ABSTRACT of HABILITATION THESIS

entitled

SUCCESSIVE APPROXIMATIONS IN CRISP CONTEXT, FUZZY CONTEXT AND SPLINES WITH OPTIMAL PROPERTIES

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The present work consisting in 5 chapters and a list of references is based on 47 selected papers of the author, written single or jointly, and on some results from the author's research monographs.(Bica-[62] and Bica-[63]), all mentioned in References.

The first chapter contains the results obtained by the author concerning the applications of the Perov's fixed point theorem to integro-differential equations (of Fredholm type and of Volterra type with constant delay) and to neutral type differential equations, including first order and second order delay differential equations and two-point boundary value problems associated to second order neutral differential equations.

The Perov's fixed point theorem was obtained in Perov-Kibenko-[201] and it is a powerful tool to investigate the existence and uniqueness of the solution of systems of differential and integral equations. Its framework is represented by generalized metric spaces with the metric taking values in the positive cone of the Euclidean space, the result being obtained by applying the fixed point technique to an operator with the spectrum included in the open unit disk of the complex field. A consequence of this fixed point theorem is the "Fiber generalized contraction theorem" obtained by Rus in [212] and [214], being useful in the proof of the smooth dependence by parameters of the solution of systems of differential and integral equations (see Rus-[212], [214], [215], Bica-[44], [52], Bica-Mureşan-Grebenişan-[50], Bica-Mureşan-[51], [60]).

In this context, the integro-differential equation

$$\begin{cases} x'(t) = f(t, x(t), x'(t-\tau)) + \int\limits_{t-\tau}^{t} g(t, s, x(s), x'(s))ds, & t \in [0, b] \\ x(t) = \varphi(t), & t \in [-\tau, 0] \end{cases}$$

is written in the equivalent form of a system of Volterra delay integral equations

$$\begin{cases} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \varphi(0) + \int\limits_0^t f(s, x(s), y(s-\tau))ds + \\ + \int\limits_0^t \left(\int\limits_{\eta-\tau}^{\eta} g(\eta, s, x(s), y(s))ds \right) d\eta \ , \\ f(t, x(t), y(t-\tau)) + \int\limits_{t-\tau}^{t} g(t, s, x(s), y(s))ds \end{pmatrix}, & t \in [0, b] \\ (x(t), y(t)) = (\varphi(t), \varphi'(t)), & t \in [-\tau, 0]. \end{cases}$$

where we have denoted $y = x'$, and the Perov's fixed point theorem is applied in the generalized metric space (X, d_B) with

$$X = C[-\tau, b] \times C[-\tau, b], \quad d_B : X \times X \longrightarrow \mathbb{R}^2,$$

$$d_B((x_1, y_1), (x_2, y_2)) = (\|x_1 - x_2\|_B \, , \|y_1 - y_2\|_B),$$

$$\|u\|_B = \max\{|u(t)| \cdot e^{-\theta(t+\tau)} : t \in [-\tau, b]\}, \quad \forall u \in C[-\tau, b], \quad \theta > 0$$

to the operator $A : X \longrightarrow X$, $A = (A_1, A_2)$,

$$(A_1(x(t), y(t)), A_2(x(t), y(t))) = (\varphi(t), \varphi'(t)), \qquad \forall t \in [-\tau, 0]$$

$$A_1(x(t),y(t)) = \varphi(0)+\int_0^t f(s,x(s),y(s-\tau))ds+\int_0^t \left(\int_{\eta-\tau}^{\eta} g(\eta,s,x(s),y(s))ds\right) d\eta$$

$$A_2(x(t),y(t)) = f(t,x(t),y(t-\tau)) + \int_{t-\tau}^{t} g(t,s,x(s),y(s))ds, \qquad \forall t \in [0,b].$$

The existence and uniqueness of the solution of the system of integral equations and Lipschitz properties for this solution and for its derivative are obtained. A technique to obtain the uniform Lipschitz property for the sequence of successive approximations is developed based on continuity and Lipschitz conditions imposed to the functions f and g. The case when $f = 0$ and g independent by t corresponds to an integro-differential equation arising in epidemiology:

$$x'(t) = \int_{t-\tau}^{t} g(s,x(s),x'(s))ds, \quad t \in \mathbb{R}$$

approached in Bica-Mureşan-[60]. Using the fiber generalized contraction theorem, the smooth dependence of the positive periodic solution by the lag τ is studied in Bica-Mureşan-[51].

Another example illustrating the effectiveness of the Perov's fixed point theorem to neutral type differential equations is the two-point boundary value problem

$$\begin{cases} y''(x) = f(x, y(x), y'(x)), & x \in [a,b] \\ y(a) = c, \quad y(b) = d \end{cases}$$

approached in Bica-[58] for the solution taking values in a Banach space X. This boundary value problem is written in the equivalent form of a system of Fredholm integral equations

$$\begin{cases} y(x) = \frac{x-a}{b-a} \cdot d + \frac{b-x}{b-a} \cdot c - \int_a^b G(x,s) \cdot f(s, y(s), z(s))\, ds \\ z(x) = \frac{d-c}{b-a} - \int_a^b \frac{\partial G}{\partial x}(x,s) \cdot f(s, y(s), z(s))\, ds, \end{cases} \quad x \in [a,b]$$

where we have denoted $z = y'$ and G is the Green's function

$$G(x,s) = \begin{cases} \frac{(s-a)(b-x)}{b-a}, & \text{if } s \le x \\ \frac{(x-a)(b-s)}{b-a}, & \text{if } s \ge x. \end{cases}$$

To this system is applied the Perov's fixed point theorem on the space $Y = C([a,b],X) \times C([a,b],X)$ with the generalized metric $d_C : Y \times Y \to \mathbb{R}^2$, defined by

$$d_C((y_1,z_1),(y_2,z_2)) = (\|y_1 - y_2\|_\infty \ , \ \|z_1 - z_2\|_\infty), \quad \text{for } (y_1,z_1),(y_2,z_2) \in Y$$

where

$$\|y\|_{\infty} = \max\{\|y(x)\|_X : x \in [a,b]\} \text{ for } y \in C([a,b],X).$$

The existence and uniqueness of the solution of this boundary value problem is studied in Bica-[58] proposing a method to approximate this solution. This method is based on the technique of successive approximations combined with a quadrature rule. The application of the method of successive approximations to differential equations begins with the works of Lindelöf and Picard (see [167] and [203]) and it is followed by D.V. Ionescu (see [149]) with the inclusion of a two-point quadrature rule (of trapezoidal or Euler-Mac Laurin type). Recent refinements of this method, inspired from the Gauss-Seidel iterations, are the waveform relaxation technique proposed by Lelarasmee et al. in [166], and the dynamic iterations method proposed by Bjørhus in [89].

We combine the method of successive approximations with the trapezoidal quadrature rule proposed by the author in [54] for Lipschitzian functions with values in Banach spaces. Are obtained in [58] sufficient conditions for the existence and uniqueness of the solution of the two-point boundary value problem and for the convergence of the approximation method.

The second chapter is dedicated to new optimal properties for cubic splines, recently obtained by the author in Bica-[73], [81], [84]. Potential applications of such results are in computer aided geometric design and in robotics, generating new types of smooth interpolation procedures. The classical optimal properties for splines are related to minimal curvature and minimal strain energy (see Ahlberg et al.-[8], de Boor-[91], Burmeister et al.-[85], Micula-[185], Wolberg-Alfy-[236], Yong-Cheng-[245]), minimal L^2-norm of the cubic spline and of its derivatives (see Kobza-[163]), and to suboptimal algorihtms for curve fitting (see Ichida et al.-[148]). Another recent minimal property for splines is connected with the idea of minimal deviation of a spline from its data polygon, generating in Floater-[132] the notion of local and global *deviation* and in Bica-[73], [81], the notion of *quadratic oscillation in average* (QOA) and partial quadratic oscillation in average (PQOA), respectively.

In order to define the QOA, we consider a partition of $[a,b]$

$$\Delta : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$$

and the given values $y_{0,}\, y_1, ..., y_n \in \mathbb{R}$. In $C[a,b]$ we can form the set $C([a,b],\Delta,y) = \{f \in C[a,b] : f(x_i) = y_i, \quad \forall i = \overline{0,n}\}$.

Definition (see Bica-[73]): The quadratic oscillation in average (QOA) of a given function $f \in C([a,b],\Delta,y)$ is the value of the functional $\rho_2 : C([a,b],\Delta,y) \to \mathbb{R}$ defined by

$$\rho_2(f) = \sqrt{\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} [f(x) - \frac{x_i - x}{h_i} \cdot y_{i-1} - \frac{x - x_{i-1}}{h_i} \cdot y_i]^2 dx}.$$

This notion can be defined for any function interpolating the points (x_i, y_i), $i = \overline{0,n}$, and for any type of spline functions, not depending by its deficiency or by its smoothness. It is easy to observe that the functional ρ_2 is positive, $\rho_2(f) \geq 0$, $\forall f \in C([a,b],\Delta,y)$ and $\rho_2(f) = 0$ if and only if f is the linear first order polynomial spline interpolating the points (x_i, y_i), $i = \overline{0,n}$ (namely, the polygonal line). Moreover, for any $\alpha \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$, we see that $\rho_2(\alpha \cdot f) = \alpha \cdot \rho_2(f)$. So,we can conclude that the QOA is a positive and homogeneous functional. In Bica-[73] we have determined a special cubic spline $s \in C^1[a,b]$ with minimal QOA, minimizing $\rho_2(s)$ in the class of Hermite type cubic splines. In the problems where we intend to minimize the quadratic oscillation only on some of the subintervals $I_i = [x_{i-1}, x_i]$, $i = \overline{1,n}$, more suitable is the notion of partial quadratic oscillation in average (PQOA).

Definition (see Bica-[81]): Let a subset $K \subset \{1,...,n\}$ be given. The partial quadratic oscillation in average of the function $f \in C([a,b],\Delta,y)$ corresponding to the subset K, is the functional $\rho(K) : C([a,b],\Delta,y) \to \mathbb{R}$ given by:

$$\rho(K)(f) = \sqrt{\sum_{i\in K} \int_{I_i} [f(x) - \frac{x_i - x}{h_i} \cdot y_{i-1} - \frac{x - x_{i-1}}{h_i} \cdot y_i]^2 dx}.$$

The QOA is now a particular case of PQOA corresponding to the set $K = \{1,...,n\}$. Because in the purpose to minimize the QOA of a spline over the all subintervals $I_i = [x_{i-1}, x_i]$, $i = \overline{1,n}$, we need the degree of freedom at least n+1, we infer that the functional QOA is suitable for splines with deficiency at least 2. For splines with deficiency 1 we can use only the functional PQOA in the problem to determine the free parameters such that the PQOA to be minimized.

In the expression of the Hermite type cubic spline

$$s(x) = \frac{(x_i - x)^2 (x - x_{i-1})}{h_i^2} \cdot m_{i-1} - \frac{(x - x_{i-1})^2 (x_i - x)}{h_i^2} \cdot m_i +$$

$$\frac{(x_i - x)^2 [2(x - x_{i-1}) + h_i]}{h_i^3} \cdot y_{i-1} + \frac{(x - x_{i-1})^2 [2(x_i - x) + h_i]}{h_i^3} \cdot y_i, \quad x \in [x_{i-1}, x_i]$$

where $h_i = x_i - x_{i-1}$, $i = \overline{1,n}$, we have determined the values $m_0, m_1, ..., m_n$ in order to minimize the quadratic oscillation in average of s. The following remarkable result was obtained.

Theorem (see Bica-[73]): For given points (x_i, y_i), $i = \overline{0,n}$, there exists an unique cubic spline of the Hermite type having minimal quadratic oscillation in average. This cubic spline $s \in C^1[a,b]$ can be determined by using an iterative algorithm. If s interpolates a function $f \in C[a,b]$, $f(x_i) = y_i$, $i = \overline{0,n}$, then its error estimation is:

$$|f(x) - s(x)| \leq \left(1 + \frac{h^3}{4\overline{h}^3}\right) \cdot \varpi(f,h), \quad \forall x \in [a,b] \tag{1}$$

where $h = \max\{h_i : i = \overline{1,n}\}$, $\overline{h} = \min\{h_i : i = \overline{1,n}\}$, $\varpi(f,h) \overset{notation}{=} \max\{\varpi(f,h_i) : i = \overline{1,n}\}$, and $\varpi(f,\delta) = \sup\{|f(t) - f(s)| : t,s \in [a,b],$

$|t-s| \leq \delta\}$ is the uniform modulus of continuity. For uniform partitions, the estimate becomes

$$|f(x) - s(x)| \leq \frac{5}{4} \cdot \varpi(f,h), \quad \forall x \in [a,b]. \tag{2}$$

So, the cubic spline obtained in the previous theorem has the smallest QOA in the class of Hermite type cubic splines. It is interesting to observe that the constant $\frac{5}{4}$ in (2) is smaller than those in the estimates for natural cubic splines, not-a-knot cubic splines, and Akima's cubic spline, in terms of the modulus of continuity. So, we can affirm that the general estimate (1) is better. Moreover, the length of graph is the smallest (excepting the case of $m_0 = m_1 = ... = m_n = 0$) for the cubic spline with minimal QOA in the class of Hermite type cubic splines, as can be observed in Bica-[81]. As it is shown in Bica-[81], the restrictions of this cubic spline s to the intervals $[x_{i-1}, x_i]$, $i = \overline{1,n}$, always preserve their third order as polynomial functions, excepting the case of collinear points. More precisely, if and only if three consecutive points $P_{i-1}(x_{i-1}, y_{i-1})$, $P_i(x_i, y_i)$, $P_{i+1}(x_{i+1}, y_{i+1})$ are collinear, then on the corresponding intervals $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ the cubic spline becomes first order polynomial and his graph it reduces to the the line joining the points P_{i-1}, P_i, P_{i+1}. So, if $n = 1$, then (after elementary calculus) we see that the cubic polynomial (1) with minimal QOA on $[a,b] = [x_0, x_1]$ it reduces to a first order polynomial, and if $n \geq 2$, then the cubic spline $s \in C^1[a,b]$ with minimal QOA has the following supplementary property: the polynomial order of its restrictions s_i, $i = \overline{1,n}$, becomes less than 3, if and only if the interpolated points $P_i(x_i, y_i)$, $i = \overline{0,n}$, are all collinear (all situated on the same line). In this case this spline it reduces to the line joining these points, the polynomial order being 1 and never 2. If $n \geq 2$ and if the points $P_i(x_i, y_i)$, $i = \overline{0,n}$ are not all collinear, then the restrictions s_i, to the intervals $[x_{i-1}, x_i]$, $i = \overline{1,n}$, of the cubic spline $s \in C^1[a,b]$ with minimal QOA, are third order cubic polynomials.

In Bica-[81] is obtained an improvement of the Akima's interpolation method near the end-points. The Akima's cubic spline (see Akima-[9]) provides a natural choice of the values m_0, $m_1, ..., m_n$ offering a suitable interpolation procedure in the smooth curve fitting. In the Akima's method, the values m_i, $i = \overline{0,n}$, are determined by using a local procedure based on geometric reasons. More exactly, for five given points $M_i(x_i, y_i)$, $i = \overline{1,5}$, are computed the slopes $p_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$, $i = \overline{1,4}$, and it is suggested the following value for the tangent in the point $M_2(x_2, y_2)$:

$$m_2 = \frac{|p_4 - p_3| \cdot p_2 + |p_2 - p_1| \cdot p_3}{|p_4 - p_3| + |p_2 - p_1|}. \tag{3}$$

This formula (3) is generalized considering the slopes $p_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$, $i = \overline{0, n-1}$ together with the derivatives

$$m_i = \frac{|p_{i+1} - p_i| \cdot p_{i-1} + |p_{i-1} - p_{i-2}| \cdot p_i}{|p_{i+1} - p_i| + |p_{i-1} - p_{i-2}|}, \quad i = \overline{2, n-2}. \tag{4}$$

In order to extend formula (4) for $i = \overline{0,n}$, the previously computed slopes are not enough and therefore, Akima proposes the construction of four new supplementary slopes $p_{-1}, p_{-2}, p_n, p_{n+1}$, based on a reasoning in the framework of a particular case (equidistant grid and exactness for second order polynomials on the end intervals $[x_0, x_1]$ and $[x_{n-1}, x_n]$):

$$p_{-1} = 2p_0 - p_1, \quad p_{-2} = 3p_0 - 2p_1, \quad p_n = 2p_{n-1} - p_{n-2}, \quad p_{n+1} = 3p_{n-1} - 2p_{n-2}.$$

Since the artificial introduction of the four slopes well performs only in the particular case of equidistant grids, this is not a strong point of the Akima's method. For this reason we have proposed, in Bica-[81], an optimal procedure for the computation of the left unspecified derivatives m_0, m_1, m_{n-1}, m_n (the other derivatives m_i, $i = \overline{2, n-2}$, being computed using (4)). This optimal procedure is based on the idea to partially minimize the PQOA of the cubic spline on the end intervals $[x_0, x_2]$ and $[x_{n-2}, x_n]$, and consequently, we consider the subset $K = \{1, 2, n-1, n\}$. The values m_0, m_1, m_{n-1}, m_n are uniquely determined such that

$$\rho(K)(s) = \sqrt{\sum_{i \in K} \int_{I_i} [s(x) - \frac{x_i - x}{h_i} \cdot y_{i-1} - \frac{x - x_{i-1}}{h_i} \cdot y_i]^2 dx}$$

to be minimized for $K = \{1, 2, n-1, n\}$.
Moreover, for the obtained cubic spline, the error estimate in the interpolation of a given function $f \in C([a, b], \Delta, y)$ is:

$$|s(x) - f(x)| \leq \left(1 + \frac{h}{4\underline{h}}\right) \cdot \varpi(f, h), \quad \forall x \in [x_2, x_{n-2}],$$

and

$$|s(x) - f(x)| \leq \left(1 + \frac{h^4}{4\underline{h}^4}\right) \cdot \varpi(f, h), \quad \forall x \in [x_0, x_2] \cup [x_{n-2}, x_n],$$

where $h = \max\{h_i : i = \overline{1, n}\}$, $\underline{h} = \min\{h_i : i = \overline{1, n}\}$, and $\varpi(f, h) = \max\{|f(u) - f(v)| : u, v \in [a, b], |u - v| \leq h\}$ is the uniform modulus of continuity. For equidistant grids the error estimate becomes

$$|s(x) - f(x)| \leq \frac{5}{4} \cdot \varpi(f, h), \quad \forall x \in [x_0, x_n],$$

being the same as in (2). So, this interpolation procedure possesses both the properties of the Akima's interpolation method of "natural" derivatives m_i, $i = \overline{2, n-2}$, and the property of minimal deviation from the polygonal line on the first two and last two subintervals.

In the third chapter we present the recent method of successive interpolations developed by the author during 2007-2013. This method differs by the collocation or spline functions methods and it is a refinement of the method of successive approximations combining the Picard's technique of successive approximations with a quadrature rule (usually the trapezoidal rule) and with an interpolation procedure applied at each iterative step. Was developed in order to extend the method of successive approximations from ordinary differential and integral equations to differential and integral equations with deviating argument and its beginning is realized in Bica-[83], [63], Bica-Oros-[56], being continued in Bica-[57],[71], [72], [79], and Bica-Curilă-[69], [70], [74], [78], [82], [85-87].

In this context, for the initial value problem (that includes the pantograph differential equation as a particular case with $a = 0$ and $\varphi(t) = \alpha t$, $\alpha \in (0,1)$)

$$\begin{cases} x'(t) = f(t, x(t), x(\varphi(t))), \quad t \in [a,b] \\ x(a) = x_0 \end{cases}$$

with $\varphi \in C[a,b]$, $a \leq \varphi(t) \leq b$ and $\varphi(t) \leq t$ for all $t \in [a,b]$, the method of successive interpolations uses in Bica-[79] the interpolation procedure of natural cubic spline generating the sequence of effective computed approximations of the solution on a uniform grid of $[a,b]$, by applying the following algorithm:

$$x_m(t_i) = x_0 + \frac{(b-a)}{2n} \cdot \sum_{j=1}^{i} [f\left(t_{j-1}, \overline{x_{m-1}(t_{j-1})} + \overline{R_{m,j-1}}, x_{m-1}(\varphi(t_{j-1}))\right) +$$

$$+ f\left(t_j, \overline{x_{m-1}(t_j)} + \overline{R_{m-1,j}}, x_{m-1}(\varphi(t_j))\right)] + R_{m,i} = x_0 + \frac{(b-a)}{2n} \cdot \sum_{j=1}^{i} [f(t_{j-1}, \overline{x_{m-1}(t_{j-1})},$$

$$, s_{m-1}(\varphi(t_{j-1}))) + f\left(t_j, \overline{x_{m-1}(t_j)}, s_{m-1}(\varphi(t_j))\right)] + \overline{R_{m,i}} = \overline{x_m(t_i)} + \overline{R_{m,i}}, \quad m \in \mathbb{N}^*,$$

where $s_{m-1} : [a,b] \to \mathbb{R}$ is the natural cubic spline interpolating the values $\overline{x_{m-1}(t_i)}$, $i = \overline{0,n}$. The restrictions $s_{m-1}^{(i)}$ of s_{m-1} to the intervals $[t_{i-1}, t_i]$, $i = \overline{1,n}$ are:

$$s_{m-1}^{(i)}(t) = \left[\frac{(t-t_{i-1})^2}{2} - \frac{(t-t_{i-1})^3}{6h} - \frac{h(t-t_{i-1})}{3}\right] \cdot M_{m-1}^{(i-1)} + \frac{t-t_{i-1}}{h} \cdot \overline{x_{m-1}(t_i)} +$$

$$+ \left[\frac{(t-t_{i-1})^3}{6h} - \frac{h(t-t_{i-1})}{6}\right] \cdot M_{m-1}^{(i)} + \frac{t_i - t}{h} \cdot \overline{x_{m-1}(t_{i-1})}, \quad t \in [t_{i-1}, t_i], \ i = \overline{1,n}$$

where $h = \frac{b-a}{n}$ and the values $M_{m-1}^{(i)}$, $i = \overline{0,n}$, are computed using a recurrent algorithm considering $a_i = 1$, $i = \overline{1,n-1}$, $b_i = c_i = \frac{1}{4}$, $i = \overline{2,n-2}$, $b_1 = c_{n-1} = 0$, $c_1 = b_{n-1} = \frac{1}{4}$

$$d_i = \frac{3}{2h^2} \cdot \left[\overline{x_{m-1}(t_{i+1})} - 2\overline{x_{m-1}(t_i)} + \overline{x_{m-1}(t_{i-1})}\right], \quad i = \overline{1,n-1},$$

and computing recurrently

$$\alpha_1 = \frac{c_1}{a_1}, \quad \omega_i = a_i - \alpha_{i-1} \cdot b_i, \quad \alpha_i = \frac{c_i}{\omega_i}, \quad i = \overline{2,n-2},$$

$$\omega_{n-1} = a_{n-1} - \alpha_{n-2} \cdot b_{n-1}, z_1 = \frac{d_1}{2}, \quad z_i = \frac{d_i - b_i \cdot z_{i-1}}{\omega_i}, \quad i = \overline{2,n-1}.$$

By backward recurrence are obtained the moments:

$M_{m-1}^{(0)} = M_{m-1}^{(n)} = 0, \quad M_{m-1}^{(n-1)} = z_{n-1}, \quad M_{m-1}^{(i)} = z_i - \alpha_i \cdot M_{m-1}^{(i+1)}, \quad i = \overline{n-2, 1}.$

The algorithm has a practical stopping criterion: for given $\varepsilon' > 0$ and given $n \in \mathbb{N}^*$, previously chosen, find the first natural number $m \in \mathbb{N}^*$ such that $\left| \overline{x_m(t_i)} - \overline{x_{m-1}(t_i)} \right| < \varepsilon'$, $\forall i = \overline{1,n}$. We stop at this step m, retaining the approximations $\overline{x_m(t_i)}$, $i = \overline{1,n}$, of the solution.
The convergence of the method is proved by providing the error estimate and showing that this method is numerically stable with respect to the choice of the initial value. This kind of numerical stability is introduced by the author and can be summarized as follows.
Consider the same initial value problem with another initial value y_0 such that $|x_0 - y_0| < \varepsilon$,

$$\begin{cases} x'(t) = f(t, x(t), x(\varphi(t))), & t \in [a,b] \\ x(a) = y_0. \end{cases}$$

and applying the same algorithm it obtains the effective computed values $\overline{y_m(t_i)}$, $i = \overline{1,n}$, $m \in \mathbb{N}^*$ with $y_m(t_i) = \overline{y_m(t_i)} + \overline{R'_{m,i}}$, $i = \overline{1,n}$, $m \in \mathbb{N}^*$.
Definition (see Bica-[79]): We say that the algorithm of successive interpolations applied to the previous initial value problem is numerically stable with respect to the initial value if there exist $p \in \mathbb{N}^*$, a sequence of continuous functions $\mu_m : [0, b-a] \to [0, \infty)$, $m \in \mathbb{N}^*$ with the property $\lim_{h \to 0} \mu_m(h) = 0$, $\forall m \in \mathbb{N}^*$ and the constants $K_1, K_2, K_3 > 0$ which not depend by h, such that

$$\left| \overline{x_m(t_i)} - \overline{y_m(t_i)} \right| \le K_1 \varepsilon + K_2 \cdot h^p + K_3 \cdot \mu_m(h),$$

for all $i = \overline{1,n}$, $m \in \mathbb{N}^*$.
Analogously is applied the method of successive interpolations to initial value problems for second order functional differential equations in Bica-[78].

Similarly, in Bica-[71], Bica-Curilă-[70], [86], the method of successive interpolations is applied to two-point boundary value problems associated to second and fourth order functional differential equations

$$\begin{cases} x''(t) = f(t, x(t), x(\varphi(t))), & t \in [a,b] \\ x(a) = c, \quad x(b) = d \end{cases}$$

and respectively,

$$\begin{cases} x^{IV}(t) = f(t, x(t), x(\varphi(t)), & t \in [a,b] \\ x(a) = c, & x(b) = d \\ x'(a) = w, & x'(b) = r, \end{cases}$$

the numerical stability being considered with respect to the choice of the boundary values. The equivalent integral equation is

$$x(t) = g(t) + \int_a^b H(t,s) \cdot f(s, x(s), x(\varphi(s))) \, ds, \quad s \in [a,b]$$

where $H(t,s)$ is the corresponding Green's function,

$$g(t) = \frac{t-a}{b-a} \cdot d + \frac{b-t}{b-a} \cdot c$$

$$H(t,s) = -G(t,s) = \begin{cases} -\frac{(s-a)(b-t)}{b-a}, & \text{if } s \le t \\ -\frac{(t-a)(b-s)}{b-a}, & \text{if } s \ge t \end{cases}$$

for the second order equation and

$$g(t) = \frac{(b-t)^2 \left[2(t-a) + (b-a)\right]}{(b-a)^3} \cdot c + \frac{(t-a)^2 \left[2(b-t) + (b-a)\right]}{(b-a)^3} \cdot d +$$

$$+ \frac{(b-t)^2 (t-a)}{(b-a)^2} \cdot w - \frac{(t-a)^2 (b-t)}{(b-a)^2} \cdot r, \quad t \in [a,b]$$

$$H(t,s) = \begin{cases} \frac{1}{6}\left(\frac{s-a}{b-a}\right)^2 \left(1 - \frac{t-a}{b-a}\right)^2 \cdot \left[\left(\frac{t-a}{b-a} - \frac{s-a}{b-a}\right) + 2\left(1 - \frac{s-a}{b-a}\right)\left(\frac{t-a}{b-a}\right)\right], & s \le t \\ \frac{1}{6}\left(\frac{t-a}{b-a}\right)^2 \left(1 - \frac{s-a}{b-a}\right)^2 \cdot \left[\left(\frac{s-a}{b-a} - \frac{t-a}{b-a}\right) + 2\left(1 - \frac{t-a}{b-a}\right)\left(\frac{s-a}{b-a}\right)\right], & s \ge t \end{cases}$$

for the fourth order equation. The technique of successive approximations is applied to this integral equation combined with the trapezoidal quadrature rule and an interpolation procedure activated at each iterative step.

The method of successive interpolations is applied to functional integral equations in Bica-Oros-[56], Bica-[57], [72], Bica-Curilă-[74], the numerical stability concept being considered with respect to the first iteration.

The fourth chapter contains the results concerning the algebraic structure of the set of fuzzy numbers, obtained by the author in Bica-[67], and the presentation of the numerical methods for nonlinear fuzzy integral equations developed in Bica-[68], [77] and Bica-Popescu-[75], [80]. In Bica-[77] it is studied the algebraic and topological structure of the set of one-sided fuzzy numbers, introducing the notion of dual decomposition of a monoid and proving that the left-sided and right-sided fuzzy numbers realizes such a dual decomposition of the additive monoid of all fuzzy numbers. An interesting interpretation of right-sided fuzzy numbers with applications in epidemiology and medical statistics is provided. The Picard's technique of successive approximations is used to construct iterative numerical methods for nonlinear Fredholm and Hammerstein-Fredholm fuzzy integral equations in Bica-[68] and Bica-Popescu-[80], and for Volterra nonlinear fuzzy integral equations with constant delay in Bica-Popescu-[75].

The thesis contains a last chapter 5, where new directions of research derived from the results presented in the first four chapters are pointed out. In this context are presented new ideas and interesting open problems concerning possible extensions of the method of successive interpolations, and some of them are focused on new arithmetic operations for fuzzy numbers with improved algebraic properties and on the possibility to apply (and to extend) the notion of quadratic oscillation in average for parametrized curves and surfaces.