$ covers the situation when at least one of u and v are crisp numbers. The scalar multiplication is defined by $\lambda \cdot u = (m_{\lambda \cdot u}, \delta_{\lambda \cdot u})$ with

$$m_{\lambda \cdot u}\,(t) = \lambda \cdot m_u\,(t)\,, \quad \delta_{\lambda \cdot u}\,(t) = |\lambda| \cdot \delta_u\,(t)\,, \quad \forall t \in [0,1] \tag{5.5}$$

for all $\lambda \in \mathbb{R}$. We see that $0 \cdot u = \overline{0}$, $\forall u \in \mathbb{R}_{\mathcal{F}}$ and the opposite of u can be defined as $-u = (-1) \cdot u = (-m_u, \delta_u)$ for all $u \in \mathbb{R}_{\mathcal{F}}$. Regarding to the scalar product defined in (5.5), the good algebraic properties in the use of the first side m_u of fuzzy numbers in MP-representation and the neutrality of the second side δ_u to the sign changing of the crisp λ, allow us to make easier the fuzzy calculus, revealing one of the advantages of the MP-representation. However, the difficulties generated by the "fuzzy zero", as was mentioned in [180], still remain. Will be a challenge to construct such operations which avoid these difficulties, the problem remaining open, and in our opinion the MP-representation will be more useful.

Why we have defined the sum as in (5.4) and not as in [170]? In the following Remark we can see the response.

Remark 149 *For the triangular fuzzy number A with the support being the interval $[1,4]$ and the core be the singleton $\{2\}$, having the level sides $A^-\,(t) = 1 + t$, $A^+\,(t) = 4 - 2t$, $t \in [0,1]$, if we use the scalar product as in [170] with $\delta_{\lambda \cdot u}\,(t) = \delta_\lambda\,(t) \vee \delta_u\,(t) = 0 \vee \delta_u\,(t) = \delta_u\,(t)\,,\ t \in [0,1]$, we see that since $\omega_A = |2.5 - 2| = 0.5$ we will have for $4 \cdot A$ that $(4 \cdot A)^-$ and $(4 \cdot A)^+$ are both strictly decreasing. So, the result of the scalar product is not a fuzzy number ! Moreover, in order to have consistent addition with this scalar product (where $\delta_{\lambda \cdot u}\,(t) = |\lambda| \cdot \delta_u\,(t)\,, \forall t \in [0,1]$) we should define the sum as in (5.4). In this way $A + A = 2 \cdot A$ (that it is natural) and using the sum from [170] we have $A + A \neq 2 \cdot A$. So, the choice of the operations was made such that the result of the addition of two fuzzy numbers to be a fuzzy number, the scalar product between any real number and any fuzzy number to be a fuzzy number, and in order to have consistency between the addition and the scalar product.*

Remark 150 *After elementary calculus we see that the sum of any two fuzzy numbers is a fuzzy number. More precisely, the sum of two symmetric fuzzy numbers is a symmetric fuzzy number. For* $u, v \in NS\mathbb{R}_{\mathcal{F}}$ *with* $m_v(t) = -m_u(t) + c$, $\forall t \in [0,1]$, $c \in \mathbb{R}$, *the sum* $u + v \in S\mathbb{R}_{\mathcal{F}}$, *and* $u + v \in NS\mathbb{R}_{\mathcal{F}}$ *elsewhere. The addition is commutative, associative, cancellative and has the neutral element* $\overline{0}$. *Moreover, the difference defined by* $u - v = u + (-1) \cdot v$ *exists everywhere in* $NS\mathbb{R}_{\mathcal{F}}$, *and for* $u \in \mathbb{R}_{\mathcal{F}} \setminus S\mathbb{R}_{\mathcal{F}}$ *we see that* $u - u$ *exists and*

$$m_{u-u}(t) = m_u(t) + (-m_u(t)) = 0, \forall t \in [0,1]$$

and, $\delta_{u-u}(t) = 2\delta_u(t)$, $\forall t \in [0,1]$. *As we have observed in Section 4.1.1, the generalized difference* $\ominus_g$ *always exists, but this generalized difference is not consistent with the usual sum defined in* $\mathbb{R}_{\mathcal{F}}$.

A metric on $NS\mathbb{R}_{\mathcal{F}}$ consistent with the MP-representation is

$$D^1(u,v) = \sup_{t\in[0,1]} |m_u(t) - m_v(t)| + \sup_{t\in[0,1]} |\delta_u(t) - \delta_v(t)|, \quad \forall u, v \in NS\mathbb{R}_{\mathcal{F}}$$

and it is easy to see that $\frac{1}{2}D_\infty(u,v) \leq D^1(u,v) \leq 2D_\infty(u,v)$, $\forall u, v \in NS\mathbb{R}_{\mathcal{F}}$.

The good properties of the operations defined above ensure us to develop the fuzzy calculus in more directions than the existing results. For instance we can define elementary fuzzy functions by $f(u) = (f(m_u), \delta_u)$, $\forall u \in NS\mathbb{R}_{\mathcal{F}}$, and since the difference $u-v$ always exists in $NS\mathbb{R}_{\mathcal{F}}$ (and it is more easy to manipulate it in calculus than the early defined differences), we can define the derivative of a function $f : [a,b] \to NS\mathbb{R}_{\mathcal{F}}$ easier than up now and develop a new theory for fuzzy calculus and fuzzy differential equations. These all thinks lead us to develop our research work in the future, in many directions of fuzzy calculus using the above defined operations with fuzzy numbers written in MP-representation.

5.3.2 Approximation of fuzzy numbers and fuzzy interpolation

Smooth approximation of fuzzy numbers

The trapezoidal approximation of fuzzy numbers is a well developed field and in order to extend the work from [24] we will consider the approximation of the level functions of a fuzzy number using smooth curves provided by cubic splines of Hermite type with minimal QOA on a certain partition of the interval $[0,1]$.

Fuzzy interpolation procedures

The study of fuzzy interpolation procedures starts in [169], [159] and [120] introducing the fuzzy variant of Lagrange polynomial interpolation and spline interpolation of fuzzy data by piecewise linear spline and not-a-knot cubic spline. Then, in [2] and [3], the complete and natural cubic spline variant interpolating fuzzy data were proposed, but due to the sign changes in the coefficients and by the lack of distributivity, these splines are not very practical and it is very difficult (or impossible) to prove the approximation theorem (as it is shown in [90]). Therefore, the fuzzy B-splines are proposed in [90] for the interpolation of fuzzy data. Recently, for the interpolation of fuzzy data, the fuzzy variant of $E(3)$ cubic splines are obtained in [34] using as basis functions the two-point interpolation Hermite type cubic polynomials.

We intend to continue the work on fuzzy interpolation by splines initiated in [76], and based on the new fuzzy arithmetic presented in Section 5.3.1, we will introduce various fuzzy spline interpolation procedures (including cubic splines) in order to improve the existing results.

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