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ABSTRACT

This thesis presents the results obtained by the author in the field of approximation trajectories of dynamical systems, after obtaining his PhD degree. In the context of an particularly intense international research area, several papers were published in ISI indexed journals with high impact factor and other papers were presented at national and international ISI conferences.

In many practical applications, technical problems are modeled by nonlinear differential equations. Because an exact solution is often very difficult or even impossible to obtain, approximate analytical solutions are particularly important for the study of dynamic systems. In the case of approximate analytical solutions, the success of a certain approximation method depends on the nonlinearities that occur in the studied problem, and thus a general algorithm for the construction of such approximate solutions do not exist in the general case.

We present approximate analytical solutions for dynamical systems modeled by nonlinear differential equations frequently encountered in engineering problems and intensely studied in recent years in the literature, namely: the multi-pantograph equation with variable coefficients, nonlinear differential equations modelling heat transfer problems, nonlinear differential equations of the Lane-Emden type, Riccati differential equations, linear differential equations with delay, Volterra-Fredholm type integral equations, the regularized long wave equation and nonlinear differential equations modelling oscillatory phenomena with periodic solutions.

Approximate analytical solutions for the above dynamical systems are obtained by original analytical methods, proposed in the most recent years in literature: Polynomial Least Square Method (PLSM), Optimal Homotopy Asymptotic Method (OHAM), Optimal Homotopy Perturbation Method (OHPM), Fourier-Least Squares Method (FLSM).

These methods, published in the literature by the author of the thesis and his collaborators, allow the approximation of the trajectories of dynamical systems in an original manner.

The approximate analytical solutions depend on an initial set of parameters, called convergence control parameters, whose optimal values are determined such that a certain minimum of the attached functional problem is achieved. Analytical approaches are presented each time accompanied by numerical
simulations and compared with results previously obtained in the literature by other methods. The strong point of the presented methods consists in their ability to obtain approximate analytical solutions which are more accurate than the ones obtained by using other methods and, at the same time, they usually require a smaller number of iterations in comparison with other methods.

The last chapter presents some research directions related to possible extensions of the methods included in the present thesis.
1. Dynamical systems modeled by ordinary differential equations

1.1. $\varepsilon$-approximate polynomial solutions for the multi-pantograph equation with variable coefficients

1.1.1. Introduction

We give an approximate polynomial solution for the multi-pantograph equation [1,2,3], which is a functional-differential equation with proportional delays:

$$u'(t) - \lambda u(t) - \sum_{i=1}^{l} \mu_i(t) u(q_i t) - f(t) = 0, \quad t \geq 0 \tag{1}$$

under the initial condition

$$u(0) = \gamma \tag{2}$$

where $\lambda, \gamma$ are real constants, $l$ is a natural number, $0 < q_i < 1$ and $\mu_i(t), f(t)$ are analytical functions.

The term "pantograph" was first used by Ockendon and Tayler in the paper [4] which discussed the collection of electrical current by the pantograph head of an electric locomotive.

Since then, due to the fact that equations of the type (1,2) have numerous practical applications in fields such as electrodynamics, nonlinear dynamical systems, astrophysics, quantum mechanics e.t.c., they were studied by many
authors and both numerical and approximate analytical solutions for such
equations were proposed.
For example, M.Z.Liu et al. [5,6,7] and Bellen et al. [8] computed
numerical solutions using Runge-Kutta-type methods, Muroya et al. [9] and
Ishiwata et al. [10] computed numerical solutions using collocation-type
method.
Sezer et al. [1,12] computed approximate analytical solutions using Taylor
approximate analytical solutions using the Variational Iteration Method, N.
Abazari et al. [3] and Keskin et al. [14] computed approximate analytical
solutions using the Differential Transformation Method and D.J.Evens et al. [15]
computed approximate analytical solutions using the Adomian decomposition
method.
Properties of the numerical and analytical solutions for such equations are
also studied extensively, for example in [16,17,18,19,20].
The method presented in the next allows us to determine an approximate
polynomial solution for the pantograph equation (1,2) and will satisfy the
following conditions:
• for a fixed $n$, $n \in N$, the error relative to the exact solution of the
  problem must be smaller than $10^{-n}$.
• the degree of the approximating polynomial must be as small as
  possible.
An approximate polynomial solution for the pantograph equation is
presented in [1], where the approximate polynomial solution is of the fifth degree
and is obtained using a method based on a Taylor matrix method.
Present method allows us to obtain third degree polynomial approximations
but with an error relative to the exact solution smaller than the error obtained in
[1] for the fifth degree polynomials.

1.1.2 Approximation method description [Polynomial least
square method (PLSM)]

We consider the operator $D(u) = u(t) - \lambda u(t) - \sum_{i=1}^{I} \mu_i(t) u(q_i t) - f(t)$.
In the process of finding approximate solutions for differential equations, we are
interested in finding approximate polynomial solutions $u_{app}$ on the $[0, b]$ interval
($b > 0$), solutions which satisfy the following condition:
$$|R(t, u_{app})| < \varepsilon$$  
(3)
together with the condition:

\[ u_{\text{app}}(0) = \gamma \]  

(4)

Here

\[ R(t, u_{\text{app}}) = D(u_{\text{app}}(t)), \ t \in [0, b] \]  

(5)

represents the error obtained by replacing the exact solution \( u \) with the polynomial approximation \( u_{\text{app}} \).

We know that there exists a sequence of polynomials \( P_n \) for which \( R(t, P_n) \) converges to 0 for \( n \to \infty \).

The following definition makes sense:

An \( \varepsilon \)-approximate polynomial solution of the problem (1,2) is an approximate polynomial solution \( u_{\text{app}} \) satisfying the relations (3, 4).

A weak \( \delta \)-approximate polynomial solution of the problem (1,2) is an approximate polynomial solution \( u_{\text{app}} \) satisfying the relation

\[ \int_a^b (R(t, u_{\text{app}}))^2 \, dt < \delta \]

together with the initial condition (4).

We will find an approximative polynomial solution of the type:

\[ \tilde{u}(t) = \sum_{k=0}^n c_k t^k \]  

(6)

where the constants \( c_0, c_1, \ldots, c_n \) are calculated using the steps outlined in the following.

By substituting the approximate solution (6) in the equation (1) we obtain the following expression:

\[ \mathcal{R}(t, c_0, c_1, \ldots, c_n) = R(t, \tilde{u}) = \tilde{u}'(t) - \tilde{\lambda} \tilde{u}(t) - \sum_{i=1}^t \mu_i(t) \tilde{u}(q_i t) - f(t) \]  

(7)

If we could find the constants \( c_0^0, c_1^0, \ldots, c_n^0 \) such that \( \mathcal{R}(t, c_0^0, c_1^0, \ldots, c_n^0) = 0 \) for any \( t \in [0, b] \) and the equivalent of (2), \( \tilde{u}(0) = \gamma \) is also satisfied, then by substituting \( c_0^0, c_1^0, \ldots, c_n^0 \) in (6) we obtain the exact solution of (1,2). In general this situation is rarely encountered in the polynomial approximation methods.

Next we will attach to the pantograph problem the following real functional:
\[ J(c_1, ..., c_n) = \int_0^b \mathbb{R}^2(t, c_0, c_1, ..., c_n) \, dt, \quad b > 0 \]  
(8)

where \( c_0 \) is computed as a function of \( c_1, ..., c_n \) by using the initial condition \( \tilde{u}(0) = \gamma \).

The values \( c_1^0, ..., c_n^0 \) which give the minimum of the functional (8) will be computed from the conditions:

\[
\begin{align*}
\frac{\partial J}{\partial c_1} &= 0 \\
\frac{\partial J}{\partial c_2} &= 0 \\
& \vdots \\
\frac{\partial J}{\partial c_n} &= 0
\end{align*}
\]
(9)

\( c_0^0 \) is again computed as a function of \( c_1^0, ..., c_n^0 \) by using the initial condition.

Using the constants \( c_0^0, c_1^0, ..., c_n^0 \) thus determined, we consider the polynomial:

\[ T_n(t) = \sum_{k=0}^{n} c_k^0 t^k \]  
(10)

The following convergence theorem holds:

**Convergence theorem.** \( \forall \varepsilon > 0, \exists n_0 \in N \) such that \( \forall n \in N, n > n_0 \) it follows that \( T_n \) is a weak \( \varepsilon \)-approximate polynomial solution of the problem (1,2).

**Remark.** Any \( \varepsilon \)-approximate polynomial solution of the problem (1,2) is also a weak \( \varepsilon^2 \cdot b \)-approximate polynomial solution, but the opposite is not always true.

Taking into account the above remark, in order to find \( \varepsilon \)-approximate polynomial solutions of the problem (1,2) we will first determine weak approximate polynomial solutions, \( \tilde{u}_{\text{app}} \). If \( |R(t, \tilde{u}_{\text{app}})| < \varepsilon \) then \( \tilde{u}_{\text{app}} \) is also a \( \varepsilon \)-approximate polynomial solution of the problem.

**Remark.** The constants \( c_0^0, c_1^0, ..., c_n^0 \) can also be determined using other methods, such as, for example, collocation-type methods. The method described above has the advantage, in comparison with other methods, that it can be
applied not only for weakly-nonlinear equations but also for strong nonlinear
differential and integro-differential equations.

1.1.3. Results and discussion

In the next we will compare results obtained using PLSM with results obtained in [1], by means of the two cases presented there. As the approximate solution given analytically in [1] consists of polynomials of 5th degree, we used the same degree when we computed our approximations.

We consider the pantograph equation:

\[
\begin{align*}
\frac{u'(t)}{2} + u(t) - \frac{q_1}{2} u(q_1 t) + \frac{q_2}{2} e^{-\gamma t} - \frac{q_3}{2} u(q_3 t) + \frac{q_4}{2} e^{-\gamma q_4 t} - \frac{q_5}{2} u(q_5 t) + \frac{q_6}{2} e^{-\gamma q_6 t} &= 0 \\
u(0) &= 1
\end{align*}
\]  
(11)

on \([0,1]\).

The exact solution of this equation is \(u_e(t) = e^{-t}\).

The approximate solution given in [1] for the equation (11) is

\[u_a(t) = 1 - t + 0.5t^2 - 0.166666666t^3 + 0.041666666t^4 - 0.008333333t^5\]

If we use a 5th degree polynomial of the form (6), the expression of the remainder (7) becomes:

\[
\begin{align*}
\Re(t, c_0, c_1, c_2, c_3, c_4, c_5) &= -0.2c_0 + c_1 + 0.13c_1t + 2c_2t + 0.313c_2t^2 + 3c_3t^2 + 0.4351c_3t^3 + \\
&+ 4c_4t^3 + 0.52513c_4t^4 + 5c_5t^4 + 0.595363c_5t^5 + 0.45e^{-0.9t} + 0.4e^{-0.8t} + 0.25e^{-0.5t} + 0.1e^{-0.2t}
\end{align*}
\]

From the initial condition we obtain \(c_0 = 1\) and by minimizing the functional (8) we obtain the coefficients:

\[
c_1^0 = -0.999978, \quad c_2^0 = 0.499673, \quad c_3^0 = -0.165106, \quad c_4^0 = 0.0384026, \quad c_5^0 = -0.00511201
\]

The 5th degree polynomial approximate analytical solution obtained using our method is:

\[\tilde{u}_{app}(t) = 1 - 0.999978t + 0.499673t^2 - 0.165106t^3 + 0.0384026t^4 - 0.00511201t^5\]

The following plot contains the graphical representation of this polynomial (green solid line) together with the graphical representation of exact solution (red dashed line). The two graphical representations are again overlapping.
Figure 1. Comparison between the exact solution (dashed line) and the 5th degree polynomial approximation (solid line).

The graphical representation of the error $R(t, \tilde{u}_{app})$ is presented in figure 2:

Figure 2. Graphical representation of the error for the case of a 5th degree polynomial approximation.

We observe that the maximal absolute value of the error in this case is less than 0.00002.

In order to make a comparison with the polynomial approximation given in [1], in the following table we listed several values on the [0, 1] interval for both approximate solutions (our 5th degree polynomial solution $\tilde{u}_{app}$ and the 5th
degree polynomial solution $u_0$ given in [1]), together with the corresponding errors ($\varepsilon_{u_{app}}$ and $\varepsilon_{u_{0}}$, respectively), computed as the absolute values for the difference between the exact solution and the polynomial approximation.

Table 1.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\tilde{u}_{app}$</th>
<th>$\varepsilon_{app}$</th>
<th>$u_{0}$</th>
<th>$\varepsilon_{u_{0}}$</th>
<th>Exact solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.904839</td>
<td>1.69423 x 10^{-2}</td>
<td>0.904837</td>
<td>7.03239 x 10^{-4}</td>
<td>0.904837</td>
</tr>
<tr>
<td>0.2</td>
<td>0.81973</td>
<td>5.25314 x 10^{-2}</td>
<td>0.819731</td>
<td>8.10876 x 10^{-4}</td>
<td>0.819731</td>
</tr>
<tr>
<td>0.3</td>
<td>0.740918</td>
<td>3.5555 x 10^{-2}</td>
<td>0.740817</td>
<td>9.52728 x 10^{-4}</td>
<td>0.740818</td>
</tr>
<tr>
<td>0.4</td>
<td>0.67032</td>
<td>2.99639 x 10^{-2}</td>
<td>0.670315</td>
<td>5.35884 x 10^{-4}</td>
<td>0.67032</td>
</tr>
<tr>
<td>0.5</td>
<td>0.606531</td>
<td>6.17275 x 10^{-2}</td>
<td>0.60651</td>
<td>9.00626 x 10^{-4}</td>
<td>0.60651</td>
</tr>
<tr>
<td>0.6</td>
<td>0.549812</td>
<td>2.58654 x 10^{-2}</td>
<td>0.549732</td>
<td>9.0006394 x 10^{-4}</td>
<td>0.549812</td>
</tr>
<tr>
<td>0.7</td>
<td>0.496565</td>
<td>3.74656 x 10^{-2}</td>
<td>0.49647</td>
<td>8.0001461 x 10^{-4}</td>
<td>0.49655</td>
</tr>
<tr>
<td>0.8</td>
<td>0.449323</td>
<td>4.99324 x 10^{-2}</td>
<td>0.449203</td>
<td>8.0000323 x 10^{-4}</td>
<td>0.449329</td>
</tr>
<tr>
<td>0.9</td>
<td>0.40657</td>
<td>1.75387 x 10^{-2}</td>
<td>0.406517</td>
<td>8.0006523 x 10^{-4}</td>
<td>0.40657</td>
</tr>
<tr>
<td>1</td>
<td>0.367879</td>
<td>1.2378 x 10^{-2}</td>
<td>0.36667</td>
<td>8.0011211 x 10^{-4}</td>
<td>0.367879</td>
</tr>
</tbody>
</table>

While close to 0 the method proposed in [1] yield accurate approximations, the farther away from 0 it goes, the larger the error obtained. Our method presents a fairly even distributed low error, and thus its overall precision is better than the one in [1]. This can also be clearly observed in the following figure, which presents the absolute values for the difference between the exact solution and the polynomial approximation $u_0$ given in [1] and for the difference between the exact solution and our polynomial approximation $\tilde{u}_{app}$. 
Figure 3. The absolute values for the difference between the exact solution and the polynomial approximation \( u_\text{a} \) given in [1] (red line) and for the difference between the exact solution and our 5th degree polynomial approximation \( \tilde{u}_{\text{app}} \) (green line)

As a remark, if we use a cubic polynomial of the form (6), by computing the minimum of the corresponding functional (8) we obtain the following cubic polynomial approximate solution of problem (11):

\[
\tilde{u}_{\text{app}}(t) = 1 - 0.99428t + 0.464971t^2 - 0.102812t^3
\]

If we plot on the same figure the function which gives the absolute value of the difference between the exact solution and this cubic polynomial approximation (in blue) and the function which gives the absolute value of the difference between the exact solution and the polynomial approximation \( u_\text{a} \) given in [1] (in red) we obtain the plot in figure 4.

From the point of view of the overall accuracy of the approximate solution, our cubic approximation seems to be comparable with the 5th degree approximate solution
Figure 4. The absolute values for the difference between the exact solution and the polynomial approximation \( u_a \) given in [1] (red) and for the difference between the exact solution and our cubic polynomial approximation \( \tilde{u}_{app} \) (blue).

We consider next the equation:

\[
\begin{aligned}
&u'(t) - \frac{1}{2} e^t u(t) - \frac{1}{2} u(t) = 0 \\
u(0) = 1
\end{aligned}
\]  

(12)

on \([0,1]\).

The exact solution of this equation is \( u_e(t) = e^t \).

The approximate solution given in [1] for the equation (12) is:

\[ u_a(t) = 1 + t + 0.5t^2 + 0.166666666t^3 + 0.041666666t^4 + 0.00833333t^5. \]

Our 5th degree polynomial approximate analytical solution is:

\[ \tilde{u}_{app}(t) = 1 + 1.00005t + 0.49229t^2 + 0.1701729t^3 + 0.03494257t^4 + 0.0138859t^5. \]

The following plot contains the graphical representation of this polynomial (green solid line) together with the graphical representation of exact solution (red dashed line). The two graphical representations are, again, practically overlapping.
Figure 5. Comparison between the exact solution (dashed line) and the 5th degree polynomial approximation (solid line).

The graphical representation of the error $R(t, \tilde{u}_{app})$ is:

Figure 6. Graphical representation of the error for the case of a 5th degree polynomial approximation.
We observe that the maximal absolute value of the error in this case is less than 0.00006.

The values for the errors $\varepsilon_{\text{app}}$ (computed as the absolute values for the difference between the exact solution and our 5th degree polynomial approximation) and $\varepsilon_{u_5}$ (computed as the absolute values for the difference between the exact solution and the 5th degree polynomial solution $u_5$ given in [1]), presented in Table 2 confirm the observations we made in Example 1, i.e. our method yield a much better overall precision. In fact for the case of the Example 2, only for values of $t$ up to 0.2 the approximation $u_5$ presents a better accuracy, while for the rest of values of $t$ up to 1 its accuracy is significantly lower that the accuracy given by $\tilde{u}_{\text{app}}$, and moreover, it gradually worsens.

### Table 2:

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\tilde{u}_{\text{app}}$</th>
<th>$\varepsilon_{\text{app}}$</th>
<th>$u_5$</th>
<th>$\varepsilon_{u_5}$</th>
<th>Exact solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.</td>
<td>1.0517</td>
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<td>1.0517</td>
<td>1.4635 x 10^{-3}</td>
<td>1.0517</td>
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<td>1.3173 x 10^{-5}</td>
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<td>9.1556 x 10^{-3}</td>
<td>1.2214</td>
</tr>
<tr>
<td>0.2</td>
<td>1.34906</td>
<td>2.52703 x 10^{-5}</td>
<td>1.34906</td>
<td>1.60702 x 10^{-3}</td>
<td>1.34906</td>
</tr>
<tr>
<td>0.3</td>
<td>1.40103</td>
<td>7.5794 x 10^{-3}</td>
<td>1.40103</td>
<td>6.40316 x 10^{-3}</td>
<td>1.40102</td>
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<tr>
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<td>1.64072</td>
<td>1.6642 x 10^{-4}</td>
<td>1.64072</td>
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</tr>
<tr>
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</tr>
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</tr>
<tr>
<td>1.</td>
<td>2.71820</td>
<td>3.9546 x 10^{-8}</td>
<td>2.71820</td>
<td>6.00001518</td>
<td>2.71820</td>
</tr>
</tbody>
</table>

The graphical representations of $\varepsilon_{\text{app}}$ (green curve) and $\varepsilon_{u_5}$ (red curve) is also presented in figure 7:
1.1.4. Conclusions

The examples presented here show that our method is a performant one for equations such as the pantograph equation. For all the examples considered, our 5th degree polynomial approximation $\tilde{u}_{\text{app}}$ presented a better overall accuracy than the 5th degree polynomial approximation $u_5$ in [1].

Moreover, this method can be easily extended for other types of equations and systems of equations, such as nonlinear systems, nonlinear differential systems, integral equations, and as such it can be considered a powerful tool for the computation of approximate solutions for nonlinear problems.
1.2. Approximate polynomial solutions for nonlinear heat transfer problems

1.2.1. Introduction

Various approximation techniques were employed to study the nonlinear heat transfer problems, the best known ones including perturbation-type methods such as the homotopy perturbation method (HPM) introduced by He ([21],[22],[23]) and the homotopy analysis method (HAM) introduced by Liao ([24],[25],[26]). These methods eliminate the necessity of a small parameter which rendered the traditional perturbation method largely inefficient for problems containing strong nonlinearities. HPM, HAM and other methods based on them ([27],[28],[29]) were successfully employed to find approximate analytical solutions for some well-known nonlinear heat transfer problems such as:

- The cooling of a lumped system involving combined modes of convection and radiation heat transfer ([28],[30],[31],[32],[33],[34],[35])
- The heat transfer with conduction in a slab of a material with temperature dependent thermal conductivity ([34],[35],[29])
- The temperature distribution equation in a thick rectangular fin radiating to free space ([32],[33],[28])

The polynomial least square method allows us to determine analytical approximate polynomial solutions for all of the above problems. We will compare our approximate solutions with approximate solutions presented in ([28],[29],[30],[32],[23],[34],[31],[35]). The computations show that our method allows us to obtain approximations with an error relative to the exact or numerical solution smaller than the errors obtained using other methods.

1.2.2 Approximation method description

The polynomial least square method (PLSM), allows us to find analytical approximate polynomial solutions for a problem consisting of a nonlinear differential equation of order $n$ and some boundary conditions:

$$u^{(n)}(t) = F \left( u^{(n-1)}(t), u^{(n-2)}(t), \ldots, u^{(1)}(t), u(t), t \right).$$ (13)

$$\alpha_{n-1} u^{(n-1)}(a) + \alpha_{n-2} u^{(n-2)}(a) + \ldots + \alpha_1 u^{(1)}(a) + \alpha_0 u(a) = 0$$
$$\beta_{n-1} u^{(n-1)}(b) + \beta_{n-2} u^{(n-2)}(b) + \ldots + \beta_1 u^{(1)}(b) + \beta_0 u(b) = 0$$ (14)
Here $F$ is a continuous function, $t \in [a,b]$ and $\alpha_i, \beta_i$ are real constants. We consider the operator:

$$D(u) = u^{(n)}(t) - F(u^{(n-1)}(t), u^{(n-2)}(t), ..., u^{(1)}(t), u(t), t).$$  \hfill (15)

If $u_{app}$ is an approximate solution of the equation (13), the error obtained by replacing the exact solution $u$ with the approximation $u_{app}$ is given by the remainder:

$$R(t, u_{app}) = D(u_{app}(t)), \quad t \in [a,b]$$ \hfill (16)

We will find approximate polynomial solutions $u_{app}$ of (13, 14) on the $[a,b]$ interval, solutions which satisfy the following conditions:

$$|R(t, u_{app})| < \varepsilon$$ \hfill (17)

\begin{align*}
\alpha_{n-1} u^{(n-1)}_{app}(a) + \alpha_{n-2} u^{(n-2)}_{app}(a) + \cdots + \alpha_1 u^{(1)}_{app}(a) + \alpha_0 u_{app}(a) &= 0 \\
\beta_{n-1} u^{(n-1)}_{app}(b) + \beta_{n-2} u^{(n-2)}_{app}(b) + \cdots + \beta_1 u^{(1)}_{app}(b) + \beta_0 u_{app}(b) &= 0
\end{align*} \hfill (18)

An $\varepsilon$-approximate polynomial solution of the problem (13, 14) is an approximate polynomial solution $u_{app}$ satisfying the relations (17, 18).

A weak $\delta$-approximate polynomial solution of the problem (13, 14) is an approximate polynomial solution $u_{app}$ satisfying the relation:

$$\int_a^b |R(t, u_{app})| \, dt \leq \delta$$

together with the initial condition (18).

We consider the sequence of polynomials $P_m(t) = a_0 + a_1 t + \ldots + a_m t^m$, $a_i \in \mathbb{R}$, $i = 0, 1, \ldots, m$ satisfying the conditions:

\begin{align*}
\alpha_{n-1} P^{(n-1)}_m(a) + \alpha_{n-2} P^{(n-2)}_m(a) + \cdots + \alpha_1 P^{(1)}_m(a) + \alpha_0 P_m(a) &= 0 \\
\beta_{n-1} P^{(n-1)}_m(b) + \beta_{n-2} P^{(n-2)}_m(b) + \cdots + \beta_1 P^{(1)}_m(b) + \beta_0 P_m(b) &= 0
\end{align*}

We say that the sequence of polynomials $P_m(t)$ is convergent to the solution of the problem (13, 14) if $\lim_{m \to \infty} D(P_m(t)) = 0$.

We will find a weak $\varepsilon$-polynomial solution of the type:
where the constants $c_0, c_1, \ldots, c_m$ are calculated using the steps outlined in the following.

- By substituting the approximate solution (19) in the equation (13) we obtain the following expression:

\[
\Re(t, c_0, c_1, \ldots, c_m) = R(t, \tilde{u}) = \tilde{u}^{(n)}(t) - F(\tilde{u}^{(n-1)}(t), \tilde{u}^{(n-2)}(t), \ldots, \tilde{u}^{(1)}(t), \tilde{u}(t), t)
\]

(20)

- If we could find the constants $c_0^0, c_1^0, \ldots, c_m^0$ such that

\[
\Re(t, c_0^0, c_1^0, \ldots, c_m^0) = 0 \quad \text{for any} \quad t \in [a, b] \quad \text{and the equivalents of (2.14):

\[
\alpha_{n-1} \tilde{u}^{(n-1)}(a) + \alpha_{n-2} \tilde{u}^{(n-2)}(a) + \ldots + \alpha_1 \tilde{u}^{(1)}(a) + \alpha_0 \tilde{u}(a) = 0,
\beta_{n-1} \tilde{u}^{(n-1)}(b) + \beta_{n-2} \tilde{u}^{(n-2)}(b) + \ldots + \beta_1 \tilde{u}^{(1)}(b) + \beta_0 \tilde{u}(b) = 0
\]

(21)

are also satisfied, then by substituting $c_0^0, c_1^0, \ldots, c_m^0$ in (19) we obtain the exact solution of (13, 14). In general this situation is rarely encountered in polynomial approximation methods.

- Next we attach to the problem (13, 14) the following real functional:

\[
J(c_2, c_3, \ldots, c_m) = \int_a^b \Re^2(t, c_0, c_1, \ldots, c_m) dt
\]

(22)

where $c_0, c_1$ are computed as functions of $c_2, c_3, \ldots, c_m$ by using the initial conditions (21).

- We compute the values of $c_2^0, c_3^0, \ldots, c_m^0$ as the values which give the minimum of the functional (22) and the values of $c_0^0, c_1^0$ again as

\[
T_n(t) = \sum_{k=0}^m c_i^0 t^k
\]

(23)

Using the constants $c_0^0, c_1^0, \ldots, c_m^0$ thus determined, we consider the polynomial:
The following convergence theorem holds:

**Theorem 1.** If the sequence of polynomials \( P_m(t) \) converges to the solution of the problem (13, 14), then the sequence of polynomials \( T_m(t) \) from (23) satisfies the property:

\[
\lim_{m \to \infty} \int_a^b R^2(t, T_m) \, dt = 0
\]

Moreover, \( \forall \epsilon > 0, \exists m_0 \in \mathbb{N} \) such that \( \forall m \in \mathbb{N}, m > m_0 \) it follows that \( T_m(t) \) is a weak-\( \epsilon \)-approximate polynomial solution of the problem (13, 14).

**Remark 1.** Any \( \epsilon \)-approximate polynomial solution of the problem (13, 14) is also a weak \( \epsilon^2 \cdot (b - a) \)-approximate polynomial solution, but the opposite is not always true. It follows that the set of weak approximate solutions of the problem (13, 14) also contains the approximate solutions of the problem.

Taking into account the above remark, in order to find \( \epsilon \)-approximate polynomial solutions of the problem (13, 14) by PLSM we will first determine weak approximate polynomial solutions, \( \tilde{u}_{app} \). If \( |R(t, \tilde{u}_{app})| < \epsilon \) then \( \tilde{u}_{app} \) is also an \( \epsilon \)-approximate polynomial solution of the problem.

**1.2.3. Results and discussion**

**1.2.3.1. The cooling of a lumped system involving combined modes of convection and radiation heat transfer**

We consider a lumped system of combined convective-radiative heat transfers. The specific heat coefficient is a linear function of temperature ([30], [31]):

\[ c = c_a (1 + \beta (T - T_a)) \]

where \( \beta \) is a constant and \( c_a \) is the specific heat at \( T_a \). The problem which describes the cooling process of the system is:

\[
\rho V c \frac{dT}{d\tau} + h A (T - T_a) + E \sigma A (T^4 - T_s^4) = 0
\]

\[ T(0) = T_i \]

In order to obtain the nondimensional form of the problem, we perform the following changes of variables:
\[ u = \frac{T}{T_i}, u_a = \frac{T_a}{T_i}, t = \frac{\tau(hA)}{\rho V c_a}, \varepsilon_1 = \beta T_i, \varepsilon_2 = \frac{E \sigma T_i^3}{h}, u_s = \frac{T_s}{T_i}. \]

Assuming for simplicity \( u_a = u_s = 0 \), we obtain the following problem:

\[ (1 + \varepsilon_1 u) \frac{du}{dt} + u + \varepsilon_2 u^4 = 0 \]

\[ u(0) = 1 \]

**Case 1:** \( \varepsilon_1 \neq 0, \varepsilon_2 \neq 0 \)

In ([30]), by using the HPM, Ganji and Rajabi computed the following approximate solution of equation (24):

\[
\begin{align*}
u_{\text{HPM}} &= \frac{3}{2} e^{-3t \varepsilon_1^2} - e^{-2t \varepsilon_1} - 2e^{-2t} \left( \varepsilon_1 - \frac{\varepsilon_2}{3} \right) \varepsilon_1 + \frac{7}{12} e^{-5t} \varepsilon_2 \varepsilon_1 + \\
&+ \frac{2}{9} e^{-7t \varepsilon_2^2} + e^{-t} \left( \varepsilon_1 - \frac{\varepsilon_2}{3} \right) + \frac{1}{3} e^{-4t} \varepsilon_2 + \frac{4}{3} e^{-4t} \left( \varepsilon_1 - \frac{\varepsilon_2}{3} \right) \varepsilon_2 + \\
&+ e^{-t} \left( - \frac{3 \varepsilon_2^2}{2} + 2 \left( \varepsilon_1 - \frac{\varepsilon_2}{3} \right) \varepsilon_1 + \frac{17 \varepsilon_2 \varepsilon_1}{12} - \frac{2 \varepsilon_2^2}{9} - \frac{4}{3} \left( \varepsilon_1 - \frac{\varepsilon_2}{3} \right) \varepsilon_2 \right) + e^{-t}
\end{align*}
\]

In ([31]), by using the HAM, Abbasbandy computed the following approximate solution of (24):

\[
\begin{align*}
u_{\text{HAM}} &= \frac{3}{2} e^{-3t \varepsilon_1^2} h^2 + \frac{2}{9} e^{-7t \varepsilon_2^2} h^2 - \frac{17}{12} e^{-5t} \varepsilon_1 \varepsilon_2 h^2 + \\
&+ e^{-2t} \left( \varepsilon_1 (h^2 + 2h) - 2 \varepsilon_1 \left( \varepsilon_1 - \frac{\varepsilon_2}{3} \right) h^2 \right) + \\
&+ e^{-4t} \left( \frac{4}{3} \left( \varepsilon_1 - \frac{\varepsilon_2}{3} \right) \varepsilon_2 h^2 + \frac{1}{2} \varepsilon_2 (-h^2 - 2h) \right) + \\
&+ e^{-t} \left( \left( \frac{\varepsilon_1^2}{2} - \frac{7 \varepsilon_2 \varepsilon_1}{12} + \frac{2 \varepsilon_2^2}{9} \right) h^2 - \left( \varepsilon_1 - \frac{\varepsilon_2}{3} \right) (h + 2) h + 1 \right)
\end{align*}
\]

Using the PLSM, we computed the following third order polynomial approximate solution of equation (24):

\[ u_{\text{PLSM}} = -0.218521 t^3 + 0.620281 t^2 - 0.954465 t + 1. \]

In order to compare our solution \( u_{\text{PLSM}} \) with the previous solutions \( u_{\text{HPM}} \) and \( u_{\text{HAM}} \), since equation (24) does not have a known exact solution, we computed for each approximate solution the relative error as the difference (in
absolute value) between the approximate solution and the numerical solution. Table 3 presents the values of the relative errors computed for a set of equidistant values of \( t \) on the \([0,1]\) interval and for the values of the parameters \( \varepsilon_1 = 1, \varepsilon_2 = 1, \text{ and } \hbar = -0.1: \)

**Table 3. Comparison of HPM, HAM and PLSM for \( \varepsilon_1 = 1, \varepsilon_2 = 1 \)**

<table>
<thead>
<tr>
<th>( t )</th>
<th>HPM</th>
<th>HAM</th>
<th>PLSM 3rd deg.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.2092</td>
<td>1.866 ( 10^{-2} )</td>
<td>1.70402 ( 10^{-3} )</td>
</tr>
<tr>
<td>0.2</td>
<td>7.24274 ( 10^{-1} )</td>
<td>5.82112 ( 10^{-3} )</td>
<td>4.66941 ( 10^{-4} )</td>
</tr>
<tr>
<td>0.3</td>
<td>4.27601 ( 10^{-1} )</td>
<td>6.18631 ( 10^{-3} )</td>
<td>1.19348 ( 10^{-3} )</td>
</tr>
<tr>
<td>0.4</td>
<td>2.47275 ( 10^{-1} )</td>
<td>1.70979 ( 10^{-2} )</td>
<td>2.12971 ( 10^{-3} )</td>
</tr>
<tr>
<td>0.5</td>
<td>1.38919 ( 10^{-1} )</td>
<td>2.68093 ( 10^{-2} )</td>
<td>1.99599 ( 10^{-3} )</td>
</tr>
<tr>
<td>0.6</td>
<td>7.49580 ( 10^{-2} )</td>
<td>3.52867 ( 10^{-2} )</td>
<td>9.44512 ( 10^{-4} )</td>
</tr>
<tr>
<td>0.7</td>
<td>3.81870 ( 10^{-2} )</td>
<td>4.25329 ( 10^{-2} )</td>
<td>5.48908 ( 10^{-4} )</td>
</tr>
<tr>
<td>0.8</td>
<td>1.78647 ( 10^{-2} )</td>
<td>4.85751 ( 10^{-2} )</td>
<td>1.78989 ( 10^{-3} )</td>
</tr>
<tr>
<td>0.9</td>
<td>7.30380 ( 10^{-3} )</td>
<td>5.34593 ( 10^{-2} )</td>
<td>1.93210 ( 10^{-3} )</td>
</tr>
</tbody>
</table>

It is easy to see that the approximate solution given by PLSM is much closer to the numerical solution that the previous ones from ([30],[31]), while, at the same time, it has a much simpler form.

The precision of our method is clearly illustrated in figure 8, which presents the relative error functions on the \([0,1]\) interval. The graphical representation of the relative error corresponding to HPM is represented by a dotted line, the one corresponding to HAM by a dashed line and the one corresponding to PLSM by a solid line.
Case 2: \( \varepsilon_1 \neq 0, \varepsilon_2 \neq 0 \)

If \( \varepsilon_2 = 0 \), the problem (24) becomes:

\[
(1 + \varepsilon_1 u) \frac{du}{dt} + u = 0
\]

\( u(0) = 1 \)

The problem (25) admits an exact solution, \( u_e \), which is the solution of the algebraic equation \( \varepsilon u + ln(u) + t = 0 \).

In ([32]), by using the HPM, Ganji computed an approximate solution of problem (25):

\[
u_{HPM} = \left( \frac{3e^{-3t}}{2} + 2e^{-2t} + \frac{e^{-t}}{2} \right) \varepsilon_1^2 + (e^{-t} - e^{-2t}) \varepsilon_1 + e^{-t}
\]

In ([33]), by using the HAM, Abbasbandy computed an approximate solution of problem (25):

\[
u_{HAM} = \frac{3}{2} e^{-3t} \varepsilon_1^2 h^2 + e^{-2t}(\varepsilon_1 h(h + 1) - 2\varepsilon_1^2 h^2) + e^{-t} \left( \frac{1}{2} \varepsilon_1^2 h^2 - \varepsilon_1 h(h + 1) \right) +
\]

\[+ e^{-2t} \varepsilon_1 h - e^{-t} \varepsilon_1 h + e^{-t}\]

In ([28]), by using the optimal homotopy asymptotic method (OHAM), Marinca and Herișanu computed an approximate solution of problem (25):
Using the PLSM, we computed the following second order polynomial approximate solution of equation (25):

\[ u_{\text{PLSM}} = 0.0690567 t^2 - 0.501923 t + 1. \]

We compare again our solution \( u_{\text{SRMM}} \) with the previous solutions given by \( u_{\text{HPM}}, u_{\text{HAM}} \) and \( u_{\text{OHAM}} \):

<table>
<thead>
<tr>
<th>( t )</th>
<th>HPM</th>
<th>HAM</th>
<th>OHAM</th>
<th>PLSM 2nd deg.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>3.35016 ( 10^{-2} )</td>
<td>3.67248 ( 10^{-5} )</td>
<td>3.70848 ( 10^{-2} )</td>
<td>1.56178 ( 10^{-3} )</td>
</tr>
<tr>
<td>0.2</td>
<td>4.34569 ( 10^{-2} )</td>
<td>1.95406 ( 10^{-3} )</td>
<td>5.36624 ( 10^{-2} )</td>
<td>8.02446 ( 10^{-5} )</td>
</tr>
<tr>
<td>0.3</td>
<td>4.02973 ( 10^{-2} )</td>
<td>4.09182 ( 10^{-2} )</td>
<td>5.65858 ( 10^{-2} )</td>
<td>1.13819 ( 10^{-3} )</td>
</tr>
<tr>
<td>0.4</td>
<td>3.07182 ( 10^{-2} )</td>
<td>5.41548 ( 10^{-3} )</td>
<td>5.11379 ( 10^{-2} )</td>
<td>1.66928 ( 10^{-3} )</td>
</tr>
<tr>
<td>0.5</td>
<td>1.88650 ( 10^{-2} )</td>
<td>5.54168 ( 10^{-3} )</td>
<td>4.11884 ( 10^{-2} )</td>
<td>1.73074 ( 10^{-3} )</td>
</tr>
<tr>
<td>0.6</td>
<td>7.17017 ( 10^{-3} )</td>
<td>4.43298 ( 10^{-3} )</td>
<td>2.94297 ( 10^{-2} )</td>
<td>1.37979 ( 10^{-3} )</td>
</tr>
<tr>
<td>0.7</td>
<td>3.06275 ( 10^{-3} )</td>
<td>2.24327 ( 10^{-3} )</td>
<td>1.76287 ( 10^{-2} )</td>
<td>6.73670 ( 10^{-4} )</td>
</tr>
<tr>
<td>0.8</td>
<td>1.12534 ( 10^{-2} )</td>
<td>7.82736 ( 10^{-4} )</td>
<td>6.85581 ( 10^{-3} )</td>
<td>3.30402 ( 10^{-4} )</td>
</tr>
<tr>
<td>0.9</td>
<td>1.72676 ( 10^{-2} )</td>
<td>4.37277 ( 10^{-3} )</td>
<td>2.32003 ( 10^{-3} )</td>
<td>1.57520 ( 10^{-3} )</td>
</tr>
</tbody>
</table>

This time we computed for each approximate solution the relative error as the difference (in absolute value) between the approximate solution and the exact solution \( u_e \).

Table 4 presents the values of the relative errors for the value of the parameters \( \nu_1 = 1, \ h = -0.6 \). The approximate solution given by PLSM is much closer to the exact solution that the previous ones from ([32],[33],[28]), while, at the same time, it has a much simpler form.

Figure 9 presents the relative error functions on the \([0,1]\) interval. The graphical representation of the relative error corresponding to HPM is represented by a dotted line, the one corresponding to HAM by a dotted-dashed line, the one corresponding to OHAM by a dashed line and the one corresponding to PLSM by a solid line.
Case 3: $\epsilon_1 = 0, \epsilon_2 \neq 0$

If $\epsilon_1 = 0$, the problem (24) becomes:

$$\frac{du}{dt} + u + \epsilon_2 u^4 = 0$$

$$u(0) = 1$$

(26)

In ([34]), by using the HPM, Rajabi, Ganji and Taherian computed an approximate solution of problem (25):

$$u_{HPM} = \frac{2}{9}(e^{-7t} + 2e^{-4t} + e^{-t})\epsilon_2^2 + \frac{1}{3}(e^{-4t} - e^{-t})\epsilon_2 + e^{-t}$$

In ([35]), by using the HAM, Domairry and Nadim computed an approximate solution of problem (26):

$$u_{HAM} = \frac{4}{45}e^{-9t}\epsilon_2^2h^2 + \frac{1}{5}e^{-4t}\epsilon_2^2h^2 - \frac{2}{15}e^{-6t}\epsilon_2h^2 - \frac{1}{5}e^{-t}\epsilon_2h^2 - \frac{2}{5}e^{-5t}\epsilon_2h^2 +$$

$$+ e^{-t} + \frac{1}{9}\epsilon_2^2h^2 + \frac{\epsilon_2h^2}{3} + \frac{2\epsilon_2h}{5}$$
Using the PLSM, we computed the following 5th order polynomial approximate solution of equation (26):

\[ u_{PLSM} = -1.1979t^5 + 3.86534t^4 - 4.98357t^3 + 3.53843t^2 - 1.92784t + 1. \]

The comparison between our solution \( u_{PLSM} \) and the previous solutions given by \( u_{HPM} \) and \( u_{HAM} \) is presented in table 3. Since no exact solution for the equation (26) is known, we computed the relative error as the difference (in absolute value) between the approximate solution and the numerical solution. Table 5 presents the values of the relative errors for the value of the parameters \( \varepsilon_2 = 1, \hbar = -0.8 \):

<table>
<thead>
<tr>
<th>( t )</th>
<th>HPM</th>
<th>HAM</th>
<th>PLSM 5th deg.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>2.24219 \times 10^{-3}</td>
<td>1.18868 \times 10^{-4}</td>
<td>6.08673 \times 10^{-5}</td>
</tr>
<tr>
<td>0.2</td>
<td>9.47661 \times 10^{-3}</td>
<td>8.71208 \times 10^{-4}</td>
<td>1.25683 \times 10^{-3}</td>
</tr>
<tr>
<td>0.3</td>
<td>1.79917 \times 10^{-2}</td>
<td>1.25548 \times 10^{-3}</td>
<td>3.12286 \times 10^{-4}</td>
</tr>
<tr>
<td>0.4</td>
<td>2.51387 \times 10^{-2}</td>
<td>1.54052 \times 10^{-3}</td>
<td>9.67612 \times 10^{-4}</td>
</tr>
<tr>
<td>0.5</td>
<td>3.00464 \times 10^{-2}</td>
<td>2.09832 \times 10^{-3}</td>
<td>1.12410 \times 10^{-3}</td>
</tr>
<tr>
<td>0.6</td>
<td>3.27857 \times 10^{-2}</td>
<td>3.04594 \times 10^{-3}</td>
<td>1.78443 \times 10^{-4}</td>
</tr>
<tr>
<td>0.7</td>
<td>3.37783 \times 10^{-2}</td>
<td>4.35783 \times 10^{-3}</td>
<td>8.14607 \times 10^{-4}</td>
</tr>
<tr>
<td>0.8</td>
<td>3.35023 \times 10^{-2}</td>
<td>5.95087 \times 10^{-3}</td>
<td>7.43428 \times 10^{-4}</td>
</tr>
<tr>
<td>0.9</td>
<td>3.23752 \times 10^{-2}</td>
<td>7.72966 \times 10^{-3}</td>
<td>3.59748 \times 10^{-4}</td>
</tr>
</tbody>
</table>

Again the approximate solution given by PLSM is much closer to the numerical solution that the previous ones from ([34],[35]), while, at the same time, it has a much simpler form.

Figure 10 presents the relative error functions on the \([0,1]\) interval. The graphical representation of the relative error corresponding to HPM is represented by a dotted line, the one corresponding to HAM by a dashed line and the one corresponding to PLSM by a solid line.
We remark the fact that, for each of the three cases of the Aplication 1, computations were also performed for other values of the parameters $\epsilon_1$ and $\epsilon_2$ and, in general, the larger the values of the parameters, the more PLSM outperforms the other methods.

Figure 10: Comparison of HPM, HAM and PLSM for $\epsilon_2 = 1$

1.2.3.2. Heat transfer with conduction in a slab of a material with temperature dependent thermal conductivity

We consider the process of one-dimensional conduction in a slab of thickness $L$, with the two faces maintained at uniform temperatures $T_1$ and $T_2$, $T_1 >> T_2$. The thermal conductivity $k$ is a linear function of temperature ([29],[34],[35]):

$$k = k_2 \left(1 + \eta (T - T_2)\right)$$

where $\eta$ is a constant and $k_2$ is the thermal conductivity at $T_2$. The problem which describes the process is:

$$\frac{d}{dx} \left(k \frac{dT}{dx}\right) = 0, \quad x \in [0, L]$$

$$T(0) = T_1, \quad T(L) = T_2$$
In order to obtain the nondimensional form of the problem, taking into account the fact that $k_1$ is the thermal conductivity at $T_1$, we introduce the dimensionless variables:

$$
\theta = \frac{T - T_2}{T_1 - T_2}, \quad y = \frac{x}{L}, \quad \varepsilon = \eta(T_1 - T_2) = \frac{k_1 - k_2}{k_2}.
$$

We obtain the following problem:

$$
- \frac{d^2 \theta}{d y^2} = \frac{\varepsilon (\frac{d \theta}{d t})^2}{1 + \varepsilon \theta}, \quad y \in [0, 1]
$$

$$
\theta(0) = 1, \quad \theta(1) = 0.
$$

Equivalently:

$$
(1 + \varepsilon \theta) \frac{d^2 \theta}{d y^2} + \varepsilon \left(\frac{d \theta}{d t}\right)^2 = 0, \quad y \in [0, 1]
$$

$$
\theta(0) = 1, \quad \theta(1) = 0.
$$

The problem (27) admits an exact solution, $\theta_e$:

$$
\theta_e = \frac{-1 + \sqrt{1 - (y - 1)\varepsilon(\varepsilon + 2)}}{\varepsilon}.
$$

In ([34]), by using the HPM, Rajabi, Ganji and Taherian computed an approximate solution of problem (27):

$$
\theta_{HPM} = -\frac{y^2 \varepsilon}{2} + \left(y^2 - \frac{y^3}{2}\right)\varepsilon^2 - \frac{y\varepsilon^2}{2} + \frac{y\varepsilon}{2} - y + 1.
$$

In ([35]), by using the HAM, Domairry and Nadim computed an approximate solution of problem (27):

$$
\theta_{HAM} = -\frac{1}{2}y^2 \varepsilon^2 h^2 + y^2 \varepsilon^2 h^2 + \frac{1}{2}y^2 \varepsilon h^2 + y^2 \varepsilon h - \frac{1}{2}y\varepsilon^2 h^2 -
$$

$$
-\frac{1}{2}y\varepsilon h^2 - y\varepsilon h - y + 1.
$$

In ([29]), by using the generalized approximation method (GAM), Khan computed an approximate solution of problem (27). The expression of the approximate solution is not given explicitly, just its values for a set of values of $y$ on the $[0, 1]$ interval, for several values of the parameter $\varepsilon$. 
Using the PLSM, we computed the following eight order polynomial approximate solution of equation (27):

\[
\theta_{PLSM} = -0.194382y^8 + 0.576857y^7 - 0.72868y^6 + 0.440894y^5 - \\
-0.173141y^4 - 0.0296625y^3 - 0.141884y^2 - 0.750001y + 1.
\]

We compare again our solution \(\theta_{PLSM}\) with the previous solutions \(\theta_{HPM}\), \(\theta_{HAM}\) and \(\theta_{GAM}\). We computed for each approximate solution the relative error as the difference (in absolute value) between the approximate solution and the exact solution \(\theta_e\). Table 6 presents the values of the relative errors for the value of the parameters \(\varepsilon = 1, \hbar = -0.5\):

<table>
<thead>
<tr>
<th>(t)</th>
<th>HPM</th>
<th>HAM</th>
<th>GAM</th>
<th>PLSM 8th deg.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.90384 (10^{-2})</td>
<td>8.65938 (10^{-5})</td>
<td>5.40617 (10^{-6})</td>
<td>5.69231 (10^{-7})</td>
</tr>
<tr>
<td>0.2</td>
<td>2.79089 (10^{-2})</td>
<td>9.11085 (10^{-3})</td>
<td>1.08915 (10^{-5})</td>
<td>2.59835 (10^{-6})</td>
</tr>
<tr>
<td>0.3</td>
<td>2.91817 (10^{-2})</td>
<td>3.06686 (10^{-4})</td>
<td>1.56862 (10^{-5})</td>
<td>1.24162 (10^{-6})</td>
</tr>
<tr>
<td>0.4</td>
<td>2.53201 (10^{-2})</td>
<td>1.32005 (10^{-3})</td>
<td>2.00531 (10^{-5})</td>
<td>4.79476 (10^{-6})</td>
</tr>
<tr>
<td>0.5</td>
<td>1.86388 (10^{-2})</td>
<td>3.01383 (10^{-3})</td>
<td>2.18301 (10^{-5})</td>
<td>3.75054 (10^{-7})</td>
</tr>
<tr>
<td>0.6</td>
<td>1.12397 (10^{-2})</td>
<td>5.23970 (10^{-3})</td>
<td>2.16974 (10^{-5})</td>
<td>5.50051 (10^{-6})</td>
</tr>
<tr>
<td>0.7</td>
<td>4.90488 (10^{-3})</td>
<td>7.52988 (10^{-3})</td>
<td>1.88752 (10^{-5})</td>
<td>4.65280 (10^{-7})</td>
</tr>
<tr>
<td>0.8</td>
<td>9.11064 (10^{-4})</td>
<td>8.91106 (10^{-3})</td>
<td>1.50641 (10^{-5})</td>
<td>6.42001 (10^{-6})</td>
</tr>
<tr>
<td>0.9</td>
<td>3.24575 (10^{-4})</td>
<td>7.55043 (10^{-3})</td>
<td>8.42510 (10^{-6})</td>
<td>2.69815 (10^{-6})</td>
</tr>
</tbody>
</table>

For the case of the Application 2, the strong nonlinearity of the equation contributed to the fact that the degree of our approximate polynomial solution must be chosen much higher that in the case of the Application 1. Still, the conclusion is the same: we can compute by PLSM an approximation better and simpler that the previous ones in ([34],[35],[29]).

Figure 11 presents the relative error functions on the \([0,1]\) interval. The graphical representation of the relative error corresponding to HAM by a dotted line, the one corresponding to GAM (obtained by interpolating the results in [29]) by a dashed line and the one corresponding to PLSM by a solid line.
The relative error corresponding to HPM is too large in comparison to the other ones, so its graphical representation was omitted from the picture.

Figure11: Comparison of HPM, HAM, GAM and PLSM for $\epsilon = 1$

1.2.3.3. Temperature distribution in a thick rectangular fin radiating in free space

We consider the nondimensional form of the temperature distribution problem for an uniformly thick rectangular fin radiating in free space with nonlinearity of high order ([32],[33],[28]):

$$\frac{d^2 \theta}{dy^2} - \epsilon \theta^4 = 0$$

(28)

$$\theta(1) = 1, \quad \frac{d\theta}{dy}(0) = 0.$$

In ([32]), by using the HPM, Ganji computed the following approximate solution of equation (28):

$$\theta_{HPM} = \left(\frac{y^2}{2} - \frac{1}{2}\right) \epsilon + \frac{1}{6} (y^4 - 6y^2 + 5) \epsilon^2 + 1.$$

In ([33]), by using the HAM, Abbasbandy computed an approximate solution of (28):
\[\theta_{\text{HAM}} = -\frac{1}{2}(y^2 - 1)\varepsilon h - \frac{1}{2}(y^2 - 1)\varepsilon(h + 1)\varepsilon + \]
\[+ \frac{1}{6}(y^4 - 6y^2 + 5)\varepsilon^2 h^2 + 1.\]

In ([28]), by using the optimal homotopy asymptotic method (OHAM), Marinca and Herișanu computed a 6\textsuperscript{th} order approximate polynomial solution of the problem (28):

\[\theta_{\text{OHAM}} = 0.000023293y^6 + 0.0202783y^4 + 0.204645y^2 + 0.775053\]

Using the PLSM, we computed the following 4\textsuperscript{th} order polynomial approximate solution of equation (28):

\[\theta_{\text{PLSM}} = 0.0696406y^4 - 0.0473588y^3 + 0.198583y^2 + 0.779135.\]

We compare again our solution \(\theta_{\text{SRM}}\) with the previous solutions \(\theta_{\text{HPM}}\), \(\theta_{\text{HAM}}\) and \(\theta_{\text{OHAM}}\):

| Table 7. Comparison of HPM, HAM, OHAM and PLSM for \(\varepsilon=1\) |
|---|---|---|---|---|
| \(t\) | HPM | HAM | OHAM | PLSM 4th deg. |
| 0.1 | 5.47359 \(10^{-1}\) | 3.39452 \(10^{-2}\) | 3.88930 \(10^{-3}\) | 8.98806 \(10^{-5}\) |
| 0.2 | 5.27037 \(10^{-1}\) | 3.32133 \(10^{-2}\) | 3.29144 \(10^{-3}\) | 2.48399 \(10^{-4}\) |
| 0.3 | 4.93714 \(10^{-1}\) | 3.19805 \(10^{-2}\) | 2.33350 \(10^{-3}\) | 3.24245 \(10^{-4}\) |
| 0.4 | 4.48203 \(10^{-1}\) | 3.02193 \(10^{-2}\) | 1.08117 \(10^{-3}\) | 2.63685 \(10^{-4}\) |
| 0.5 | 3.91624 \(10^{-1}\) | 2.78745 \(10^{-2}\) | 3.56592 \(10^{-4}\) | 8.1475 \(10^{-5}\) |
| 0.6 | 3.25387 \(10^{-1}\) | 2.48429 \(10^{-2}\) | 1.80809 \(10^{-3}\) | 1.25384 \(10^{-4}\) |
| 0.7 | 2.51159 \(10^{-1}\) | 2.09446 \(10^{-2}\) | 3.00939 \(10^{-3}\) | 2.73888 \(10^{-4}\) |
| 0.8 | 1.70822 \(10^{-1}\) | 1.58783 \(10^{-2}\) | 3.56040 \(10^{-3}\) | 2.72332 \(10^{-4}\) |
| 0.9 | 8.64095 \(10^{-2}\) | 9.15554 \(10^{-3}\) | 2.85895 \(10^{-3}\) | 1.19728 \(10^{-4}\) |

Since no exact solution for the equation (28) is known, we computed the relative error as the difference (in absolute value) between the approximate solution and the numerical solution. Table 7 presents the values of the relative errors for the value of the parameters \(\varepsilon=1, h = -0.4\). Again, the approximate
solution given by PLSM is closer to the numerical solution that the previous ones from ([32],[33],[28]), while, at the same time, it has a simpler form.

Figure 12 presents the relative error functions on the [0,1] interval. The graphical representation of the relative error corresponding to HAM is represented by a dotted line, the one corresponding to OHAM by a dashed line and the one corresponding to PLSM by a solid line. As in the case of the previous application, the relative error corresponding to HPM is too large in comparison to the other ones, so its graphical representation was omitted from the picture.

![Figure 12](image)

Figure 12: Comparison of HPM, HAM, OHAM and PLSM for $\varepsilon = 1$

1.2.4. Conclusions

The polynomial least square method (PLSM) is a straightforward and efficient method to compute approximate polynomial solutions for nonlinear heat transfer problems.

The applications presented clearly illustrate the accuracy of the method, since for all the problems we were able to compute better approximations than the ones computed in previous papers. We remark again that while, due to the number of problems solved, we only included the cases were the $\epsilon$ parameters were equal to one, computations performed for other values of the parameters show that PLSM still has the advantage over the other methods and, moreover, the larger the values of $\epsilon$, the more PLSM outperforms the other methods.
1.3. Approximate polynomial solutions of the nonlinear Lane–Emden type equations arising in astrophysics

1.3.1. Introduction

The general form of the Lane-Emden equations studied in the next is:

\[ y''(x) + \frac{N}{x} y'(x) + f(x, y) = g(x) \]  \hspace{1cm} (29)

where \( N, x > 0 \) and \( f(x, y) \) is a nonlinear function. The equations (1) are usually presented together with initial conditions of the type:

\[ y(0) = \alpha, \quad y'(0) = \beta \]  \hspace{1cm} (30)

Historically, the first equation of this type is the standard Lane-Emden equation, obtained by taking in (1,2) \( N = 2, \; f(x, y) = y^M, \; g(x) = 0 \) and \( \alpha = 1, \beta = 0 \):

\[ y''(x) + \frac{2}{x} y'(x) + y^M = 0 \]  \hspace{1cm} (31)

\[ y(0) = 1, \quad y'(0) = 0 \]  \hspace{1cm} (32)

The standard Lane-Emden equation is named after the astrophysicists Jonathan Lane and Robert Emden who were the first to study it ([36],[37],[38],[39]). It represents a dimensionless form of Poisson’s equation for the gravitational potential of a Newtonian self-gravitating, spherically symmetric, polytropic fluid. For this reason the equation (31, 32) is also called the polytropic differential equation and it gives a useful approximation for self-gravitating gaseous spheres such as stars. Only the cases \( M = 0, \; M = 1 \) and \( M = 5 \) can be solved analytically ([38]) while others were originally solved numerically.

Beside the standard Lane-Emden equations, choosing particular functions and constants in the general equation (29, 30) lead to equations describing various physical phenomena such as the gravitational potential of the degenerate white-dwarf stars, the isothermal gas spheres or the thermionic currents, equations which usually do not have exact analytic solutions.

Due to their multiple applicability, the Lane-Emden type equations were extensively studied. Qualitative studies include papers such as [40], [41], [42], [43], [44]. In order to find approximate solutions of these equations, many methods were proposed, such as:
• Runge-Kutta type methods - Horedt, 1986 ([45]).
• Perturbation techniques (based on an artificial parameter $\delta$) - Bender et al., 1989 ([46]).
• Adomian decomposition methods - Shawagfeh, 1993 ([47]); Wazwaz, 2001 ([48]).
• Quasilinearization - Mandelzweig and Tabakin, 2001 ([49]).
• Variational principles - He, 2003 ([50]).
• Homotopy analysis method - Liao, 2003 ([51]); Singh et al., 2009 ([52]).
• Legendre wavelets - Yousefii, 2006 ([53]).
• Homotopy perturbation method - Chowdhury and Hashim, 2007 ([54]); Yildirim and Ozis, 2008 ([55]).
• Hybrid functions - Marzban et al., 2008 ([56]).
• Series expansion - Ramos, 2008 ([57]); VanGorder and Vajravelu, 2008 ([58]).
• Variational iteration method - Dehghan and Shakeri, 2008 ([59]).
• Rational Legendre pseudospectral approach - Parand et al., 2009 ([60]).
• Hermite function collocation method - Parand et al., 2010 ([61]).
• Lagrangian interpolation method - Parand et al., 2010 ([62]).
• Differential transform method - Mukherjee et al., 2011 ([63]).
• Bernstein operational matrix of integration - Kumar et al., 2011 ([64]); Pandey and Kumar, 2012 ([65]).
• Optimal homotopy asymptotic method - Iqbal and Javed, 2011 ([66]).
• Jacobi matrix method - Eslahchi et al., 2012 ([67]).
• Boubaker polynomials expansion scheme - Boubaker and Van Gorder, 2012 ([68]).
• Modified Legendre-spectral method - Rismani and Monfared, 2012 ([69]).
• Legendre operational matrix of integration - Pandey et al., 2012 ([70]).
• Birkhoff interpolation method - Dehghan et al., 2013 ([71]).
• B-spline expansion methods - Lakestani and Dehghan, 2013 ([72]).
We will use (PLSM [73], [74]) which allows us to determine analytical approximate polynomial solutions for Lane-Emden type equations. We will compare our approximate solutions with approximate solutions presented in previous papers. The extensive computations performed show that PLSM allows us to obtain approximations with an error relative to the exact or numerical solution smaller than the errors obtained using other methods.

1.3.2. Approximation method description

We apply the PLSM to the general problem (29, 30), for which (31, 32) is a particular case.

For the problem (29, 30) we consider the operator:

\[ D(y) = y''(x) + \frac{N}{x} y'(x) + f(x, y) - g(x). \]  

(33)

If \( y_{app} \) is an approximate solution of the equation (29), the error obtained by replacing the exact solution \( y \) with the approximation \( y_{app} \) is given by the remainder:

\[ R(x, y_{app}) = D(y_{app}(x)), \quad x \in [0, b] \]  

(34)

We will find approximate polynomial solutions \( y_{app} \) of (29, 30) on the \([0, b]\) interval, solutions which satisfy the following conditions:

\[ |R(x, y_{app})| < \varepsilon \]  

(35)

\[ y_{app}(0) = \alpha, \quad y_{app}'(0) = \beta \]  

(36)

An \( \varepsilon \)-approximate polynomial solution of the problem (29, 30) an approximate polynomial solution \( y_{app} \) satisfying the relations (35, 36).

A weak \( \delta \)-approximate polynomial solution of the problem (29, 30) an approximate polynomial solution \( y_{app} \) satisfying the relation:

\[ \int_0^b |R(x, y_{app})| dx \leq \delta \]

together with the initial conditions (36).

We consider the sequence of polynomials \( P_m(x) = a_0 + a_1 x + ... + a_m x^m \), \( a_i \in \mathbb{R}, \quad i = 0,1,...,m \) satisfying the conditions:

\[ P_m(0) = \alpha, \quad P_m'(0) = \beta \]
We call the sequence of polynomials \( P_m(x) \) convergent to the solution of the problem (29,30) if \( \lim_{m \to +\infty} D(P_m(x)) = 0 \).

We will find a weak \( \varepsilon \)-polynomial solution of the type:

\[
\tilde{y}(x) = \sum_{k=0}^{m} c_k x^k, \tag{37}
\]

where the constants \( c_0, c_1, \ldots, c_m \) are calculated using the following steps:

- By substituting the approximate solution (37) in the equation (29) we obtain the following expression:

\[
\Re(x, c_0, c_1, \ldots, c_m) = R(x, \tilde{y}) = \tilde{y}'(x) + \frac{N}{x} \tilde{y}(x) + f(x, \tilde{y}) - g(x) \tag{38}
\]

If we could find the constants \( c_0^0, c_1^0, \ldots, c_m^0 \) such that \( \Re(x, c_0^0, c_1^0, \ldots, c_m^0) = 0 \) for any \( x \in [0, b] \) and the equivalents of (30):

\[
\tilde{y}(0) = \alpha, \quad \tilde{y}'(0) = \beta \tag{39}
\]

are also satisfied, then by substituting \( c_0^0, c_1^0, \ldots, c_m^0 \) in (37) we obtain the exact solution of (29, 30). In general this situation is rarely encountered in polynomial approximation methods.

- Next we attach to the problem (29, 30) the following real functional:

\[
J(c_2, c_3, \ldots, c_m) = \int_{0}^{b} \Re^2(x, c_0, c_1, \ldots, c_m) dx \tag{40}
\]

where \( c_0, c_1 \) are computed as functions of \( c_2, c_3, \ldots, c_m \) by using the initial conditions (39).

- We compute the values of \( c_2^0, c_3^0, \ldots, c_m^0 \) as the values which give the minimum of the functional (40) and the values of \( c_0^0, c_1^0 \) again as functions of \( c_2^0, c_3^0, \ldots, c_m^0 \) by using the initial conditions.

- Using the constants \( c_0^0, c_1^0, \ldots, c_m^0 \) thus determined, we consider the polynomial:

\[
T_m(x) = \sum_{k=0}^{m} c_k^0 x^k \tag{41}
\]

The following convergence theorem holds:
Theorem 1 \textit{If the sequence of polynomials } \( P_m(x) \) \textit{converges to the solution of the problem (29, 30), then the sequence of polynomials } \( T_m(x) \) \textit{from (41) satisfies the property:}

\[ \lim_{m \to \infty} \int_0^b R^2(x, T_m)dx = 0 \]

Moreover, \( \forall \varepsilon > 0, \exists m_0 \in \mathbb{N} \) such that \( \forall m \in \mathbb{N}, m > m_0 \) it follows that \( T_m(x) \) is a weak \( \varepsilon \)-approximate polynomial solution of the problem (29, 30).

Based on the way the coefficients of polynomial \( T_m(x) \) are computed and taking into account the relations (38-41), the following inequality holds:

\[ 0 \leq \int_0^b R^2(x, T_m(x))dx \leq \int_0^b R^2(x, P_m(x))dx, \ \forall m \in \mathbb{N}. \]

It follows that:

\[ 0 \leq \lim_{m \to \infty} \int_0^b R^2(x, T_m(x))dx \leq \lim_{m \to \infty} \int_0^b R^2(x, P_m(x))dx = 0. \]

We obtain:

\[ \lim_{m \to \infty} \int_0^b R^2(x, T_m(x))dx = 0. \]

From this limit we obtain that \( \forall \varepsilon > 0, \exists m_0 \in \mathbb{N} \) such that \( \forall m \in \mathbb{N}, m > m_0 \) it follows that \( T_m(x) \) is a weak \( \varepsilon \)-approximate polynomial solution of the problem (29, 30) q.e.d.

Any \( \varepsilon \)-approximate polynomial solution of the problem (29, 30) is also a weak \( \varepsilon^2 \cdot b \)-approximate polynomial solution, but the opposite is not always true. It follows that the set of weak approximate solutions of the problem (29, 30) also contains the approximate solutions of the problem.

Taking into account the above remark, in order to find \( \varepsilon \)-approximate polynomial solutions of the problem (29, 30) by PLSM we will first determine weak approximate polynomial solutions, \( \tilde{y}_{\text{app}} \). If \( |R(x, \tilde{y}_{\text{app}})| < \varepsilon \) then \( \tilde{y}_{\text{app}} \) is also an \( \varepsilon \)-approximate polynomial solution of the problem.
1.3.3. Results and discussion

1.3.3.1. The standard Lane-Emden equation

For the standard Lane-Emden equation (31, 32), only the cases $M = 0$, $M = 1$ and $M = 5$ have an analytical solution ([3]) while for other values of $M$ the solution must be found using numerical or approximation techniques. While PLSM gave accurate approximation for all the values of $M$ tested, in the following we present the results for the cases $M = 0$, $M = 3$ and $M = 5$.

**Case 1: $M = 0$**

The analytical solution of (31, 32) in the case $M = 0$ is ([38]):

$$y_e = -\frac{1}{6} x^2 + 1$$

Since the solution is a polynomial, we expected that by using PLSM we would be able to find the exact solution or, at least, a very accurate approximation. Indeed, by choosing a second order polynomial (37) we obtain the approximate solution:

$$y_{PLSM} = -0.16666666666666666 x^2 + 1.$$

The absolute error, computed as the difference (in absolute value) between the exact solution and our approximate solution is practically zero in any point of the $[0,1]$ interval. Also zero are the values of the norms

$$\|y_e - y_{PLSM}\|_1, \|y_e - y_{PLSM}\|_2 \text{ and } \|y_e - y_{PLSM}\|_\infty$$

computed on the $[0,1]$ interval.

**Case 2: $M = 3$**

The standard Lane-Emden equation with $M = 3$ can be used as a simplified model for main sequence stars like our Sun. It corresponds to the Eddington standard model of stellar structure, which assumes that all the mass and energy are concentrated at the center of the star. Since in this case there is no analytic solution, various methods were used to find approximate ones.

Using PLSM we obtained the following approximate polynomial solution:

$$y_{PLSM} = 0.0000275109 x^9 - 0.00029439 x^8 + 0.000944262 x^7 - 0.000283224 x^6 - 0.0036385 x^5 - 0.00003678 x^4 + 0.0250058 x^3 - 4.2899107 x^2 - 0.166667 x^1 + 1.$$
In order to compare our solution $y_{PLSM}$ with previous solutions, we computed for each approximate solution absolute errors as differences (in absolute value) between values of the approximate solution and of the numerical solution obtained by using a fourth order Runge-Kutta method. In the computation of this numerical solution, in order to avoid problems due to the singularity in origin, we used a modified first integration step as presented in [10].

The methods included in our comparison are: Adomian decomposition methods ADM ([48]), Series expansion SEM ([57], [58], same result), Homotopy analysis method HAM ([58]), Rational Legendre pseudospectral approach RLM ([60]), Hermite function collocation method HCM ([61]), Lagrangian interpolation method LIM ([62]), Bernstein operational matrix of integration BOM ([65]), Boubaker polynomials expansion scheme BPES ([68]), Modified Legendre-spectral method LSM ([69]).

Table 8 presents the values of the absolute errors corresponding to these methods computed for a set of values of $x$ on the $[0,1]$ interval. Since most of the results of previous computations were presented as tables corresponding to the values 0, 0.1, 0.5 and of $x$, we give the results in the same format.
Table 8. Comparison of absolute errors of the approximate solutions for the standard Lane-Emden equation, case $M = 3$

<table>
<thead>
<tr>
<th></th>
<th>$x = 0$</th>
<th>$x = 0.1$</th>
<th>$x = 0.5$</th>
<th>$x = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ADM</td>
<td>$9.56918 \times 10^{-9}$</td>
<td>$4.00551 \times 10^{-8}$</td>
<td>$3.74814 \times 10^{-5}$</td>
<td></td>
</tr>
<tr>
<td>SEM</td>
<td>$9.88098 \times 10^{-15}$</td>
<td>$3.04346 \times 10^{-9}$</td>
<td>$1.12406 \times 10^{-5}$</td>
<td></td>
</tr>
<tr>
<td>HAM</td>
<td>$8.05795 \times 10^{-5}$</td>
<td>$5.92549 \times 10^{-5}$</td>
<td>$2.23547 \times 10^{-2}$</td>
<td></td>
</tr>
<tr>
<td>RLM</td>
<td>$8.21569 \times 10^{-7}$</td>
<td>$4.95014 \times 10^{-6}$</td>
<td>$1.19461 \times 10^{-5}$</td>
<td></td>
</tr>
<tr>
<td>HCM</td>
<td>$1.37043 \times 10^{-6}$</td>
<td>$3.02006 \times 10^{-6}$</td>
<td>$2.02141 \times 10^{-6}$</td>
<td></td>
</tr>
<tr>
<td>LIM</td>
<td>$1.28296 \times 10^{-5}$</td>
<td>$1.80699 \times 10^{-5}$</td>
<td>$5.68589 \times 10^{-7}$</td>
<td></td>
</tr>
<tr>
<td>BOM</td>
<td>$9.56918 \times 10^{-9}$</td>
<td>$5.51303 \times 10^{-11}$</td>
<td>$1.41136 \times 10^{-9}$</td>
<td></td>
</tr>
<tr>
<td>BPES</td>
<td>$1.41704 \times 10^{-5}$</td>
<td>$1.7907 \times 10^{-4}$</td>
<td>$1.60524 \times 10^{-2}$</td>
<td></td>
</tr>
<tr>
<td>LSM</td>
<td>$6.91793 \times 10^{-11}$</td>
<td>$4.48697 \times 10^{-11}$</td>
<td>$8.86429 \times 10^{-11}$</td>
<td></td>
</tr>
<tr>
<td>PLSM</td>
<td>$1.88153 \times 10^{-11}$</td>
<td>$1.17604 \times 10^{-11}$</td>
<td>$4.70624 \times 10^{-13}$</td>
<td></td>
</tr>
</tbody>
</table>

It is easy to see that the approximate solution given by PLSM is closer to the numerical solution than the previous ones.

**Case 3: $M = 5$**

The analytical solution of (3,4) in the case $M = 5$ is ([38]):

$$y_e = \left(1 + \frac{x^2}{3}\right)^{\frac{1}{2}}$$

The approximate polynomial solution obtained using PLSM is:

$$y_{PLSM} = -0.0153664 x^5 + 0.0495644 x^4 - 0.00158565 x^3 - 0.166588 x^2 + 1.$$
We compare our solution with solutions computed using the following methods: Homotopy perturbation method HPM ([54]), Variational iteration method VIM ([59]), Optimal homotopy asymptotic method OHAM ([66]), Boubaker polynomials expansion scheme BPES ([68]).

Table 9 presents the values of the absolute errors (as the difference in absolute value between the approximate solution and the exact solution) corresponding to these methods computed for a set of values of $x$ on the $[0,1]$ interval.

Table 9. Comparison of absolute errors of the approximate solutions for the standard Lane-Emden equation, case $M = 5$

<table>
<thead>
<tr>
<th></th>
<th>x = 0</th>
<th>x = 0.1</th>
<th>x = 0.5</th>
<th>x = 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>HPM</td>
<td>0</td>
<td>$3.36569 \times 10^{-11}$</td>
<td>$1.22677 \times 10^{-5}$</td>
<td>$2.59948 \times 10^{-3}$</td>
</tr>
<tr>
<td>OHAM</td>
<td>0</td>
<td>$4.01135 \times 10^{-5}$</td>
<td>$3.56912 \times 10^{-4}$</td>
<td>$4.49742 \times 10^{-4}$</td>
</tr>
<tr>
<td>BPES</td>
<td>0</td>
<td>$5.22512 \times 10^{-4}$</td>
<td>$1.30811 \times 10^{-2}$</td>
<td>$8.26346 \times 10^{-2}$</td>
</tr>
<tr>
<td>PLSM</td>
<td>0</td>
<td>$1.46725 \times 10^{-7}$</td>
<td>$3.56494 \times 10^{-6}$</td>
<td>$5.07745 \times 10^{-7}$</td>
</tr>
</tbody>
</table>

It is easy to see that the approximate solution given by PLSM is closer to the exact solution.

Since the full expressions of the approximate solutions obtained using HPM and OHAM were given (in [54] and [66], respectively), we were able to illustrate the precision of our method in Figure 13, which presents the absolute error functions on the $[0,1]$ interval. The graphical representation of the absolute error corresponding to HPM is represented by a dotted line, the one corresponding to OHAM by a dashed line and the one corresponding to PLSM by a solid line. The full expressions of the approximate solutions obtained using BPES is not presented in [68] (only tabular data), but the absolute error corresponding to BPES is too large in comparison to the other ones anyway.
We also computed the norms $\|y_e - y_{app}\|_1$, $\|y_e - y_{app}\|_2$ and $\|y_e - y_{app}\|_\infty$ corresponding to HPM, OHAM and PLSM on the $[0,1]$ interval. Table 10 presents these norms together with the corresponding results from [59] obtained using the Variational iteration method VIM.

Table 10. Comparison of $\|y_e - y_{app}\|_1$, $\|y_e - y_{app}\|_2$ and $\|y_e - y_{app}\|_\infty$ norms for the standard Lane-Emden equation, case $M = 5$

<table>
<thead>
<tr>
<th></th>
<th>$|y_e - y_{app}|_1$</th>
<th>$|y_e - y_{app}|_2$</th>
<th>$|y_e - y_{app}|_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>HPM</td>
<td>3.01812 $10^{-4}$</td>
<td>6.46559 $10^{-4}$</td>
<td>2.59948 $10^{-3}$</td>
</tr>
<tr>
<td>OHAM</td>
<td>2.61883 $10^{-4}$</td>
<td>3.12201 $10^{-4}$</td>
<td>5.7839 $10^{-4}$</td>
</tr>
<tr>
<td>VIM</td>
<td>2.12689 $10^{-5}$</td>
<td>4.40431 $10^{-5}$</td>
<td>1.68821 $10^{-4}$</td>
</tr>
<tr>
<td>PLSM</td>
<td>1.68593 $10^{-6}$</td>
<td>2.10769 $10^{-6}$</td>
<td>3.91469 $10^{-6}$</td>
</tr>
</tbody>
</table>

1.3.3.2. The white-dwarf equation

The white-dwarf equation is obtained from (29, 30) for $\alpha = 1, \beta = 0, N = 2$, $f(x,y) = (y^2 - C)^2$ and $g(x) = 0$, and it can be used to
model the gravitational potential of a degenerate white-dwarf star ([38]). The equation and corresponding initial conditions are:

\[ y''(x) + \frac{2}{x} y'(x) + \left( y^2 - C \right)^{\frac{3}{2}} = 0 \]  
\[ y(0) = 1, \quad y'(0) = 0 \]

The approximate polynomial solution obtained using PLSM for the case \( C = 0.2 \) is:

\[ y_{PLSM} = -0.00346042 \, x^5 + 0.0170652 \, x^4 + 0.00107582 \, x^3 - 0.119688 \, x^2 + 1. \]

In Table 11 and Figure 14 we compare this solution with approximate solutions for the equation (42, 43) previously computed using the following methods: Homotopy perturbation method HPM ([54]), Modified homotopy analysis method MHAM ([52]) and Rational Legendre pseudospectral approach RLM ([60]). The comparison presents the absolute errors as the difference (in absolute value) between the approximate solution and the numerical solution obtained by using a fourth order Runge-Kutta method and, again, emphasises the accuracy of our method.

**Table 11. Comparison of absolute errors of the approximate solutions for the white-dwarf equation, case \( C = 0.2 \)**

<table>
<thead>
<tr>
<th></th>
<th>x = 0.1</th>
<th>x = 0.5</th>
<th>x = 1</th>
<th>x = 1.5</th>
<th>x = 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>HPM</td>
<td>9.11 (10^{-7})</td>
<td>2.77 (10^{-5})</td>
<td>5.68 (10^{-6})</td>
<td>3.99 (10^{-3})</td>
<td>3.92 (10^{-2})</td>
</tr>
<tr>
<td>MHAM</td>
<td>3.19 (10^{-6})</td>
<td>1.96 (10^{-3})</td>
<td>3.00 (10^{-2})</td>
<td>1.42 (10^{-1})</td>
<td>4.18 (10^{-1})</td>
</tr>
<tr>
<td>RLM</td>
<td>6.18 (10^{-6})</td>
<td>5.55 (10^{-4})</td>
<td>2.42 (10^{-3})</td>
<td>1.39 (10^{-3})</td>
<td>7.30 (10^{-3})</td>
</tr>
<tr>
<td>PLSM</td>
<td>3.16 (10^{-6})</td>
<td>1.60 (10^{-5})</td>
<td>6.34 (10^{-5})</td>
<td>1.07 (10^{-4})</td>
<td>7.30 (10^{-6})</td>
</tr>
</tbody>
</table>
We remark that the solution obtained in [52] using the Modified homotopy analysis method MHAM, which is a seventh degree polynomial, is actually the same solution previously obtained in [48] using an Adomian decomposition method, in [51] using a Homotopy analysis method and in [57] using a Series expansion method.

1.3.3.3. The isothermal gas spheres equation

The isothermal gas spheres equation ([37]) is obtained from (29, 30) for $\alpha = 0, \beta = 0, N = 2, f(x, y) = e^y$ and $g(x) = 0$. The equation and corresponding initial conditions are:

$$y''(x) + \frac{2}{x}y'(x) + e^y = 0$$
$$y(0) = 0, \quad y'(0) = 0 \quad (45)$$

The approximate polynomial solution obtained using PLSM is:

$$y_{PLSM} = -0.00031099x^6 - 0.00034799x^5 + 0.0085476x^4 - 0.00005321x^3 - 0.166663x^2.$$ 

In Table 12 we compare this polynomial approximate solution with approximate solutions for the equation (44, 45) previously computed using the
following methods: Modified homotopy analysis method MHAM ([52]), Bernstein operational matrix of integration BOM [65]), Legendre operational matrix of integration LOM ([70]) and Hermite function collocation method HCM ([61]). The comparison presents the absolute errors as the difference (in absolute value) between the approximate solution and the numerical solution obtained by using a fourth order Runge-Kutta method.

Table 12. Comparison of absolute errors of the approximate solutions for the isothermal gas spheres equation

<table>
<thead>
<tr>
<th></th>
<th>x = 0</th>
<th>x = 0.1</th>
<th>x = 0.5</th>
<th>x = 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>MHAM</td>
<td>0</td>
<td>1.37161 $10^{-12}$</td>
<td>1.43237 $10^{-7}$</td>
<td>3.47563 $10^{-5}$</td>
</tr>
<tr>
<td>BOM</td>
<td>6.10622 $10^{-11}$</td>
<td>1.43208 $10^{-7}$</td>
<td>3.47563 $10^{-5}$</td>
<td></td>
</tr>
<tr>
<td>LOM</td>
<td>9.24804 $10^{-18}$</td>
<td>5.61062 $10^{-10}$</td>
<td>8.12396 $10^{-6}$</td>
<td>4.94344 $10^{-4}$</td>
</tr>
<tr>
<td>HCM</td>
<td>0</td>
<td>5.84939 $10^{-7}$</td>
<td>5.57708 $10^{-7}$</td>
<td>4.96176 $10^{-7}$</td>
</tr>
<tr>
<td>PLSM</td>
<td>0</td>
<td>1.08584 $10^{-9}$</td>
<td>3.17338 $10^{-8}$</td>
<td>3.30731 $10^{-10}$</td>
</tr>
</tbody>
</table>

Again we remark that the solution obtained in [52] using a Modified homotopy analysis method MHAM was previously obtained in [48] using a variant of the Adomian decomposition method ADM, in [51] using a Homotopy analysis method HAM, in [54] using a Homotopy perturbation method HPM and in [57] using a Series expansion method SEM. Figure 15 presents the comparison between this solution and our solution, both sixth degree polynomials. While very close to zero the MHAM solution is a little more accurate, overall the PLSM solution is clearly better.
1.3.3.4. Richardson’s theory of thermionic currents

If in (29,30) we replace $\alpha = 0, \beta = 0, N = 2, f(x, y) = e^{-y}$ and $g(x) = 0$ we obtain the equation used by Richardson in his theory of thermionic currents, which studies the density and electric force of an electron gas in the neighborhood of a hot body in thermal equilibrium. The equation and corresponding initial conditions are:

$$y''(x) + \frac{2}{x} y'(x) + e^{-y} = 0 \quad (46)$$

$$y(0) = 0, \quad y'(0) = 0 \quad (47)$$

The approximate polynomial solution obtained using PLSM is:

$$y_{PLSM} = -0.000850028x^6 + 0.000553538x^5 - 0.00869272x^4$$
$$+ 0.0000928951x^3 - 0.166673x^2.$$
In Table 13 and Figure 16 we compare our solution with the approximate solutions for the equation (46, 47) previously computed using the Homotopy perturbation method HPM ([54]). The comparison presents the absolute errors as the difference between the approximate solution and the numerical solution obtained by using a fourth order Runge-Kutta method. As in the case of the previous section, the overall accuracy of the PLSM solution is clearly better.

Table 13. Comparison of absolute errors of HPM and PLSM for Richardson’s equation

<table>
<thead>
<tr>
<th></th>
<th>x = 0</th>
<th>x = 0.1</th>
<th>x = 0.5</th>
<th>x = 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>HPM</td>
<td>0</td>
<td>6.27835 $10^{-13}$</td>
<td>1.48709 $10^{-7}$</td>
<td>4.03943 $10^{-5}$</td>
</tr>
<tr>
<td>PLSM</td>
<td>0</td>
<td>2.95629 $10^{-9}$</td>
<td>4.56525 $10^{-8}$</td>
<td>7.91836 $10^{-10}$</td>
</tr>
</tbody>
</table>

Figure 16: Comparison of absolute errors of HPM and PLSM for Richardson’s equation
1.3.3.5. Non-homogeneous Lane-Emden type equation

Choosing in
\[(29, 30) \alpha = 0, \beta = 0, N = 8, \ f(x, y) = xy \ \text{and} \ g(x) = x^5 - x^4 + 44x^2 - 30x\]
we obtain the following non-homogeneous problem ([61]):
\[y''(x) + \frac{2}{x} y'(x) + xy = x^5 - x^4 + 44x^2 - 30x \quad (48)\]
\[y(0) = 0, \ y'(0) = 0 \quad (49)\]

The analytical solution of (20,21) is ([61]):
\[y_e = x^4 - x^3\]

The approximate polynomial solution obtained using PLSM is:
\[y_{PLSM} = x^4 - x^3 - 1.98745 \cdot 10^{-10} x^2.\]

In Table 14 we compare this polynomial approximate solution with approximate solutions for the equation (48, 49) previously computed using the Hermite function collocation method HCM ([61]). The comparison presents the absolute errors as the difference (in absolute value) between the approximate solution and the exact solution.

Table 14. Comparison of absolute errors of the approximate solutions for the non-homogeneous problem (48, 49)

<table>
<thead>
<tr>
<th>x = 0</th>
<th>x = 2</th>
<th>x = 4</th>
<th>x = 6</th>
<th>x = 8</th>
<th>x = 10</th>
</tr>
</thead>
<tbody>
<tr>
<td>HCM</td>
<td>0</td>
<td>1.74 10^{-7}</td>
<td>3.68 10^{-8}</td>
<td>4.74 10^{-7}</td>
<td>9.36 10^{-6}</td>
</tr>
<tr>
<td>PLSM</td>
<td>0</td>
<td>1.27 10^{-12}</td>
<td>2.09 10^{-11}</td>
<td>6.37 10^{-12}</td>
<td>1.32 10^{-11}</td>
</tr>
</tbody>
</table>

The polynomial least square method (PLSM) is presented as a straightforward and efficient method to compute approximate polynomial solutions for Lane-Emden-type differential equations.

The applications presented clearly illustrate the accuracy of the method, since for all the problems we were able to compute better approximations than the ones computed in previous papers. Moreover, we mention that for the case of
most of the previous solutions, the approximation computed using PLSM was not only more precise but also presented a simpler form, i.e. less terms or smaller polynomial degree.

1.4. Approximate polynomial solutions for Riccati differential equations

1.4.1. Introduction

We consider the following class of Riccati differential equations:

\[ u'(t) = p(t) + q(t)u(t) + r(t)u^2(t), \quad 0 \leq t \leq b, \]  
\[ u(0) = \alpha \]

where \( p, q, r : [0, b] \rightarrow \mathbb{R} \) are given continuous functions on \([0, b]\) and \( \alpha \) is an arbitrary constant.

These equations play an important role in applied sciences and are closely related to optimal control theory. It is well known that finding exact solutions of Riccati differential equations is possible only in some cases, for example, if a particular solution of the equation is known or if the equation has constant coefficients. This justifies the need to resort to approximate methods for computing approximate solutions, solutions which could offer important information about the phenomena studied.

In order to obtain approximate solutions of Riccati differential equations several methods have been proposed, including: the Adomian Decomposition Method ([75]), the Homotopy Perturbation Method ([76]), the Taylor Matrix Method ([77]), the New Homotopy Perturbation Method ([78]), the Cubic B-spline Scaling Functions Method and the Chebyshev Cardinal Functions Method ([79]), the Bessel Collocation Method ([80]).

1.4.2. Approximation method description

For the problem (50,51) we consider the operator

\[ D(u) = u'(t) - p(t) - q(t)u(t) - r(t)u^2(t). \]
If \( u_\alpha \) is an approximate solution of the equation (1), the error obtained by replacing the exact solution \( u \) with the approximation \( u_\alpha \) is given by the remainder

\[
R(t, u_\alpha) = \mathcal{N}(u_\alpha(t)), \quad t \in [0, b]
\]

We will find approximate polynomial solutions \( u_\alpha \) of (50,51) on the \([0, b]\) interval, solutions which satisfy the following conditions:

\[
|R(t, u_\alpha)| \ll \varepsilon \quad \text{(52)}
\]

\[
u_\alpha(0) = \alpha. \quad \text{(53)}
\]

An \( \varepsilon \)-approximate polynomial solution of the problem (50,51) an approximate polynomial solution \( u_\alpha \) satisfying the relations (52,53).

A weak \( \delta \)-approximate polynomial solution of the problem (50,51) an approximate polynomial solution \( u_\alpha \) satisfying the relation \( \int_0^b |R(t, u_\alpha)| \, dt \leq \delta \)

together with the initial conditions (53).

We consider the sequence of polynomials

\[
H_{n}(\ell) = a_0 + a_1 \ell + \ldots + a_m \ell^m, \quad a_i \in \mathbb{R}, \quad i = 0, 1, \ldots, m
\]

satisfying the conditions

\[
H_{n}(0) = \alpha.
\]

We call the sequence of polynomials \( H_{n}(\ell) \) convergent to the solution of the problem (50,51) if

\[
\lim_{n \to \infty} D(H_{n}(\ell)) = 0.
\]

We will find a weak \( \varepsilon \)-polynomial solution of the type:

\[
\tilde{u}(\ell) = \sum_{k=0}^{m} c_k \ell^k, \quad \text{(54)}
\]

where the constants \( c_0, c_1, \ldots, c_m \) are calculated using the following steps:

- By substituting the approximate solution (54) in the equation (50) we obtain the following expression:

\[
\mathcal{R}(t, c_0, c_1, \ldots, c_m) = \mathcal{R}(t, \tilde{u}) = \tilde{u}'(t) - p(t) - q(t) u(t) - r(t) u^2(t) \quad \text{(55)}
\]

- Next we attach to the problem (50,51) the following real functional:

\[
\int_0^b \mathcal{R}(t, c_0, c_1, \ldots, c_m) \, dt
\]

where \( c_0 \) is computed as a function of \( c_1, c_2, c_3, \ldots, c_m \) by using the initial conditions.
• We compute the values of $c_2^0, c_3^0, c_2^0, \ldots, c_m^0$ as the values which give the minimum of the functional (2.56) and the value of $c_0$ again as function of $c_2^0, c_3^0, c_2^0, \ldots, c_m^0$ by using the initial conditions.

• Using the constants $c_2^0, c_3^0, \ldots, c_m^0$ thus determined, we consider the polynomial:

$$T_m(t) = \sum_{k=0}^{m} c_k^0 t^k$$

The following convergence theorem holds:

**Theorem 1.** If the sequence of polynomials $P_m(t)$ converges to the solution of the problem (50, 51), then the sequence of polynomials $T_m(t)$ from (57) satisfies the property:

$$\lim_{m \to \infty} \int_{a}^{b} R^2(t, T_m)dt = 0.$$  

Moreover, $\forall \epsilon > 0, \forall m_0 \in N$ such that $\forall m \in N, m > m_0$ it follows that $T_m(t)$ is a weak $\epsilon$-approximate polynomial solution of the problem (50, 51).

**Remark 1.** Any $\epsilon$-approximate polynomial solution of the problem (50, 51) is also a weak $\epsilon^2 \cdot b$-approximate polynomial solution, but the opposite is not always true. It follows that the set of weak approximate solutions of the problem (50, 51) also contains the approximate solutions of the problem.

Taking into account the above remark, in order to find $\epsilon$-approximate polynomial solutions of the problem (50,51) by PLSM we will first determine weak approximate polynomial solutions, $\overline{u}_a$.

If $|R(t, \overline{u}_a)| < \epsilon$ then $\overline{u}_a$ is also an $\epsilon$-approximate polynomial solution of the problem.

**1.4.3. Results and discussion**

We consider the Riccati differential equation:

$$\begin{align*} 
  u'(t) + u^2(t) &= 1, \quad 0 \leq t \leq 1 \\
  u(0) &= 0 
\end{align*}$$  

(58)
The equation (58) has been studied in the recent papers [77] and [80] and its exact solution is

\[ u_e(t) = \frac{e^{2t} - 1}{e^{2t} + 1}. \]

In [77] the Taylor Matrix Method is used to compute an approximate solution of the equation (58) and in [80] a collocation method based on the Bessel functions of first kind is used to compute an approximate solution of (58).

The 5th degree polynomial approximate solution obtained by using PLSM is:

\[ \tilde{u}_5(t) = -0.0156299 t^5 + 0.183861 t^4 - 0.424355 t^3 + 0.0189407 t^2 + 0.998777 t \]

The 8th degree polynomial approximate solution obtained by using PLSM is:

\[ \tilde{u}_8(t) = 0.00878043 t^8 - 0.0136428 t^7 - 0.0642412 t^6 + 0.177039 t^5 - 0.015732 t^4 - 0.330355 t^3 - 0.000261078 t^2 + 1.00001 t \]

In Table 15 we present the comparison of the absolute errors, computed as the difference in absolute value between the exact solution and the approximate solution, corresponding to \( |u_e - u_{\text{Taylor}}| \) from [77], to \( |u_e - u_{\text{Bessel}}| \) from [80] and to \( |u_e - \tilde{u}| \) obtained by using PLSM.

Table 15. Comparison of the absolute errors of the approximate solutions for problem (58)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>t = 0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>t = 0.2</td>
<td>6.79775096 \times 10^{-7}</td>
<td>5.173418178 \times 10^{-5}</td>
<td>3.213141641 \times 10^{-5}</td>
</tr>
<tr>
<td>t = 0.4</td>
<td>8.308774478 \times 10^{-5}</td>
<td>2.59692815 \times 10^{-5}</td>
<td>1.940166889 \times 10^{-5}</td>
</tr>
<tr>
<td>t = 0.6</td>
<td>1.310433002 \times 10^{-3}</td>
<td>4.065700692 \times 10^{-5}</td>
<td>1.213233671 \times 10^{-5}</td>
</tr>
<tr>
<td>t = 0.8</td>
<td>8.987229732 \times 10^{-3}</td>
<td>1.239019267 \times 10^{-5}</td>
<td>2.528296905 \times 10^{-5}</td>
</tr>
<tr>
<td>t = 1</td>
<td>3.040504404 \times 10^{-2}</td>
<td>7.514143492 \times 10^{-6}</td>
<td>7.862531504 \times 10^{-6}</td>
</tr>
</tbody>
</table>
In Figure 17 and Figure 18 we plotted the absolute error functions corresponding to the approximate polynomial solution of the 5th degree and of the the 8th degree, respectively. Both the figures and Table 15 show that PLSM presents the most accurate solution.

Figure 17: Comparison of the absolute error functions of Eq. (58) for 5th degree polynomials

![Figure 17](image)

Figure 18. Comparison of the absolute error functions of Eq. (58) for 8th degree polynomials

Be the Riccati differential equation:

$$\begin{cases} u'(t) - 2u(t) + u^2(t) = 1, & 0 \leq t \leq 1 \\ u(0) = 0 \end{cases} \tag{59}$$

The exact solution of Riccati differential equation (58) is

$$u_0(t) = 1 + \sqrt{2} \tanh(\sqrt{2}t + \frac{\log(-1 + e^t)}{2}).$$

Using different methods of approximation, approximate solutions for (58) were computed by several authors. Thus, in [75] the Adomian Decomposition Method was used, in [76] the Homotopy Perturbation Method was used, in [78], the New Homotopy Perturbation Method was used, in [79] the Cubic B-spline Scaling Functions Method and the Chebyshev Cardinal
Functions Method were used and in ([80]) the Bessel Collocation Method was used.

The 5th degree polynomial approximate solution obtained by using PLSM method for the equation (58) is:

$$\hat{u}_5(t) = 0.604006t^5 - 1.90983t^4 + 1.1586t^3 + 0.825387t^2 + 1.01135t$$

In Figure 19 we present the comparison of the absolute errors corresponding to solutions obtained by using several of the methods mentioned above. It is easy to see that the solution computed by PLSM is again the most accurate.

The 12th degree polynomial approximate solution obtained by using PLSM method for the equation (58) is:

$$\hat{u}_{12}(t) = -0.212167t^{12} + 1.1659t^{11} - 2.46403t^{10} + 2.37379t^9 - 1.09975t^8 + 0.624072t^7 - 0.238695t^6 - 0.468825t^5 + 0.331542t^4 + 0.33301t^3 + 1.00001t^2 + 1.0$$

The graphical representation of the error for the case of this polynomial approximation is plotted in Fig. 20 and we can see that the absolute value of the error in this case is less than $6 \times 10^{-9}$. Since the absolute value of the error obtained in [79] is less than $3 \times 10^{-9}$ we can conclude that PLSM can compute a more accurate solution for Eq. (58) than the method from [79].

![Figure 19. Comparison of the absolute error functions of Eq. (58) for 5th degree polynomials](image)

Figure 19. Comparison of the absolute error functions of Eq. (58) for 5th degree polynomials
1.4.4. Conclusions

The applications presented in this paper demonstrate that PLSM performs very well for the case of Riccati differential equations.

The comparison of the the absolute errors obtained by using PLSM with the ones obtained in previous papers confirms that our approximate solutions are much closer to the exact solution and thus emphasizes the high accuracy of PLSM. Moreover, the fact that the error does not increase as we leave the vicinity of the initial point 0, as it happens in the case of the other methods, confirms the advantage of PLSM.
1.5. Analytical approximate solutions for a general class of nonlinear delay differential equations

1.5.1. Introduction

Delay differential equations are frequently used to model real-life phenomena in various fields such as mechanics, biology, computer science, chemistry etc. Some of the recent studies involving delay differential equations include topics as varied as: epidemic models that describe the fraction of a population infected by a virus ([83]), complex oscillator network ([84]) and neural networks ([85]).

It is known that the computation of exact solutions for delay differential equations is only possible in particular cases. It follows that in most cases, in order to obtain informations about the phenomena modeled, the computation of approximate solutions becomes a necessity.

We used the polynomial least squares method (PLSM) to compute approximate solutions for the following class of nonlinear delay differential equations:

\[ F(x^{(n)}(h_0(t)), x^{(n-1)}(h_{n-1}(t)), x^{(n-2)}(h_{n-2}(t)), \ldots, x^{(1)}(h_1(t)), x(h_0(t)), t) = 0 \]  

(60)

together with the initial conditions

\[ \sum_{i=0}^{n-1} r_{ij} x^{(i)}(\alpha) = s_j, \quad j = 1, 2, \ldots, n. \]

(61)

Here \( F \) is a function which satisfies such conditions as necessary to ensure that the problem (60, 61) admits a unique solution, \( r_{ij}, s_j \) are real constants and the functions \( h_k(t), \quad k = 0, 1, \ldots, n \) are polynomial functions in the \( t \) variable, \( t \in [\alpha, \beta] \).

Among the methods recently used to compute approximate solutions for various delay differential equations of the (60) type we mention:

• In [86] a numerical approximation based on the Bessel functions of the first kind was applied to an equation (60) of the Riccati type.

• In [87] a two-stage order-one Runge-Kutta method was applied to an equation (60) of the neutral-function differential type. The same equation was studied in [88] and [89] by using the one-leg \( \theta \)-method, in [90] by using the
Variational Iteration Method, in [91] by using the Homotopy Perturbation Method and in [92] by using a method based on Chebyshev polynomials.

- In [93] the Homotopy Perturbation Method was applied to an equation (60) of the pantograph type. The same equation was studied in [94] by using the Variational Iteration Method.

- In [95] the Jacobi rational-Gauss collocation method was applied to an equation (60) of the generalized pantograph type. The same equation was studied in [96] by using the Taylor series method, in [97] by using a Chebyshev method and in [98] by using a Hermite collocation method.

- In [99] the Variational Iteration Method and the Adomian Decomposition Method were applied for the case of a nonlinear time-delay model in biology which is also a particular case of the equation (60).

- In [100] and [101] local fractional methods (local fractional variation iteration method and local fractional Adomian decomposition method, respectively) were applied.

### 1.5.2. Approximation method description

The polynomial least squares method (PLSM), allows us to find analytical approximate polynomial solutions for the problem (60, 61), and we compare our approximate solutions with approximate solutions presented in [86-99].

Let $D$ be the operator associate to the differential equation (60):

$$D(x) = F\left(x^{(n)}(h_{n}(t)), x^{(n-1)}(h_{n-1}(t)), x^{(n-2)}(h_{n-2}(t)), \ldots, x^{(1)}(h_{1}(t)), x(h_{0}(t)), t\right) \quad (62)$$

The error obtained by replacing the exact solution $x$ with an approximate solution $x_{a}$ is given by the so-called remainder:

$$R(t,x_{a}) = D(x_{a}(t)), \quad t \in [\alpha, \beta] \quad (63)$$

As a consequence, we will search for approximate polynomial solutions $x_{a}$ of (60, 61) on the $[\alpha, \beta]$ interval, solutions which satisfy the following conditions:

$$\left| R(t,x_{a}) \right| < \epsilon, \quad \epsilon \in \mathbb{R}_{+} \quad (64)$$

$$\sum_{i=0}^{n-1} r_{ij} x^{(i)}(\alpha) = s_{j}, \quad j = 1,2,\ldots,n \quad (65)$$

- An $\epsilon$-approximate polynomial solution of the problem (60, 61) is an approximate polynomial solution $x_{a}$ satisfying the relations (64, 65).
- **Weak δ-approximate polynomial solution** of the problem (60, 61) is an approximate polynomial solution \( x_a \) satisfying the relation:

\[
\int_a^b R^2(t, x_a) dt \leq \delta, \delta \in R,
\]

together with the initial condition (65).

Taking into account the way the problem (60, 61) is defined, from the Weierstrass approximation theorem it follows that there exists the sequence of polynomials \( P_m(t) = b_0 + b_1 t + \ldots + b_m t^m \), \( b_i \in R \), \( i = 0, 1, \ldots, m \) satisfying the conditions:

\[
\sum_{j=0}^{n-1} f_j P_m^{(j)}(\alpha) = s_j, \quad j = 1, 2, \ldots , n
\]

such that the sequence of polynomials \( P_m(t) \) is convergent to the solution of the problem (60, 61) i.e. \( \lim_{m \to \infty} D(P_m(t)) = 0 \).

Any \( \varepsilon \)-approximate polynomial solution of the problem (60, 61) is also a weak \( \varepsilon^2 \cdot (\beta - \alpha) \)-approximate polynomial solution, but the opposite is not always true. It follows that the set of weak approximate solutions of the problem (60, 61) also contains the approximate solutions of the problem.

**Theorem 1** The problem (60, 61) admits a sequence of weak approximate polynomial solutions.

Taking into account the definition, we will find a weak \( \varepsilon \)-polynomial solution of the type:

\[
\tilde{x}(t) = \sum_{k=0}^{m} c_k t^k, \quad m > n,
\]

(66)

where the constants \( c_0, c_1, \ldots, c_m \) are calculated using the following steps:

1. In the first step we substitute the approximate solution (66) in the equation (60) and obtain the remainder:

\[
\mathcal{M}(t, c_0, c_1, \ldots, c_m) = R(t, \tilde{x}) = \sum_{i=0}^{m} F_i(\tilde{x}^{(i)}(h_0(t)), \tilde{x}^{(n-1)}(h_{n-1}(t)), \tilde{x}^{(n-2)}(h_{n-2}(t)), \ldots, \tilde{x}^{(1)}(h_1(t)), \tilde{x}(h_0(t)), t)
\]

(67)
2. Next we compute \( c_0, c_1, \ldots, c_{n-1} \) as functions of \( c_n, \ldots, c_m \) by using the initial conditions:

\[
\sum_{j=0}^{n-1} f_j \tilde{x}^{(i)}(\alpha) = s_j, \quad j = 1, 2, \ldots, n
\]  

(68)

3. We attach to the problem (60, 61) the following real functional:

\[
J(c_n, \ldots, c_m) = \int_{\alpha}^{\beta} R^2(t, c_0, c_1, \ldots, c_m) dt
\]  

(69)

4. Next we compute the values of \( c_n^0, \ldots, c_m^0 \) as the values which give the minimum of the functional (69) and the values of \( c_n^0, \ldots, c_m^0 \) again as functions of \( c_n, \ldots, c_m \) by using the initial conditions.

5. By using the constants \( c_0^0, c_1^0, \ldots, c_m^0 \) thus determined, we consider the polynomial:

\[
T_m(t) = \sum_{k=0}^{m} c_k^0 \cdot t^k
\]  

(70)

Based on the way the coefficients of polynomial \( T_m(t) \) are computed and taking into account the relations (67-70), the following inequality holds:

\[
0 \leq \int_{\alpha}^{\beta} R^2(t, T_m(t)) dt \leq \int_{\alpha}^{\beta} R^2(t, P_m(t)) dt, \quad \forall m \in \mathbb{N}.
\]

It follows that:

\[
0 \leq \lim_{m \to \infty} \int_{\alpha}^{\beta} R^2(t, T_m(t)) dt \leq \lim_{m \to \infty} \int_{\alpha}^{\beta} R^2(t, P_m(t)) dt = 0.
\]

We obtain:

\[
\lim_{m \to \infty} \int_{\alpha}^{\beta} R^2(t, T_m(t)) dt = 0.
\]

From this limit we obtain that \( \forall \epsilon > 0, \exists m_\epsilon \in \mathbb{N} \) such that \( \forall m \in \mathbb{N}, m > m_\epsilon \). It follows that \( T_m(t) \) is a weak \( \epsilon \) -approximate polynomial solution of the problem (60, 61) q.e.d.
As a consequence of the second remark, in order to find \( \varepsilon \)-approximate polynomial solutions of the problem (60, 61) by PLSM we will first determine weak approximate polynomial solutions, \( \tilde{x}_a \) following the steps 1 to 5 previously described. If \( |R(t, \tilde{x}_a)| < \varepsilon \) then \( \tilde{x}_a \) is also an \( \varepsilon \)-approximate polynomial solution of the problem.

1.5.3. Results and discussion

1.5.3.1. Nonlinear Riccati equation

Our first test problem is the following Cauchy problem:

\[
\begin{align*}
\dot{x}(t-2) - t^2 \cdot x(2 \cdot t - 3) - x^2(t-1) + 5t^4 - 20t^3 + 19t^2 - 2t - 3 &= 0 \\
x(0) &= -2
\end{align*}
\]

The exact solution of this problem is \( x_e(t) = t^2 - t - 2 \).

In [86], by using a numerical approximation based on the Bessel functions of the first kind, Yüzbaşı computed the following approximate solution of equation (71):

\[
x_{BES} = (0.1 \cdot 10^{-19})t^3 + t^2 - 1.0000000000000001t - 2
\]

The maximum absolute error of this approximation is reported as \( 1.2171 \cdot 10^{-19} \).

Using the steps described in the previous section we performed the following computations:

- We computed a polynomial solution of the form:

  \[
x_{PLSM} = c_0 + c_1 \cdot t + c_2 \cdot t^2
\]

- Taking into account the fact that by using the initial condition \( c_0 \) must be equal to \(-2\), the functional (10) corresponding to the problem (71) is:

\[
J(c_1, c_2) = \frac{c_1^4}{5} - \frac{2c_1^3 c_2}{3} + \frac{6c_1^3}{5} + \frac{6c_1^2 c_2^2}{7} - \frac{134c_1^2 c_2}{21} + \frac{701c_1^2}{105} - \frac{c_1 c_2^2}{2} + \frac{107c_1 c_2^2}{14}
\]

\[
- \frac{1049c_1 c_2}{70} + \frac{1963c_1}{210} + \frac{c_2^4}{9} - \frac{25c_1^3}{9} + \frac{1206c_2^2}{35} - \frac{18341c_2}{315} + \frac{8543}{315}
\]
• To find the minimum of this functional we compute the stationary points as the solutions of the system:

\[
\begin{align*}
\frac{\partial J}{\partial c_1} &= 0 \\
\frac{\partial J}{\partial c_2} &= 0
\end{align*}
\]

Since the only stationary point is \( c_1 = -1, c_2 = 1 \) and it is easy to show that this point is indeed a minimum, we obtain the following polynomial approximate solution of the equation (12):

\[ x_{PLSM} = -2 - t + t^2, \]

which is actually the exact solution of the problem.

We remark that while in this simple case we were able to compute the exact minimum of the functional (69), in most of the applications the direct computation of the minimum is not possible and some numerical techniques are employed.

1.5.3.2. Second-order neutral functional-differential equation with proportional delays

Our second test problem is the following Cauchy problem:

\[
\begin{align*}
x''(t) - \frac{3}{4} x(t) - x\left(\frac{t}{2}\right) - x'\left(\frac{t}{2}\right) - \frac{1}{2} x''\left(\frac{t}{2}\right) + t^2 + t - 1 &= 0, \quad 0 \leq t \leq 1 \\
x(0) &= 0, x'(0) &= 0
\end{align*}
\]

The exact solution of this problem is \( x_*(t) = t^2 \).

In [87], Bellen and Zennaro used a two-stage order-one Runge–Kutta method to compute a numerical solution of the problem (72) and the absolute error of their approximation is of the order of \( 10^{-3} \). In [88] and [89], Wang et al. used the one-leg \( \theta \)-method to compute approximate solutions of (72) and the absolute error of their best approximation is of the order of \( 10^{-3} \). In [90], Chen and Wang used the Variational Iteration Method to compute approximate solutions of (72) and the absolute error of their best approximation is of the order of \( 10^{-6} \). In [91], Biazar and Ghanbari used the Homotopy Perturbation Method
(HPM) to compute approximate solutions of equation (72) and the absolute error of their best approximation is of the order of $10^{-6}$. In [92], Sedaghat, Ordokhani and Dehghan used a method based on Chebyshev polynomials to compute a numerical solution of the problem (72) and the absolute error of their approximation is of the order of $10^{-17}$.

Using our method we performed the following computations:

- We compute a polynomial solution of the form:
  \[ x_{PLSM} = c_0 + c_1 \cdot t + c_2 \cdot t^2 \]

- Taking into account the initial conditions we obtain the following values of the constants : $c_0 = 0, c_1 = 0$. In this case the functional (69) corresponding to the problem (72) is:
  \[ J(c_2) = \frac{11 c_2^2}{30} - \frac{11 c_2}{15} + \frac{11}{30} \]

- To find the minimum of this functional we compute the stationary points as the solutions of the equation $J'(c_2) = 0$. The only stationary point is $c_2 = 1$ and it is easy to show that this point is indeed a minimum.

- We obtain the following polynomial approximate solution of equation (72):
  \[ x_{PLSM} = t^2 \]

Again we obtained the exact solution of the problem.

### 1.5.3.3. Pantograph-type nonlinear equation

Our third test problem is the following Cauchy problem:

\[
\begin{align*}
&x'''(t) - e^{-t} x'(t) - \frac{1}{5} x(t) + 2 x(t) \left(\frac{t}{3}\right) - x^2(t) - \left(-\frac{1}{4} \sin^2 \left(\frac{t}{3}\right) + \frac{4}{9} \sin \left(\frac{t}{3}\right)\right) \\
&+ \sin \left(\frac{t}{9}\right) \exp(-t) \left(\frac{1}{6} \cos \left(\frac{t}{3} - \frac{1}{15}\right) - \frac{1}{6} \sin \left(\frac{t}{3} - \frac{1}{10}\right)\right) \left(-\frac{1}{9} \cos^2 \left(\frac{t}{2}\right) + \frac{1}{4} \cos \left(\frac{t}{2}\right)\right) \\
&+ \frac{2}{3} \cos \left(\frac{t}{6}\right) - \frac{1}{3} \sin \left(\frac{t}{3}\right) \cos \left(\frac{t}{2}\right) + t \left(\frac{1}{2} \sin \left(\frac{t}{3}\right) - \frac{1}{3} \cos \left(\frac{t}{2}\right)\right) = 0
\end{align*}
\] (73)

\[
\begin{align*}
3 x(0) + 6 x'(0) &= 2 \\
-2 x(0) + x'(0) &= -\frac{1}{2}
\end{align*}
\]
The exact solution of this problem is:

\[ x_0(t) = \frac{1}{2} \sin \left( \frac{t}{3} \right) + \frac{1}{3} \cos \left( \frac{t}{2} \right) . \]

In [93], Shakeri and Dehghan used the Homotopy Perturbation Method (HPM) to compute approximate solutions \( x_{HPM} \) of equation (73).

In [94], Yildirim, Kocak and Tutkun used the Variational Iteration Method (VIM) to compute approximate solutions \( x_{VIM} \) of equation (73).

Using our method we obtained the following polynomial approximate solution of equation (73):

\[
x_{PLSM} = -3.8337691 \cdot 10^{-6} \cdot t^5 + 0.00088928 \cdot t^4 -
- 0.00309479 \cdot t^3 - 0.0416659 \cdot t^2 + \frac{1}{6} \cdot t + \frac{1}{3} .
\]

Table 16 presents the comparison between the absolute errors (as the difference in absolute value between the approximate solution and the exact solution) corresponding to the approximate solution \( x_{HPM} \) from [93], to the approximate solution \( x_{VIM} \) from [94] and to our approximate solution \( x_{PLSM} \).

The absolute errors corresponding to the approximate solution \( x_{VIM} \) are not explicitly given in [94], but we extracted some approximate values from the figure presented in the paper (figure 2b).

<table>
<thead>
<tr>
<th>( t )</th>
<th>VIM</th>
<th>HPM</th>
<th>PLSM</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>5 \cdot 10^{-5}</td>
<td>1.34 \cdot 10^{-5}</td>
<td>1.02 \cdot 10^{-8}</td>
</tr>
<tr>
<td>0.4</td>
<td>1 \cdot 10^{-4}</td>
<td>5.13 \cdot 10^{-5}</td>
<td>6.14 \cdot 10^{-8}</td>
</tr>
<tr>
<td>0.6</td>
<td>5 \cdot 10^{-5}</td>
<td>6.26 \cdot 10^{-6}</td>
<td>8.92 \cdot 10^{-8}</td>
</tr>
<tr>
<td>0.8</td>
<td>1 \cdot 10^{-4}</td>
<td>2.21 \cdot 10^{-5}</td>
<td>1.02 \cdot 10^{-7}</td>
</tr>
<tr>
<td>1</td>
<td>5 \cdot 10^{-4}</td>
<td>3.69 \cdot 10^{-5}</td>
<td>1.53 \cdot 10^{-7}</td>
</tr>
</tbody>
</table>
It is easy to see that the approximate solution given by PLSM is much closer to the exact solution than the previous ones from [93] and [94]. We mention the fact that our solution is not only more precise but, at the same time, it has a much simpler form.

1.5.3.4. Generalized pantograph-type equation

Our third test problem is the following Cauchy problem:

\[
\begin{align*}
    x'''(t) + x(t) + x(t-0.3) - e^{-t+0.3} \\
    x(0) = 1, x'(0) = -1, x''(0) = 1, \quad 0 \leq t \leq 1
\end{align*}
\]  
(74)

The exact solution of this problem is

\[ x_e(t) = e^{-t}. \]

In [95], Doha, Bhrawy, Baleanu and Hafez used the Jacobi rational-Gauss collocation method (JRC) to compute approximate solutions \( x_{\text{JRC}} \) of equation (74). In [96], Sezer and Akyuz-Dascioglu used the Taylor series method (TM) to compute approximate solutions \( x_{\text{TM}} \) of equation (74). In [97], Ozturk and Gulsu used a Chebyshev method (CM) to compute approximate solutions \( x_{\text{CM}} \) of equation (74). In [98], Yalcinbas, Aynigul and Sezer used a Hermite collocation method (HCM) to compute approximate solutions \( x_{\text{HCM}} \) of the same equation.

Using our method we obtained the following polynomial approximate solution of equation (74):

\[ x_{\text{PLSM}} = -1.6853299 \cdot 10^{-6} \cdot t^9 + 0.0000227752 \cdot t^8 - 0.000196271 \cdot t^7 + 0.00138757 \cdot t^6 \\
- 0.00833288 \cdot t^5 + 0.0416666 \cdot t^4 - 0.166667 \cdot t^3 + \frac{t^2}{2} - t + 1 \]

Table 17 presents the comparison between the absolute errors (as the difference in absolute value between the approximate solution and the exact solution) corresponding to the approximate solution \( x_{\text{TM}} \) from [96], to the approximate solution \( x_{\text{CM}} \) from [97], to the approximate solution \( x_{\text{HCM}} \) from
[98], to the (best) approximate solution $x_{JRC}$ from [95], as given in [95], together with the errors corresponding to our approximate solution $x_{PLSM}$.

Table 17. Comparison of TM, CM, HCM, JRC and PLSM for equation (74)

<table>
<thead>
<tr>
<th>$t$</th>
<th>TM</th>
<th>CM</th>
<th>HCM</th>
<th>JRC</th>
<th>PLSM</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.2</td>
<td>$8.54 \times 10^{-8}$</td>
<td>$3.70 \times 10^{-7}$</td>
<td>$6.200 \times 10^{-9}$</td>
<td>$3.605 \times 10^{-8}$</td>
<td>$6.255 \times 10^{-13}$</td>
</tr>
<tr>
<td>0.4</td>
<td>$5.36 \times 10^{-6}$</td>
<td>$2.38 \times 10^{-5}$</td>
<td>$5.760 \times 10^{-8}$</td>
<td>$9.299 \times 10^{-9}$</td>
<td>$7.193 \times 10^{-12}$</td>
</tr>
<tr>
<td>0.6</td>
<td>$5.95 \times 10^{-5}$</td>
<td>$5.97 \times 10^{-6}$</td>
<td>$1.796 \times 10^{-7}$</td>
<td>$3.503 \times 10^{-10}$</td>
<td>$1.839 \times 10^{-11}$</td>
</tr>
<tr>
<td>0.8</td>
<td>$3.26 \times 10^{-4}$</td>
<td>$3.48 \times 10^{-5}$</td>
<td>$3.735 \times 10^{-7}$</td>
<td>$8.345 \times 10^{-9}$</td>
<td>$3.341 \times 10^{-11}$</td>
</tr>
<tr>
<td>1</td>
<td>$1.21 \times 10^{-3}$</td>
<td>$2.03 \times 10^{-4}$</td>
<td>$6.368 \times 10^{-7}$</td>
<td>$1.161 \times 10^{-8}$</td>
<td>$5.642 \times 10^{-11}$</td>
</tr>
</tbody>
</table>

Again it is easy to see that the approximate solution given by PLSM is much closer to the exact solution than the previous ones.

1.5.3.5. Nonlinear time-delay model in biology

Our next test problem is:

$$\begin{cases}
x'(t) - 2 \cdot x(t) \left(1 - \frac{x(t-0.1)}{0.5}\right) = 0 \\
x(0) = 1
\end{cases} \quad (75)$$

The exact solution of this problem is not known.

In [99], Dehghan and Salehi used the Variational Iteration Method (VIM) and the Adomian Decomposition Method (ADM) to compute approximate solutions $x_{VIM}$ and $x_{ADM}$ of equation (75).

Using our method we obtained the following polynomial approximate solution of equation (75):

$$x_{PLSM} = -2.57841 \cdot t^7 + 15.8186 \cdot t^6 - 39.5946 \cdot t^5 + 52.059 \cdot t^4 - 38.6665 \cdot t^3 + 16.3752 \cdot t^2 - 3.89839 \cdot t + 1$$

Table 18 present the comparison between the absolute errors (as the difference in absolute value between the approximate solution and the numerical
solution presented in [99]) corresponding to the approximate solutions $x_{ADM}$ and $x_{VIM}$ from [99] and to our approximate solution $x_{PLSM}$.

Table 18. Comparison of VIM, HPM and PLSM for equation (75)

<table>
<thead>
<tr>
<th>$t$</th>
<th>ADM</th>
<th>VIM</th>
<th>PLSM</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.05</td>
<td>$1.04 \cdot 10^{-1}$</td>
<td>-</td>
<td>$6.33 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>0.1</td>
<td>$1.38 \cdot 10^{-1}$</td>
<td>-</td>
<td>$7.86 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>0.15</td>
<td>$1.31 \cdot 10^{-1}$</td>
<td>-</td>
<td>$7.13 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>0.2</td>
<td>$1.26 \cdot 10^{-1}$</td>
<td>-</td>
<td>$5.83 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>0.25</td>
<td>$1.25 \cdot 10^{-1}$</td>
<td>-</td>
<td>$4.47 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>0.3</td>
<td>$1.26 \cdot 10^{-1}$</td>
<td>-</td>
<td>$3.27 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>0.35</td>
<td>$1.27 \cdot 10^{-1}$</td>
<td>-</td>
<td>$2.32 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>0.4</td>
<td>$1.29 \cdot 10^{-1}$</td>
<td>-</td>
<td>$1.66 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>0.5</td>
<td>-</td>
<td>$1.59 \cdot 10^{-2}$</td>
<td>$1.07 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>1</td>
<td>-</td>
<td>$4.3 \cdot 10^{-3}$</td>
<td>$7.41 \cdot 10^{-4}$</td>
</tr>
<tr>
<td>1.5</td>
<td>-</td>
<td>$8.2 \cdot 10^{-4}$</td>
<td>$6.14 \cdot 10^{-4}$</td>
</tr>
</tbody>
</table>

The approximate solution given by PLSM is closer to the numerical solution than the previous ones from [99].

1.5.3.6. Scalar differential equation with several delays

Our last test problem is:

$$\begin{cases} x'(t) + \frac{0.2}{\pi} x(t - \pi) \sin^2(t) + \frac{0.2}{\pi} x(t - 2\pi) \cos^2(t) = 0 \\ x(0) = 1 \end{cases}$$ (76)

The exact solution of this problem is not known. In [102], Berezansky and Braverman studied the existence of positive solutions for equations of the type

$$x'(t) + \sum_{k=0}^{m} q_k(t)x(h_k(t)) = 0$$
In the case of (76) it was shown that there exists indeed such a positive solution, but the solution was not effectively computed.

Using our method on the interval $[0,1]$ we obtained the following polynomial approximate solution of equation (76):

$$x_{\text{PLSM}} = -7.0502 \cdot 10^{-6} \cdot t^5 - 0.0000361911 \cdot t^4 + 0.000175863 \cdot t^3$$
$$+ 0.000388792 \cdot t^2 - 0.00895158 \cdot t + 0.1$$

The error obtained by replacing the approximate solution back in the equation and computing the remainder is of the order $10^{-7}$.

It is easy to see that the solution is positive on the interval where the computation was performed.

1.5.4. Conclusions

The polynomial least squares method (PLSM) was presented as a straightforward and efficient method to compute approximate polynomial solutions for nonlinear delay differential equations.

The applications presented clearly illustrate the accuracy of the method. Indeed, for the equations of the type (60) considered, namely the equations (71 - 76), the solutions obtained by using PLSM are more precise than the ones previously computed by using other methods.

1.6. Polynomial least squares method for the solution of nonlinear Volterra-Fredholm integral equations

1.6.1. Introduction

We consider nonlinear integral Volterra-Fredholm equations of the general form:

$$y(t) = f(t) + \lambda_1 \cdot \int_a^t k_1(t,s) \cdot g_1(s, y(s)) \, ds +$$
$$+ \lambda_2 \cdot \int_a^b k_2(t,s) \cdot g_2(s, y(s)) \, ds$$

(77)
where $a, b, \lambda_1, \lambda_2$ are constants and $f, k_1, k_2, g_1, g_2$ are functions assumed to have suitable derivatives on the $[a, b]$ interval.

Equations of this type are frequently used to model applications from various fields of science such as elasticity, electricity and magnetism, fluid dynamics, the dynamic of populations, mathematical economics etc.

In general, the exact solution of these nonlinear integral equations can not be found and thus it is often necessary to find approximate solutions for such equations. In this regard, many approximation techniques were employed over the years. Some of the approximation methods employed in recent years include:

- Collocation methods ([103], [114])
- Taylor expansion methods ([104])
- Rationalized Haar functions method ([105], [107])
- Chebyshev polynomials method ([106], [116])
- Triangular functions (TF) method ([108])
- Sinc approximation method ([109], [111], [115])
- Fixed point method ([110])
- Quasilinearization methods ([112])
- Bernstein polynomials methods ([113],[118])
- Radial basis functions method ([117])
- Bernoulli matrix method ([119])

The Polynomial least square method (PLSM), allows us to determine analytical approximate polynomial solutions for nonlinear integral equations. We will compare approximate solutions obtained using PLSM with approximate solutions computed recently for several test problems. If the exact solution of the test problem is polynomial, PLSM is able to find the exact solution. If not, PLSM allows us to obtain approximations with an error relative to the exact solution smaller than the errors obtained using other methods. In most cases, the approximate solutions obtained are not only more precise but also present a simpler expression in comparison to previous ones.
1.6.2. Approximation method description

In the following we generalize PLSM for the case of nonlinear integral equations of the mixed Volterra-Fredholm type.

We consider the following operator, corresponding to the equation (103):

\[
D(y) = y(t) - f(t) - \lambda_1 \cdot \int_a^t k_1(t, s) \cdot g_1(s, y(s))ds - \\
\lambda_2 \cdot \int_a^b k_2(t, s) \cdot g_2(s, y(s))ds
\]  

We also consider the so-called remainder associated to the equation (77), defined as the error obtained by replacing the exact solution \(y\) with an approximate solution \(y_{app}\):

\[
R(t, y_{app}) = D(y_{app}(t)), \ t \in [a, b]
\]

Before we present the actual steps of the method, we introduce the following types of solutions:

- An \(\varepsilon\)-approximate polynomial solution of the equation (77) an approximate polynomial solution \(y_{app}\) satisfying the relation (3).

- A weak \(\delta\)-approximate polynomial solution of the equation (77) an approximate polynomial solution \(y_{app}\) satisfying the relation:

\[
\int_a^b R^2(t, y_{app})dt \leq \delta.
\]

We also consider the following type of convergence:

We consider the sequence of polynomials

\[
P_m(t) = a_0 + a_1t + \ldots + a_m t^m, \ a_i \in \mathbb{R}, \ i = 0, 1, \ldots, m
\]

the sequence of polynomials \(P_m(t)\) convergent to the solution of the equation (77) if

\[
\lim_{m \to \infty} D(P_m(t)) = 0.
\]

The aim of PLSM is to find a weak \(\varepsilon\)-polynomial solution of the type:

\[
\tilde{y}(t) = \sum_{k=0}^m c_k \cdot t^k, \ m > n,
\]

The values of the constants \(c_0, c_1, \ldots, c_m\) are calculated using the following steps:

**Step 1** - By substituting the approximate solution (80) in the equation (77) we obtain the following expression:
\[
R(t, c_0, c_1, \ldots, c_m) = R(t, \tilde{y}) = \\
= \tilde{y}(t) - f(t) - \lambda_1 \cdot \int_a^b k_1(t, s) \cdot g_1(s, \tilde{y}(s)) \, ds - \\
- \lambda_2 \cdot \int_a^b k_2(t, s) \cdot g_2(s, \tilde{y}(s)) \, ds
\]  

(81)

- **Step 2** - Next we attach to the equation (77) the following real functional:

\[
J(c_0, c_1, \ldots, c_m) = \int_a^b R^2(t, c_0, c_1, \ldots, c_m) \, dt
\]  

(82)

- **Step 3** - We compute \(c^0_0, c^0_1, \ldots, c^0_m\) as the values which give the minimum of the functional (82). We remark that the computation of the minimum can be performed in many ways and some examples are presented in the next section.

- **Step 4** - Using the constants \(c^0_0, c^0_1, \ldots, c^0_m\) determined in the previous step, we consider the polynomial:

\[
T_m(t) = \sum_{k=0}^m c^0_k t^k
\]  

(83)

The following convergence theorem holds:

**Theorem 1** If the sequence of polynomials \(P_m(t)\) converges to the solution of the equation (1), then the sequence of polynomials \(T_m(t)\) from (83) satisfies the property:

\[
\lim_{m \to \infty} \int_a^b R^2(t, T_m) \, dt = 0
\]

Moreover, \(\forall \varepsilon > 0, \exists m_0 \in \mathbb{N}\) such that \(\forall m \in \mathbb{N}, m > m_0\) it follows that \(T_m(t)\) is a weak \(\varepsilon\)-approximate polynomial solution of the equation (77). Based on the way the coefficients of polynomial \(T_m(t)\) are computed and taking into account the relations (80-83), the following inequality holds:

\[
0 \leq \int_a^b R^2(t, T_m(t)) \, dt \leq \int_a^b R^2(t, P_m(t)) \, dt, \quad \forall m \in \mathbb{N}.
\]

It follows that:

\[
0 \leq \lim_{m \to \infty} \int_a^b R^2(t, T_m(t)) \, dt \leq \lim_{m \to \infty} \int_a^b R^2(t, P_m(t)) \, dt = 0.
\]
We obtain:
\[ \lim_{m \to \infty} R^2(t, T_m(t))dt = 0. \]

From this limit we obtain that \( \forall \epsilon > 0, \exists m_0 \in \mathbb{N} \) such that \( \forall m \in \mathbb{N}, m > m_0 \) it follows that \( T_m(t) \) is a weak \( \epsilon \) -approximate polynomial solution of the equation (77) q.e.d.

- **Step 5** - Taking into account the fact that any \( \epsilon \)-approximate polynomial solution of the equation (77) is also a weak \( \epsilon^2 \cdot (b-a) \) - approximate polynomial solution (but the opposite is not always true), it follows that the set of weak approximate solutions of the equation (77) also contains the approximate solutions of the equation. As a consequence, in order to find \( \epsilon \) -approximate polynomial solutions of the equation (77) by PLSM we will first compute weak approximate polynomial solutions, \( \tilde{y}_{app} \). If \( |R(t, \tilde{y}_{app})| < \epsilon \) then \( \tilde{y}_{app} \) is also an \( \epsilon \)-approximate polynomial solution of the problem.

### 1.6.3. Results and discussion

Our first application is a simple nonlinear Fredholm integral equation ([111],[119]):

\[ y(t) = 1 - \frac{5}{12} \cdot t + \int_0^t s \cdot y^2(s) ds \quad (84) \]

This equation is obtained from (77) by choosing the constants \( a = 0, b = 1, \hat{\lambda}_1 = 0, \hat{\lambda}_2 = 1 \) and the functions

\[ f(t) = 1 - \frac{5}{12} \cdot t, \quad k_2(t,s) = t \cdot s, \quad g_2(s, y(s)) = y^2(s). \]

The exact solution of the equation (84) presented in [111],[119] is \( y_e(t) = 1 + \frac{t}{3} \). In [111] approximate solutions of (84) were computed using two Sinc-collocation-type methods and in [119] the exact solution \( y_e(t) \) was determined using a Bernoulli matrix method.
Using PLSM we found two exact solutions of (84):

\[ y_{PLSM_1} = 1 + \frac{t}{3} \]
\[ y_{PLSM_2} = 1 + t \]  \hspace{1cm} (85)

Our second application is a nonlinear Volterra integral equation ([106], [116]):

\[ y(t) = \frac{1}{15} \left( 2t^6 - 5t^4 + 15t^2 - 8t - 20 \right) + \int_{-1}^{t} (t-2s)y^2(s)ds \]  \hspace{1cm} (86)

This equation is obtained from (77) by choosing the constants \( a = -1, \lambda_1 = 1, \lambda_2 = 0 \) and the functions

\[ f(t) = \frac{1}{15} \left( 2t^6 - 5t^4 + 15t^2 - 8t - 20 \right), \; k_i(t,s) = t - 2s, \; g_i(s,y(s)) = y^2(s). \]

The exact solution of of the equation of (86) is \( y_e(t) = t^2 - 1 \). In [106] and [116] approximate solutions of (86) were computed using approximations methods based on Chebyshev polynomials. The absolute errors of the approximate solutions obtained are of the order of \( 10^{-2} \) in [106] and of \( 10^{-15} \) in [116].

In the following we will compute the exact solution of the problem (86) using PLSM.

The solution obtained using PLSM is in fact the exact solution of (86):

\[ y_{PLSM} = t^2 - 1. \] \hspace{1cm} (87)

The third application is a nonlinear mixed Volterra-Fredholm integral equation ([107], [108], [117]):

\[ y(t) = -\frac{1}{30}t^6 + \frac{1}{3}t^4 - \frac{1}{3}t^2 + \frac{5}{3}t - \frac{5}{4} + \int_0^t (t-s)(y(s))^2 ds + \int_0^t (t+s)y(s)ds \] \hspace{1cm} (88)

The exact solution of of the equation is \( y_e = t^2 - 2 \). In [107] approximate solutions of (88) were computed using a Rationalized Haar functions method, in [10] approximate solutions of (88) were computed using a Triangular functions method and in [117] approximate solutions of (88) were computed using a
Radial basis functions method. The values of the absolute errors of the approximate solutions obtained varied from $10^{-3}$ to $10^{-15}$ but none of these methods could find the exact solution.

The solution obtained using PLSM is the exact solution of (88):

$$y_{PLSM} = t^2 - 2$$ (89)

The next application is the nonlinear Volterra-Fredholm integral equation ([105], [114]):

$$y(t) = 2 \cos(t) - 2 + 3 \int_0^t \sin(t - s) y^2(s)ds +$$

$$+ \frac{6}{7 - 6 \cos(1)} \int_0^1 (1 - s) \cos^2(s)(s + y(s))ds$$ (90)

The exact solution of the equation (90) is $y_e = \cos(t)$. Approximate solutions for this equation were computed in [105] using the Rationalized Haar functions method RHM and in [114] using a composite collocation method CCM consisting of a hybrid of block-pulse functions and Lagrange polynomials. The solution in [105] contained sixteen terms while the one in [114] is a piecewise polynomial solution consisting of two polynomials of fourth degree.

Using the PLSM we computed a seventh order polynomial approximate solution of (90). We obtained the approximate solution:

$$y_{PLSM} = 0.000103064070567775t^7 - 0.00157260234848809t^6$$

$$+ 0.000179224005722049t^5 + 0.0415635195906221t^4 + 0.0000354244251299783t^3$$

$$- 0.500006936162663t^2 + 7.11478876757492 \cdot 10^{-7}t + 1.00000000094979$$

Table 19 presents the comparison of the absolute errors corresponding to the three approximate solutions $y_{RHM}$ ([105]), $y_{CCM}$ ([114]) and $y_{PLSM}$. The approximate solution given by PLSM is much closer to the exact solution and has a simpler form.
Table 19. Comparison of the absolute errors of the approximate solutions for problem (90)

<table>
<thead>
<tr>
<th></th>
<th>RHM</th>
<th>CCM</th>
<th>PLSM</th>
</tr>
</thead>
<tbody>
<tr>
<td>t = 0</td>
<td>5⋅10⁻⁴</td>
<td>3.1021⋅10⁻⁵</td>
<td>9.4979⋅10⁻¹⁰</td>
</tr>
<tr>
<td>t = 0.2</td>
<td>5⋅10⁻⁶</td>
<td>3.2341⋅10⁻⁶</td>
<td>3.1009⋅10⁻⁰⁸</td>
</tr>
<tr>
<td>t = 0.4</td>
<td>1⋅10⁻⁴</td>
<td>1.9092⋅10⁻⁵</td>
<td>3.7752⋅10⁻⁰⁸</td>
</tr>
<tr>
<td>t = 0.6</td>
<td>1⋅10⁻⁴</td>
<td>1.5029⋅10⁻⁵</td>
<td>4.9982⋅10⁻⁰⁸</td>
</tr>
<tr>
<td>t = 0.8</td>
<td>5⋅10⁻⁵</td>
<td>3.6499⋅10⁻⁶</td>
<td>7.0526⋅10⁻⁰⁸</td>
</tr>
<tr>
<td>t = 1</td>
<td>1⋅10⁻⁴</td>
<td>2.4290⋅10⁻⁵</td>
<td>1.0015⋅10⁻⁰⁷</td>
</tr>
</tbody>
</table>

1.6.4. Conclusions

The test problems solved illustrate the accuracy of the method, since for all the problems we were able to compute better approximations than the ones computed in previous papers, and in some cases, the exact solutions were found. Moreover, the expressions of the approximations computed by PLSM is also simpler than the expressions of the approximations computed by using other methods.

1.7. Optimal homotopy asymptotic method applied to the steady flow of a fourth-grade fluid past a porous plate

1.7.1. Introduction

The Optimal Homotopy Asymptotic Method (OHAM), is proposed in [141] and used to the steady flow of a fourth-grade fluid. This approach does not depend upon any small/large parameters. This method provides us with a convenient way to control the convergence of approximation series and adjust
convergence regions when necessary. The series solution is developed and the recurrence relations are given explicitly. Considerable attention has been paid in recent years to problems of flow of non-Newtonian fluids. The governing equations are very complicated and highly nonlinear as compared to those for Newtonian fluids. There are few analytic solutions of the equations involving Newtonian fluids and such solutions become even rarer when equations for non-Newtonian fluids are considered. There is no single model available which clearly exhibits the properties of all non-Newtonian fluids. The governing differential equation is highly nonlinear and it is not easy to obtain an analytic solution using traditional methods. Also, the order of the differential equation in the case of fourth-grade fluid is higher than that of the Navier–Stokes equation. The no-slip boundary condition is sufficient for a Newtonian fluid but for a fourth-grade one it may not be sufficient and, therefore, one needs additional conditions at the boundary.

Consider the steady flow of a fourth-grade fluid past a porous plate. If $T$ is the Cauchy stress tensor, $V$ is the velocity field, $\rho$ is the fluid density, we choose the x-axis parallel to the plate and the y-axis normal to it, the velocity field depending only on $y$. The flow past a porous plate with suction or injection and with a uniform stream at infinity has a two-dimensional structure. The complete set of governing equations of the fourth-grade fluid consists of the incompressibility conditions[119], [120] and [121]

$$\nabla V = 0$$

and the momentum equation

$$\frac{dV}{dt} = \frac{1}{\rho} \nabla T$$

After other simple manipulations, from Eqs.(1) and (2) we obtain

$$-\rho V_0 \frac{du}{dy} = \mu \frac{d^2 u}{dy^2} - \alpha_1 V_0 \frac{d^3 u}{dy^3} + \beta_2 V_0 \frac{d^4 u}{dy^4} + 6(\beta_2 + \beta_3) \left( \frac{du}{dy} \right)^2 \frac{d^2 u}{dy^2} - \gamma_1 V_0 \frac{d^5 u}{dy^5}$$

$$-2(3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8) V_0 \frac{d}{dy} \left[ \left( \frac{du}{dy} \right)^2 \frac{d^2 u}{dy^2} \right]$$

(93)
where \( \alpha(i = 1, 2), \beta(i = 1, 2, 3), \gamma(i = 1, 2, ..., 8) \) are material constants and \( \mu \) is the coefficient of viscosity, \( V_0 > 0 \) is the suction velocity and \( V_0 < 0 \) corresponds to the injection velocity.

The boundary conditions on \( u \) are

\[
    u(0) = 0; \quad u(y) \to u_0 \text{ as } y \to \infty \tag{94}
\]

where \( u_0 \) is the mainstream velocity.

Introducing the quantities

\[
    \bar{u} = \frac{u}{u_0}, \quad \bar{v} = \frac{u_0^2}{\nu}, \quad \bar{v}_0 = \frac{v_0}{u_0}, \quad \bar{\alpha}_1 = \frac{\alpha_i u_0^2 v_0^2}{\rho v_0^3}, \quad \bar{\beta}_1 = \frac{\beta_i u_0^2 v_0^4}{\rho v_0^3}, \quad \bar{\gamma}_1 = \frac{\gamma_i u_0^2 v_0^6}{\rho v_0^3} \tag{95}
\]

and a new variable

\[
    \tau = \nu_0 y \tag{96}
\]

the boundary value problem takes the form

\[
\frac{d^2 u}{d \tau^2} + \frac{du}{d \tau} - \alpha_1 \frac{d^3 u}{d \tau^3} + \beta_1 \frac{d^4 u}{d \tau^4} - \gamma_1 \frac{d^5 u}{d \tau^5} + \beta \left( \frac{du}{d \tau} \right)^2 \frac{d^2 u}{d \tau^2} - \gamma \left[ 2 \frac{d^3 u}{d \tau^3} \left( \frac{du}{d \tau} \right)^2 + \left( \frac{du}{d \tau} \right)^2 \frac{d^3 u}{d \tau^3} \right] = 0 \tag{97}
\]

\[
    u(0) = 0; \quad u(\infty) = 1 \tag{98}
\]

For simplicity we omitted the bars of the non-dimensional quantities.

The fifth-order differential equation (97) subject to the boundary conditions (98) can be treated using OHAM.

**1.7.2. Approximation method description (Optimal Homotopy Asymptotic Method (OHAM))**

We apply OHAM to the following differential equation:

\[
L(u(\tau)) + g(\tau) + N(u(\tau)) = 0, \quad B(u) = 0 \tag{99}
\]
where $L$ is a linear operator, $\tau$ denotes an independent variable, $u(\tau)$ is an unknown function, $g(\tau)$ is a known function, $N(u(\tau))$ is a nonlinear operator and $B$ is a boundary operator. By means of OHAM one first constructs a family of equations:

$$(1 - p)[L(\phi(\tau, p)) + g(\tau)] = H(p)[L(\phi(\tau, p)) + g(\tau) + N(\phi(\tau, p))], B(\phi(\tau, p)) = 0$$ (100)

where $p \in [0, 1]$ is an embedding parameter, $H(p)$ is a nonzero auxiliary function for $p \neq 0$ and $H(0) = 0$, $\phi(\tau, p)$ is an unknown function. Obviously, when $p = 0$ and $p = 1$ it holds that

$$\phi(\tau, 0) = u_0(\tau), \phi(\tau, 1) = u(\tau)$$ (101)

Thus, as $p$ increases from 0 to 1, the solution $\phi(\tau, p)$ varies from $u_0(\tau)$ to the solution $u(\tau)$, where $u_0(\tau)$ is obtained from Eq. (10) for $p = 0$:

$$L(u_0(\tau)) + g(\tau) = 0, B(u_0) = 0$$ (102)

We choose the auxiliary function $H(p)$ in the form

$$H(p) = pC_1 + p^2C_2 + \cdots$$ (103)

where $C_1, C_2, \ldots$ are constants which can be determined later.

Expanding $\phi(\tau, p, C_i)$ in a series with respect to $p$, one has

$$\phi(\tau, p, C_i) = u_0(\tau) + \sum_{k \geq 1} u_k(\tau, C_i) p^k, \quad i = 1, 2, \ldots$$ (104)

Substituting Eq. (104) into Eq. (100), collecting the same powers of $p$ to zero, we obtain

$$L(u_i(\tau)) = C_i N_0(u_0(\tau)), B(u_i) = 0$$ (105)

$$L(u_k(\tau) - u_{k-1}(\tau)) = C_k N_0(u_0(\tau)) +$$

$$+ \sum_{i = 1}^{k-1} C_i [L(u_{k-i}(\tau)) + N_{k-i}(u_0(\tau), u_i(\tau), \ldots, u_{k-i}(\tau))], k = 2, 3, \ldots, B(u_k) = 0, C_k = K(t)$$ (106)

where $N_i, i \geq 0$, are the coefficients of $p^i$ in the nonlinear operator $N$: 
\( N(u(t)) = N_0(u_0(t)) + pN_1(u_0(t), u_1(t)) + p^2N_2(u_0(t), u_1(t), u_2(t)) + \ldots \) \tag{107} 

and \( K(\tau) \) is a function which will be defined later.

It should be emphasized that the \( u_k \) for \( k \geq 0 \) are governed by the linear Eq. (102), (105) and (106) and with the linear boundary conditions that come from original problem, which can be easily solved.

The convergence of the series (104) depends upon the auxiliary constants \( C_1, C_2, \ldots \). If it is convergent at \( p=1 \), one has

\[
u(\tau, C_i) = u_0(\tau) + \sum_{k=1}^{\infty} u_k(\tau, C_i) \tag{108}\]

Generally speaking, the solution of Eq. (99) can be determined approximately in the form

\[
u^m = u_0(\tau) + \sum_{k=1}^{m} u_k(\tau, C_i) \tag{109}\]

We note that the last coefficient \( C_m \) can be function of \( \tau \).

Substituting Eq. (109) into Eq. (99), there results the following residual:

\[
R(\tau, C_i) = L(\nu^m(\tau, C_i)) + g(\tau) + N(\nu^m(\tau, C_i)) \tag{110}
\]

If \( R(\tau, C_i)=0 \) then \( \nu^m(\tau, C_i) \) happens to be the exact solution. Generally such a case will not arise for nonlinear problems, but we can minimize the functional

\[
J(C_1, C_2, \ldots, C_m) = \int_a^b R^2(\tau, C_1, C_2, \ldots, C_m) d\tau \tag{111}
\]

where \( a \) and \( b \) are two values, depending on the given problem. The unknown constants \( C_i (i=1, 2, \ldots, m) \) can be identified from the conditions

\[
\frac{\partial J}{\partial C_1} = \frac{\partial J}{\partial C_2} = \ldots = \frac{\partial J}{\partial C_m} = 0 \tag{112}
\]

With these constants known, the approximate solution (of order \( m \)) (109) is well determined. The constants \( C_1, C_2, \ldots \) can be determined in other forms (collocation, Ritz method etc.).

It is easy to observe that the so-called Homotopy Perturbation Method (HPM) proposed by He\[122\] and \[123\] is only a special case of Eq. (100) when \( H(p) = -p \), and the Homotopy Analysis Method (HAM) proposed by Liao\[124\] and \[125\] is also another special case of Eq. (100) when
where the parameter $h$ is determined from so-called “$h$-curves”. It can be observed that the method proposed in this work generalizes these two methods using the special (more general) auxiliary function $H(p)$.

1.7.3. Results and discussion

1.7.3.1. Application of OHAM to the steady flow of fourth-grade fluid

According to Eqs. (97) and (98), we choose the linear operator

$$L(\phi(\tau, p)) = \frac{\partial^2 \phi(\tau, p)}{\partial \tau^2} + \frac{\partial \phi(\tau, p)}{\partial \tau}$$

and we define a nonlinear operator

$$N(\phi(\tau, p)) = -\alpha \frac{\partial^4 \phi(\tau, p)}{\partial \tau^4} + \beta \frac{\partial^3 \phi(\tau, p)}{\partial \tau^3} - \gamma \frac{\partial^2 \phi(\tau, p)}{\partial \tau^2} + \beta \left(\frac{\partial \phi(\tau, p)}{\partial \tau}\right)^2 \frac{\partial^2 \phi(\tau, p)}{\partial \tau^2}$$

$$- \gamma \left[2 \frac{\partial \phi(\tau, p)}{\partial \tau} \left(\frac{\partial^2 \phi(\tau, p)}{\partial \tau^2}\right)^2 + \left(\frac{\partial \phi(\tau, p)}{\partial \tau}\right)^2 \left(\frac{\partial^3 \phi(\tau, p)}{\partial \tau^3}\right)\right] = 0$$

The initial conditions are

$$\phi(0, p) = 0, \phi(\infty, p) = 1$$

Zeroth-order problem given by Eq. (102)$g(\tau)=0$:

$$\frac{d^2 u_0(\tau)}{d \tau^2} + \frac{du_0(\tau)}{d \tau} = 0, u_0(0) = 0, u_0(\infty) = 1.$$  

It is obtained that

$$u_0(\tau) = 1 - e^{-\tau}$$

First-order problem given by Eq. (105):
The solution of Eq. (118) is given by

\[ u_i(\tau) = \frac{C_1}{6}(\beta + 3\gamma)(e^{-\tau} - e^{-3\tau}) + C_1(\alpha_i + \beta_i + \gamma_i)\tau e^{-\tau}. \] (119)

Second-order problem given by Eq. (106) for \( k = 2 \):

\[
\frac{\partial^2 u_k(\tau)}{\partial \tau^2} + \frac{\partial u_k(\tau)}{\partial \tau} - \frac{\partial^2 u_k(\tau)}{\partial \tau^2} + C_1 \left[ -\alpha \frac{\partial^2 u_k(\tau)}{\partial \tau^2} + \beta \frac{\partial^3 u_k(\tau)}{\partial \tau^3} - \gamma \frac{\partial^3 u_k(\tau)}{\partial \tau^3} \right]
\]

\[ u_k(0) = 0, u_k(\infty) = 0 \]

Choosing \( K(\tau) = C_2 + C_3 e^{-2\tau} \) (\( C_2 \) and \( C_3 \) are constants), Eq. (120) has the solution
Substitution of Eqs.(27), (29) and (31) into Eq. (19) yields the second-order approximate solution \((m=2)\) for Eqs.(7) and (8):

\[
\bar{u}^2(\tau) = 1 + (A\tau^2 + B\tau + C)e^{-\tau} + (D\tau + E)e^{-3\tau} + Fe^{-5\tau}
\]

where

\[
A = \frac{C_e}{2} (\alpha + \beta + \gamma)^2
\]

\[
B = (\alpha + \beta + \gamma) \left[ C_e^2 + 2C_e + C_e^2 (2\alpha + 3\beta + 4\gamma) - \frac{C_e}{6} (\beta + 3\gamma) \right]
\]

\[
C = \frac{\beta + 3\gamma}{6} \left[ C_e^2 + 2C_e + C_e^2 \left( \frac{9}{2} \alpha + \frac{27}{2} \beta + \frac{81}{2} \gamma - \frac{\beta + 3\gamma}{2} \right) \right] + \frac{\alpha + \beta + \gamma}{6} \left[ C_e^2 (4\beta + 15\gamma) + C_e \right] - \frac{5C_e^2}{12} (\beta + 3\gamma)(\alpha + \beta + \gamma) + \frac{\beta + 3\gamma}{40} (5\beta C_e^2 + 5\gamma C_e^2 + 2C_e) - 1
\]

\[
D = \frac{C_e}{2} (\beta + 3\gamma)(\alpha + \beta + \gamma)
\]

\[
E = \frac{\beta + 3\gamma}{6} \left[ C_e^2 + 2C_e + C_e^2 \left( \frac{9}{2} \alpha + \frac{27}{2} \beta + \frac{81}{2} \gamma - \frac{\beta + 3\gamma}{2} \right) \right] - \frac{\alpha + \beta + \gamma}{6} \left[ C_e^2 (4\beta + 15\gamma) + C_e \right] + \frac{5(\beta + 3\gamma)}{12} C_e (\alpha + \beta + \gamma)
\]

\[
F = \frac{\beta + 3\gamma}{40} (5\beta C_e^2 + 5\gamma C_e^2 + 2C_e)
\]

(123)

Substituting the approximate solution of the second-order (32) into Eq. (20) yields the residual and the functional \(J\), respectively:
The constants $C_1$, $C_2$, $C_3$ result from the conditions (22). In the particular cases:

(a) $\alpha_1 = -0.5; \beta_1 = 1; \gamma_1 = 0.3; \beta = 1; \gamma = 1$,

it is obtained that

$C_1 = -0.12454086; C_2 = -0.14906175; C_3 = -0.08325414$ ,

and the approximate solution of second order in the form

$$u(\tau) = 1 + (-0.00496333673 \cdot \tau^2 - 0.274671983803 \cdot \tau - 1.027435052878) e^{-\tau} +$$

$$+ (0.02481668364 \cdot \tau + 0.02629465182) e^{3\tau} + 0.00114041055 e^{5\tau}.$$  

(b) $\alpha_1 = -0.5; \beta_1 = 1; \gamma_1 = 0.3; \beta = 0.5; \gamma = 0.5$,

it is obtained that

$C_1 = -0.12454086; C_2 = -0.14906175; C_3 = -0.08325414$,

and the approximate solution of second order in the form

$$u(\tau) = 1 + (-0.00246192499 \cdot \tau^2 - 0.166375477221 \cdot \tau - 1.01683012456) e^{-\tau} +$$

$$+ (0.006154812473 \cdot \tau + 0.01695236848) e^{3\tau} + 0.000169356021 e^{5\tau}.$$  

The residual $R(\tau)$ has maximum magnitude 0.06 in case (a) and 0.03 in case (b), which proves the accuracy of the approximate solution.

1.7.4. Conclusions

The OHAM has a distinct advantage over usual approximation methods in that the approximate solution obtained here is valid not only for weakly nonlinear equations, but also for strongly nonlinear ones. The convergence and
low error are remarkable. In this method we control the convergence using a number $k$ of auxiliary constants $C_1, C_2, \ldots$, which are optimally determined. In this respect the functional $J$ given by (21) must be minimized, which means that conditions (22) must be satisfied. This condition allows the determination of the parameters $C_1, C_2, \ldots$, and with these known, the approximate solution is well determined. The error obtained for the solution is remarkably low.

1.8. Approximate periodic solutions for oscillatory phenomenamodeled by nonlinear differential equations

1.8.1. Introduction

Oscillatory phenomena are frequently encountered in various fields of science such as, for example, physics, molecular biology and many branches of engineering. These oscillatory phenomena can be modelled using nonlinear differential equations. Nonlinear differential equations are one of the most important mathematical tools required for understanding these oscillatory phenomena present in everyday life. As is known, finding exact solutions of nonlinear differential equations is possible only in some particular cases. This justifies the need to resort to approximate methods for the computation of approximate periodic solutions, solutions which in turn could provide important information about the phenomena studied.

In the present paper we apply the Fourier least square method (FLSM) for the computation of approximate periodic solutions for oscillatory phenomena modelled by nonlinear differential equations of the type:

\[ x^{(n)}(t) = F(x^{(n-1)}(t), x^{(n-2)}(t), \ldots, x^{(1)}(t), x(t), t) \]  \hspace{1cm} (124)

with the initial conditions:

\[ x^{(k)}(0) = A_k, \quad k \in 0,1,2,\ldots,n-1 \]  \hspace{1cm} (125)

where $F$ is a nonlinear continuous function, $t \in \mathbb{R}$, $A_k \in \mathbb{R}$.

The equation (124) is a very general one, being able to model a large class of oscillatory phenomena. As a consequence, since our method can be applied for equation (124), it follows that it can be considered a powerful and useful method.

We remark that recently there has been much interest in finding approximate periodic solutions of nonlinear differential equations of the type
Among the methods used to compute such approximate periodic solutions we mention:  

The Homotopy Perturbation Method ([160],[161], [162] [163], [165]), the Variational Formulation Method ([164], [165], [166]), the Harmonic Balance Methods ([167], [168], [169], [170], [171], [172]), the Quasilinearization technique ([173]), the Reproducing Kernel Space method ([174]), the Adomian Decomposition Method ([175]), the Parameter-Expansion Method ([176], [177]), the Variational Iteration methods ([178], [179], [180]), the Energy Balance Method ([179], [181]), the Amplitude-Frequency Formulation ([163], [177]), the Homotopy Analysis Method ([182], [183], [184]), the Max-Min approach ([177]), the Optimal Homotopy Asymptotic Method ([185]), the Residue Harmonic Balance Method ([186]), the Enhanced Cubication Method ([187]), the Linearisation Method ([188]), Perturbation methods ([189]), numerical methods ([190], [191], [192], [193], [194]).

If the problem (124, 125) admits a periodic solution, FLSM allows us to determine an accurate approximate solution of this problem. In order to test the accuracy of the method, we apply it to several well-known examples of nonlinear equations and compare the approximate solutions obtained with this method with approximate solutions obtained by other methods.

1.8.2. Approximation method description(The Fourier-Least Squares Method)

We suppose that the problem (124, 125) admits a periodic solution with the period \( T \) and corresponding frequency \( \omega = \frac{2\cdot \pi}{T} \). We consider the operator:

\[
D(x) = x^{(n)}(t) - F(x^{(n-1)}(t), x^{(n-2)}(t), \ldots, x^{(1)}(t), x(t), t) \tag{126}
\]

If \( \tilde{x} \) is an approximate periodic solution of eq. (124), we evaluate the error obtained by replacing the exact solution \( x \) with the approximate one \( \tilde{x} \) as the remainder:

\[
R(t, \tilde{x}) = D(\tilde{x}(t)), \quad t \in R \tag{127}
\]

We call an \textbf{Fourier-function} a function of the form:

\[
f(t) = \sum_{k=0}^{m} \left[ a_k \cdot \cos(k \cdot \omega \cdot t) + b_k \cdot \sin(k \cdot \omega \cdot t) \right]
\]
where \( a_k, b_k, \omega \in \mathbb{R} \), \( m \in \mathbb{N} \).

We will find approximate Fourier-solutions \( \tilde{x} \) of the problem (124, 125) on \( \mathbb{R} \) which satisfy the following conditions:

\[
|R(t, \tilde{x})| < \varepsilon \quad (128)
\]

\[
\tilde{x}(0) = A_0, \quad \tilde{x}^{(1)}(0) = A_1 \quad (129)
\]

where:

\[
\tilde{x} = \sum_{k=0}^{p}[\tilde{a}_k \cdot \cos(k \cdot \omega \cdot t) + \tilde{b}_k \cdot \sin(k \cdot \omega \cdot t)] \quad (130)
\]

We call a \( \varepsilon \)-approximate Fourier-solution of the problem (124, 125) a Fourier-function \( \tilde{x} \) satisfying the relations (128, 129).

We call a weak \( \delta \)-approximate Fourier-solution of the problem (124, 125) a Fourier-function \( \tilde{x} \) satisfying the relation:

\[
\int_{0}^{2\pi/\omega} \varphi(t, \tilde{x}) dt \leq \delta
\]

together with the initial conditions (129).

We call a Fourier-sequence a sequence of Fourier-functions \( \{s_p(t)\}_{p \in \mathbb{N}} \),

\[
s_p(t) = \sum_{k=0}^{p}[a_{p_k} \cdot \cos(k \cdot \omega_p \cdot t) + b_{p_k} \cdot \sin(k \cdot \omega_p \cdot t)]
\]
\[
a_{p_k}, b_{p_k} \in \mathbb{R}, \quad b_{p0} = 0.
\]

We consider a Fourier-sequence satisfying the conditions:

\[
s_p(0) = A_0, \quad s_p^{(1)}(0) = A_1
\]

We call the Fourier-sequence \( \{s_p(t)\}_{p \in \mathbb{N}} \), convergent to the solution of the problem (124,125) if:

\[
\lim_{p \to \infty} R(t, s_p(t)) = 0.
\]

We will find a weak \( \varepsilon \)-approximate Fourier-solution of the type (7) where \( p \geq 1 \) and the constants \( \tilde{a}_p, \tilde{b}_p, \tilde{\omega} \) are calculated using the following steps:

• By substituting the approximate solution (130) in the equation (134) we obtain the following expression:

\[
R(t, \tilde{a}_1, \ldots, \tilde{a}_p, \tilde{b}_1, \ldots, \tilde{b}_p, \tilde{\omega}) = R(t, \tilde{x}) = \tilde{x}^{(n)}(t) - F(\tilde{x}^{(n-1)}(t), \tilde{x}^{(n-2)}(t), \ldots, \tilde{x}^{(1)}(t), \tilde{x}(t), t) \quad (131)
\]
• We attach to the problem (134, 135) the following real functional:

\[ J(\tilde{a}_1, ..., \tilde{a}_p, \tilde{b}_2, ..., \tilde{b}_p, \tilde{\omega}) = \int_{0}^{2\pi/\omega} R^2(t, \tilde{a}_0, \tilde{a}_1, ..., \tilde{a}_p, \tilde{b}_1, ..., \tilde{b}_p, \tilde{\omega}) dt \]  (132)

where \( \tilde{a}_0, \tilde{b}_1 \) are computed as functions of \( \tilde{a}_1, ..., \tilde{a}_p, \tilde{b}_2, ..., \tilde{b}_p \) by using the initial conditions (139).

• We compute the values of \( a^0_1, ..., a^0_p, b^0_2, ..., b^0_p, \omega^0 \) as the values which give the minimum of the functional (132) and the values of \( a^0_0, b^0_1 \) again as functions of \( a^0_1, ..., a^0_p, b^0_2, ..., b^0_p \) by using the initial conditions (129).

• Using the constants \( a^0_0, a^0_1, ..., a^0_p, b^0_1, ..., b^0_p, \omega^0 \) thus determined, we consider the Fourier-sequence:

\[ s_p(t) = \sum_{k=0}^{p} [a_{pk} \cdot \cos(k \cdot \omega_p \cdot t) + b_{pk} \cdot \sin(k \cdot \omega_p \cdot t)] \]  (133)

with \( a_{pk} = a_k^0 \) for \( k = 0, ..., p \), \( b_{pk} = b_k^0 \) for \( k = 1, ..., p \) and \( \omega_p = \omega^0 \)

(where \( b_0^0 = 0 \)).

The following convergence theorem holds:

**Theorem 1** If the problem (124, 125) admits a periodic solution, then the Fourier-sequence \( s_p(t) \), satisfies the property:

\[ \lim_{p \to \infty} \int_{0}^{2\pi/\omega} R^2(t, s_p(t)) dt = 0 \]

Moreover, \( \forall \epsilon > 0, \exists p_0 \in \mathbb{N} \) such that \( \forall p \in \mathbb{N}, p > p_0 \) it follows that \( s_p(t) \) is a weak \( \epsilon \) -approximate Fourier-solution of the problem (124, 125).

From the fact that the problem (124, 125) admits a periodic solution it follows that the series:

\[ \sum_{k=0}^{\infty} [a_k \cdot \cos(k \cdot \omega \cdot t) + b_k \cdot \sin(k \cdot \omega \cdot t)] \]

exists and its sequence of partial sums \( f_p \).
\[ f_p(t) = \sum_{k=0}^{p} [a_k \cdot \cos(k \cdot \omega \cdot t) + b_k \cdot \sin(k \cdot \omega \cdot t)] \]

converges to the solution of the problem (124, 125), i.e.:
\[
\lim_{p \to \infty} R(t, f_p(t)) = 0
\]

Based on the way the Fourier-function \( s_p(t) \) is computed and taking into account the relations (130-133), the following inequality holds:
\[
0 \leq \int_0^{2\pi} R^2(t, s_p(t))dt \leq \int_0^{2\pi} R^2(t, f_p(t))dt, \ \forall p \in \mathbb{N}.
\]
It follows that:
\[
0 \leq \lim_{p \to \infty} \int_0^{2\pi} R^2(t, s_p(t))dt \leq \lim_{p \to \infty} \int_0^{2\pi} R^2(t, f_p(t))dt = 0, \ \forall p \in \mathbb{N}.
\]
we obtain:
\[
\lim_{p \to \infty} \int_0^{2\pi} R^2(t, s_p(t))dt = 0.
\]
From this limit we obtain that \( \forall \epsilon > 0, \exists p_0 \in \mathbb{N} \) such that \( \forall p \in \mathbb{N}, \ p > p_0 \) it follows that \( s_p(t) \) is a weak \( \epsilon \)-approximate Fourier-solution of the problem (124, 125) q.e.d.

Any \( \epsilon \)-approximate Fourier-solution of the problem (124, 125) is also a weak approximate Fourier-solution, but the opposite is not always true. It follows that the set of weak approximate Fourier-solutions of the problem (124, 125) also contains the approximate Fourier-solutions of the problem.

Taking into account the above remark, in order to find \( \epsilon \)-approximate Fourier-solutions of the problem (124, 125) by the Fourier-Least Squares Method we will first determine weak approximate Fourier-solutions, \( \tilde{x} \). If \( |R(t, \tilde{x})| < \epsilon \) then \( \tilde{x} \) is also an \( \epsilon \)-approximate Fourier-solution of the problem.
1.8.3. Results and discussion

The test problems included in this section are the Duffing oscillator (two cases, an autonomous one and one involving integral forcing terms) and the Jerk equations.

These problems were extensively studied over the years, and various solutions, both approximate analytical ones and numerical ones were proposed.

The qualitative properties of these oscillators were also extensively studied. Stability and bifurcation studies for the Duffing oscillators include, among many others, [195], [196], [197]. For Jerk-type equations a comprehensive bifurcation study can be found in [198] and a study of the limit cycles can be found in [199]. Following the computations presented in these papers, corresponding conclusions can be drawn for the problems studied in the sections 1.8.3.1.-1.8.3.3. For example, in the case of the autonomous Duffing oscillator (134), for the values of the parameters considered in the computations ($\varepsilon = 1.25$ and $A = 2$), a quick computation similar to the one in [195] indicates that the only equilibrium point is the origin, which is a center; similar computations can be performed for the other problems.

In the following we compute approximate solutions for the Duffing oscillator and the Jerk equations and compare our results with similar analytical approximations previously computed by using other methods.

1.8.3.1. The autonomous Duffing oscillator

Our first test problem is the autonomous Duffing oscillator:

\[
\begin{align*}
    x^{(2)}(t) + x(t) + \varepsilon \cdot x^3(t) &= 0 \\
    x(0) &= A, \quad x^{(1)}(0) = 0
\end{align*}
\]  

(134)

The Duffing oscillator is extensively studied in literature and some relatively recent results are presented in [171] and [178]. In [171] approximate solutions are computed using the rational harmonic balance method (RHB) and in [178] approximate solutions of (134) are computed using a variational iteration procedure (VI). The approximate frequency of the oscillations obtained by using these methods is compared with the exact period known in the literature.

In the following, in order to obtain our approximation of the frequency and of the solution, we will perform the steps described in the previous section. We will perform in detail our computations for the values $\varepsilon = 1.25$ and $A = 2$. 
Since the computations of the minimum of the functional (132) is relatively difficult for large values of $p$ in (130), we will actually use an iterative procedure, starting with $p = 1$ and increasing the value until we achieve the desired precision.

**Approximate solution for $p = 1$**

Taking into account these considerations, first we choose the approximate solution (140) of the form:

$$\tilde{x}_1(t) = \tilde{a}_0 + \tilde{a}_i \cdot \cos(\tilde{a}_i \cdot t) + \tilde{b}_i \cdot \sin(\tilde{a}_i \cdot t)$$

In Step 1, the expression (141) becomes:

$$\mathcal{R}(\tilde{a}_0, \tilde{a}_i, \tilde{b}_i, \tilde{a}_0) = \tilde{a}_i \tilde{a}_0^2 \cos(\tilde{a}_i \cdot t) - \tilde{b}_i \tilde{a}_0^2 \sin(\tilde{a}_i \cdot t) +\frac{5}{4}(\tilde{a}_i \cos(\tilde{a}_i \cdot t) + \tilde{b}_i \sin(\tilde{a}_i \cdot t) + \tilde{a}_0)^3 + \tilde{a}_i \cos(\tilde{a}_i \cdot t) + \tilde{b}_i \sin(\tilde{a}_i \cdot t) + \tilde{a}_0$$

Taking into account the initial conditions $\tilde{x}(0) = A$, $\tilde{x}^{(1)}(t) = 0$ we obtain the relations:

$$\tilde{a}_0 = -\tilde{a}_i + 2, \quad \tilde{b}_i = 0$$

Replacing these values, the corresponding functional (132) from Step 2 is:

$$J(\tilde{a}_i, \tilde{a}_0) = \left(36864\pi + 5775\pi\tilde{a}_0^6 + 128\pi\tilde{a}_0^2\tilde{a}_i^4 - 37800\pi\tilde{a}_0^2\tilde{a}_i^6 + 107800\pi\tilde{a}_0^4\tilde{a}_i^2\right) \frac{1}{128\pi\tilde{a}_0^4} + \left(-172800\tilde{a}_0\tilde{a}_i^3 - 16\left(75\pi\tilde{a}_0^4 - 240\pi\tilde{a}_0^2\tilde{a}_i^3 + 256\pi\tilde{a}_0^2\tilde{a}_i^5\right)\tilde{a}_0^2 + 167424\pi\tilde{a}_0^2\tilde{a}_i^2 - 98304\pi\tilde{a}_0^4\right) \frac{1}{128\pi\tilde{a}_0^4}$$

In Step 3 we must compute the minimum of $J$ with respect to $\tilde{a}_i$ and $\tilde{a}_0$. For relatively simple problems such as this it is possible to compute directly the critical points of $J$ and subsequently select the value corresponding to the minimum.

In general, the critical points corresponding to the functional (132) $J(\tilde{a}_0, \tilde{a}_i, \tilde{b}_i, \tilde{a}_0, \tilde{b}_i, \tilde{a}_0)$ are the solution of the system:
for \( p = 1 \), we obtain our first approximation:

\[
\tilde{x}_1(t) = 1.9586125349 \cos(2.1643274538t) + 0.041387465104
\]

In Figure 26 we present the comparison between our \( \tilde{x}_1 \) approximate solution (solid line) and the numerical solution obtained by using a fourth order Runge-Kutta method (dotted line).

\[
\begin{align*}
\frac{\partial J}{\partial \tilde{a}_i} &= 0 \\
\vdots \\
\frac{\partial J}{\partial \tilde{a}_p} &= 0 \\
\frac{\partial J}{\partial b_1} &= 0 \\
\vdots \\
\frac{\partial J}{\partial b_p} &= 0 \\
\frac{\partial J}{\partial \tilde{a}_1} &= 0
\end{align*}
\]  

(135)
As we already saw, the approximate value of the frequency obtained here is \( \tilde{\omega}_1 = \omega_1^0 = 2.16342748538 \) and it is already close to the exact value which in this case is \( \omega_{\text{ex}} = 2.150416169536 \).

---

**Approximate solution for** \( p > 1 \)

As the following results will show, the Newton method is able to find approximate solutions of (135) which can lead to highly accurate approximate solutions of the problem (134).

For \( p = 2 \) the approximate solution (130) has the form:

\[
\tilde{x}_2(t) = \tilde{a}_0 + \tilde{a}_1 \cdot \cos(\tilde{\omega}_2 \cdot t) + \tilde{b}_1 \cdot \sin(\tilde{\omega}_2 \cdot t) + \tilde{a}_2 \cdot \cos(2 \cdot \tilde{\omega}_2 \cdot t) + \tilde{b}_2 \cdot \sin(2 \cdot \tilde{\omega}_2 \cdot t)
\]

After we compute the corresponding expressions of

\[ \Re\{t, \tilde{a}_0, \tilde{a}_1, \tilde{a}_2, \tilde{b}_1, \tilde{b}_2, \tilde{\omega}_2\} \]

and \( J(\tilde{a}_1, \tilde{a}_2, \tilde{b}_2, \tilde{\omega}_2) \) and of the system (135) (all too large to insert here), we apply the Newton’s method taking as the starting point of the iteration \((\tilde{a}_{1,s}, \tilde{a}_{2,s}, \tilde{b}_{2,s}, \tilde{\omega}_{2,s})\) where \( \tilde{a}_{1,s} \) and \( \tilde{\omega}_{2,s} \) are the values of \( a_1^0 \) and \( \omega_1^0 \) computed for the previous approximation \( p = 1 \), namely \( \tilde{a}_{1,s} = 1.95861253449 \) and \( \tilde{\omega}_{2,s} = 2.16342748538 \). In order for the sequence of approximations given by Newton’s method to converge to the solution(s) of the system (135), \( \tilde{a}_{2,s} \) and \( \tilde{b}_{2,s} \) will take successively values on a given grid of the type \( G = I_{\tilde{a}_2} \times I_{\tilde{b}_2} \) where \( I_i \) is a division of an symmetric interval centered in zero.

For all the test problems included in this paper and for all the values of \( p \) tested, a grid of the form \( G = \{-1, 0.9, -0.8, ..., 1\} \times \{-1, 0.9, -0.8, ..., 1\} \) (i.e. from -1 to 1 with the step 0.1) is large enough in the sense that if the starting point \((s_0, s_1)\) scans \( G \) we can obtain using Newton’s method the desired solutions of (135).

In the particular case of the problem (134) it was actually sufficient to choose \( \tilde{a}_{2,s} = 0 \) and \( \tilde{b}_{2,s} = 0 \) and the sequence converged to the minimum of \( J \).

For \( p = 3 \) we repeat the same procedure: we compute the corresponding expressions of

\[ \Re\{t, \tilde{a}_0, \tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \tilde{\omega}_3\} \]

and \( J(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{b}_2, \tilde{b}_3, \tilde{\omega}_3) \) and of the system (135) and we apply the Newton’s method taking as the starting point of the iteration \((\tilde{a}_{1,s}, \tilde{a}_{2,s}, \tilde{a}_{3,s}, \tilde{b}_{2,s}, \tilde{b}_{3,s}, \tilde{\omega}_{3,s})\) where \((\tilde{a}_{1,s}, \tilde{a}_{2,s}, \tilde{b}_{3,s}, \tilde{\omega}_{3,s})\) are the
minimum values \( (a_1^0, a_2^0, b_2^0, \omega_0^0) \) computed for the previous approximation \( p = 2 \) and \( a_{3,s} = b_{3,s} = 0 \).

The process can be carried on for increased values of \( p \) until the desired accuracy is reached.

Thus, for \( p = 7 \) we obtained for the approximate frequency the value \( \omega_7^0 = 2.15041853722 \) and the corresponding approximate solution (130) has the form:

\[
\tilde{x}_7(t) = 1.0618674103207715 \cdot 10^{-08} + 1.930547907025286 \cdot t \\
+ 1.4239017145731592 \cdot 10^{-09} \cdot \cos(8.601674148870872 \cdot t) \\
+ 7.582384931426247 \cdot 10^{-05} \cdot \cos(15.052929760524027 \cdot t)
\]

In Figure 28 we present the comparison between the \( \tilde{x}_7 \) approximate solution (solid line) and the numerical solution obtained by using a fourth order Runge-Kutta method (dotted line).

![Figure 28: Comparison between \( \tilde{x}_7 \) (solid line) and numerical solution (dotted line)](image-url)
In [178] approximate solutions for the problem (134) were computed for the cases \( \varepsilon = 1.25, A = 2 \) (studied above) and \( \varepsilon = 250, A = 2 \).

For the case \( \varepsilon = 1.25, A = 2 \), the approximate solution from [178] is:

\[
x_{\text{VI}}(t) = 1.989351555 \cdot \cos(2.15031 \cdot t) + 0.008334403 \cdot \cos(6.45093 \cdot t) \\
+ 0.002206664 \cdot \cos(10.75155 \cdot t) + 0.000053689 \cdot \cos(15.05217 \cdot t) \\
+ (2.6 \cdot 10^{-7}) \cdot \cos(19.35279 \cdot t)
\]

Table 29 presents the comparison of the absolute errors (computed as the difference in absolute value between the approximate solution and the corresponding numerical solution given by the Runge-Kutta method) corresponding to the approximate solutions \( x_{\text{VI}} \) ([178]) and \( \tilde{x} \) for the case \( \varepsilon = 1.25, A = 2 \).

### Table 29. Comparison of the absolute errors of the approximate solutions for problem (144) in the case \( \varepsilon = 1.25, A = 2 \)

<table>
<thead>
<tr>
<th>t</th>
<th>( x_{\text{VI}} ) (( 10^{-5} ))</th>
<th>( \tilde{x} ) (( 10^{-7} ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>t = 0</td>
<td>5.34289999 998 \cdot 10^{-5}</td>
<td>0.00000000000</td>
</tr>
<tr>
<td>t = 0.4</td>
<td>8.81935946 956 \cdot 10^{-2}</td>
<td>3.92745813 649 \cdot 10^{-7}</td>
</tr>
<tr>
<td>t = 0.8</td>
<td>3.40733004 552 \cdot 10^{-2}</td>
<td>1.66866686 269 \cdot 10^{-6}</td>
</tr>
<tr>
<td>t = 1.2</td>
<td>5.63129918 651 \cdot 10^{-2}</td>
<td>5.23059939 095 \cdot 10^{-6}</td>
</tr>
<tr>
<td>t = 1.6</td>
<td>1.96411171 977 \cdot 10^{-2}</td>
<td>1.89375331 883 \cdot 10^{-6}</td>
</tr>
<tr>
<td>t = 2</td>
<td>7.93642203 903 \cdot 10^{-2}</td>
<td>6.81658032 309 \cdot 10^{-6}</td>
</tr>
<tr>
<td>t = 2.4</td>
<td>8.23170554 343 \cdot 10^{-2}</td>
<td>1.36265840 209 \cdot 10^{-5}</td>
</tr>
<tr>
<td>t = 2.8</td>
<td>1.50931697 081 \cdot 10^{-2}</td>
<td>9.46274851 943 \cdot 10^{-6}</td>
</tr>
</tbody>
</table>
For the case $\varepsilon = 250, A = 2$, the approximate solution from [178] is:

$$x_{VI}(t) = 1.977507645 \cdot \cos(26.8026 \cdot t) + 0.018844215 \cdot \cos(80.4078 \cdot t) + 0.003619986 \cdot \cos(134.013 \cdot t) + 0.000027438 \cdot \cos(187.6182 \cdot t) + 0.000164633 \cdot \cos(241.2234 \cdot t)$$

For this case, the approximate solution computed using our method is:

$$\tilde{x}_7(t) = 3.22758664474 \cdot 10^{-08} + 1.91015330733 \cdot \cos(26.810864231 \cdot t) + 6.9616769357 \cdot 10^{-08} \cdot \cos(53.6216924619 \cdot t) + 0.0859766881274 \cdot \cos(80.4325386 \cdot t) + 7.78739552572 \cdot 10^{-09} \cdot \cos(107.243384924 \cdot t) + 0.00371019879647 \cdot \cos(134.054231155 \cdot t) + 7.82719602479 \cdot 10^{-10} \cdot \cos(160.865077386 \cdot t) + 0.0001596952849 \cdot \cos(187.675923617 \cdot t)$$

Table 30 presents the comparison of the absolute errors (computed as the difference in absolute value between the approximate solution and the corresponding numerical solution given by the Runge-Kutta method) corresponding to the approximate solutions $x_{VI}$ ([178]) and $\tilde{x}_7$ for the case $\varepsilon = 250, A = 2$.

<table>
<thead>
<tr>
<th></th>
<th>$x_{VI}$</th>
<th>$\tilde{x}_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t = 0$</td>
<td>$1.6391700000 \cdot 10^{-4}$</td>
<td>$0.0000000000$</td>
</tr>
<tr>
<td>$t = 0.04$</td>
<td>$9.94430509 \cdot 655 \cdot 10^{-2}$</td>
<td>$1.5943294392 \cdot 10^{-6}$</td>
</tr>
<tr>
<td>$t = 0.08$</td>
<td>$1.01602511 \cdot 979 \cdot 10^{-1}$</td>
<td>$2.50754265 \cdot 524 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>$t = 0.12$</td>
<td>$1.84880609 \cdot 089 \cdot 10^{-3}$</td>
<td>$1.44364392 \cdot 085 \cdot 10^{-6}$</td>
</tr>
<tr>
<td>$t = 0.16$</td>
<td>$9.38684852 \cdot 713 \cdot 10^{-2}$</td>
<td>$2.52297827 \cdot 337 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>$t = 0.2$</td>
<td>$1.00379880 \cdot 594 \cdot 10^{-1}$</td>
<td>$4.54516346 \cdot 458 \cdot 10^{-5}$</td>
</tr>
</tbody>
</table>
Finally, in Table 31 we compare the approximate values for the frequency $\omega_i^0$ computed using our method with approximate values computed in [178] ($\omega_{VI}$) and [171] ($\omega_{RHB}$). The comparison is made by means of the percentage error, which for a given approximate error $\omega_{approx}$ is defined as

$$e_{\omega_{approx}} = 100 \times \frac{|\omega_{exact} - \omega_{approx}|}{\omega_{exact}},$$

where $\omega_{exact}$ is the corresponding exact error. It is easy to see that our approximations are far better than the ones previously computed and they remain accurate even for the case of a very strong nonlinearity.

Table 31. Comparison of approximate errors for the frequency $\omega$ for problem (134)

<table>
<thead>
<tr>
<th>$\epsilon \cdot A^2$</th>
<th>5</th>
<th>100</th>
<th>1000</th>
<th>10000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_{exact}$</td>
<td>2.1504161695</td>
<td>8.5335861889</td>
<td>26.8107384581</td>
<td>84.7274799361</td>
</tr>
<tr>
<td>$\omega_{VI}$</td>
<td>2.15031</td>
<td>x</td>
<td>26.8026</td>
<td>84.7015</td>
</tr>
<tr>
<td>$e_{\omega_{VI}}$ (%)</td>
<td>0.004937</td>
<td>x</td>
<td>0.0303552</td>
<td>0.0306629</td>
</tr>
<tr>
<td>$\omega_{RHB}$</td>
<td>2.15045</td>
<td>8.53402</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>$e_{\omega_{RHB}}$ (%)</td>
<td>0.0015</td>
<td>0.0051</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>$\omega_{7}$</td>
<td>2.15041853722</td>
<td>8.53361825153</td>
<td>26.810846231</td>
<td>84.7278228489</td>
</tr>
<tr>
<td>$e_{\omega_{7}}$ (%)</td>
<td>0.0001101034</td>
<td>0.0003757217</td>
<td>0.0004019763</td>
<td>0.0004047244</td>
</tr>
</tbody>
</table>
1.8.3.2. The Duffing equation involving integral forcing terms

Our second test problem is:

\[
\begin{cases}
    x^{(2)}(t) + x(t) \cdot x^{(1)}(t) + \int_{0}^{t} s x^2(s) \, ds = f(t), & 0 < t < 1 \\
    x(0) - x^{(1)}(0) = 0, \\
    x(1) + x^{(1)}(1) = 0
\end{cases}
\]  \quad (136)

where

\[
f(t) = -3t - 3t^2 + \frac{5t^3}{2} + \frac{2t^4}{3} - \frac{t^5}{4} - \frac{2t^6}{5} + \frac{t^7}{6}.
\]

The problem (146) ([173], [174]) is a version of the well-known Duffing equation involving both integral and non-integral forcing terms with separated boundary conditions. This equation has been studied in a series of recent papers including [173] and [174].

In [173], the authors applied a generalized quasilinearization technique to prove the existence and uniqueness of the solution of Duffing equation involving both integral and non-integral forcing terms. They showed that there are sequences of approximate solutions converging monotonically and quadratically to the unique solution of the problem.

In [174], the authors gave a representation of exact solution and approximate solution of Duffing equation involving both integral and non-integral forcing terms in the reproducing kernel space (RKS). They represented the exact solution in the form of a series and they showed that the n-term approximation of the exact solution converges to the exact solution.

Next we present our results for (136) using FLSM. Also, we will compare these results with those obtained in [174].

Thus, for \( p = 3 \) we obtained that the approximate periodic solution (130) has the form:

\[
\tilde{\varphi}_3(t) = 1.45453 \sin(t) - 0.26409 \sin(2t) + 0.0245502 \sin(3t) + 2.6625 \cos(t) - 0.169569 \cos(2t) + 0.00174065 \cos(3t) - 1.49467
\]

Since in [174] only the numerical results are presented while the expression of approximate solution expression is not, we can not perform a direct graphical comparison of our approximate solution with the corresponding solution from [174].

Therefore, in Table 32 we present the comparison of several values of the absolute errors (computed as the difference in absolute value between the exact solution and the approximate solution) corresponding to the approximate
solutions $x_{RKS}$ from [174] and to our approximate solutions for $t = 0.1, 0.2, ..., 1$, as given in [174].

Table 32. Comparison of the absolute errors of the approximate solutions for problem (136)

<table>
<thead>
<tr>
<th>$t$</th>
<th>$x_{RKS}$</th>
<th>$\tilde{x}_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>9.46600$\times 10^{-6}$</td>
<td>9.85192$\times 10^{-8}$</td>
</tr>
<tr>
<td>0.2</td>
<td>9.77708$\times 10^{-6}$</td>
<td>9.2974$\times 10^{-8}$</td>
</tr>
<tr>
<td>0.3</td>
<td>9.81938$\times 10^{-6}$</td>
<td>1.58651$\times 10^{-7}$</td>
</tr>
<tr>
<td>0.4</td>
<td>9.66575$\times 10^{-6}$</td>
<td>5.76015$\times 10^{-8}$</td>
</tr>
<tr>
<td>0.5</td>
<td>9.37204$\times 10^{-6}$</td>
<td>2.07316$\times 10^{-7}$</td>
</tr>
<tr>
<td>0.6</td>
<td>8.97794$\times 10^{-6}$</td>
<td>5.9859$\times 10^{-8}$</td>
</tr>
<tr>
<td>0.7</td>
<td>8.50917$\times 10^{-6}$</td>
<td>1.55722$\times 10^{-7}$</td>
</tr>
<tr>
<td>0.8</td>
<td>7.97984$\times 10^{-6}$</td>
<td>9.17107$\times 10^{-8}$</td>
</tr>
<tr>
<td>0.9</td>
<td>7.39469$\times 10^{-6}$</td>
<td>9.65862$\times 10^{-8}$</td>
</tr>
<tr>
<td>1</td>
<td>6.75105$\times 10^{-6}$</td>
<td>7.84328$\times 10^{-10}$</td>
</tr>
</tbody>
</table>

1.8.3.3. The Jerk equation containing velocity-cubed and velocity times displacement-squared

Our last test is a Jerk nonlinear equation, which describes several physical using mechanical oscillations of the third order 3. The most general
form of the Jerk nonlinear equations, which contains the third temporal derivative of displacement, is:

$$x^{(3)} + \alpha \cdot x \cdot x^{(1)} + x^{(2)} + \beta \cdot x^{(1)} \cdot x^{(2)} + \delta \cdot x^{2} \cdot x^{(1)} + \varepsilon \cdot x^{(1)^{3}} + \gamma \cdot x^{(1)} = 0 \quad (137)$$

where the parameters $\alpha, \beta, \delta, \varepsilon$ and $\gamma$ are constants.

Nonlinear Jerk equations (137) are intensely studied by several authors in the literature and some recent results are presented in [162], [167], [168], [186]. In this paper we consider the Jerk equation involving velocity-cubed and velocity times displacement-squared given by:

$$\begin{cases} x^{(3)} + x^{2} \cdot x^{(1)} + x^{(1)^{3}} = 0, \\ x(0) = 0, x^{(1)}(0) = B, x^{(2)}(0) = 0. \end{cases} \quad (138)$$

Recently, Ma et al. in [162], using the Homotopy Perturbation Method, obtained high-order analytic approximate periods and periodic solutions of the Jerk equation (138). In [167] and [168], Gottlieb used the lowest-order Harmonic Balance Method to determine analytical approximations to the periodic solution of the Jerk equations. Also, Leung et al. in [186] obtained approximations for the angular frequency and the limit cycle for (138) based on the Residue Harmonic Balance approach.

For $B = 0.5$, the approximate periodic solution $x_{HPM}$ from [162] is:

$$x_{HPM}(t) = 0.8708027 \sin(0.61924 \cdot t) - 0.0106486 \sin(1.85772 \cdot t)$$

Applying FLSM we computed an approximate periodic solution of the problem (138) of the same order (containing terms up to $\sin(3 \cdot \omega \cdot t)$):

$$\tilde{x}_{3}(t) = 0.847907380098 \cdot \sin(0.615691681067t) - (3.59275929188 \cdot 10^{-5}) \cdot \sin(1.2313836213t) - 0.0119135817912 \cdot \sin(1.8470750432 \cdot t).$$

The approximate frequency and period are

$$\omega_{3}^{0} = 0.615691681067, \quad T_{3}^{0} = 10.2050839736674$$

with an error of $e_{3}^{0} = -0.0555984645279972$.

In Figure 29, we can visualize and compare our approximate solution (solid line), the approximate solution $x_{HPM}$ from [162](dashed line) and the numerical solution obtained by using a fourth order Runge-Kutta method (dotted line).
Figure 29: Comparison between our solution \( \tilde{x}_3 \) (solid line), solution \( x_{HPM} \) in [162] (dashed line) and numerical solution (dotted line)

Table 33 presents the comparison of the absolute errors (computed as the difference in absolute value between the approximate solution and the corresponding numerical solution given by the Runge-Kutta method) corresponding to the approximate solution \( x_{HPM} \) from [162] and our approximate solution \( \tilde{x}_3 \) for the case \( B = 0.5 \).

**Table 33. Comparison of the absolute errors corresponding to the solution \( x_{HPM} \) from [162] and to our solution \( \tilde{x}_3 \)**

<table>
<thead>
<tr>
<th>( t )</th>
<th>( x_{HPM} )</th>
<th>( \tilde{x}_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t = 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( t = 1.0 )</td>
<td>( 1.72354316 , 028 \cdot 10^{-2} )</td>
<td>( 2.11130244 , 099 \cdot 10^{-4} )</td>
</tr>
<tr>
<td>( t = 2.0 )</td>
<td>( 2.33072876 , 71 \cdot 10^{-2} )</td>
<td>( 1.31078376 , 172 \cdot 10^{-4} )</td>
</tr>
<tr>
<td>( t = 3.0 )</td>
<td>( 1.79768181 , 549 \cdot 10^{-2} )</td>
<td>( 3.46325784 , 747 \cdot 10^{-2} )</td>
</tr>
<tr>
<td>$t$</td>
<td>$x_{RHB}$ (computed)</td>
<td>$x_{RHB}$ (numerical)</td>
</tr>
<tr>
<td>------</td>
<td>---------------------</td>
<td>---------------------</td>
</tr>
<tr>
<td>4.0</td>
<td>4.72752471 $124 \cdot 10^{-3}$</td>
<td>8.73925906 $836 \cdot 10^{-4}$</td>
</tr>
<tr>
<td>5.0</td>
<td>1.46107699 $703 \cdot 10^{-2}$</td>
<td>1.42164922 $819 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>6.0</td>
<td>3.05145610 $435 \cdot 10^{-2}$</td>
<td>1.51326363 $254 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>7.0</td>
<td>3.04290934 $289 \cdot 10^{-2}$</td>
<td>8.83072635 $498 \cdot 10^{-4}$</td>
</tr>
<tr>
<td>8.0</td>
<td>1.45339383 $549 \cdot 10^{-2}$</td>
<td>7.00497125 $303 \cdot 10^{-4}$</td>
</tr>
<tr>
<td>9.0</td>
<td>6.86925713 $324 \cdot 10^{-3}$</td>
<td>2.00167777 $983 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>10.0</td>
<td>2.91276307 $352 \cdot 10^{-2}$</td>
<td>2.84571545 $126 \cdot 10^{-3}$</td>
</tr>
</tbody>
</table>

Also for $B = 0.5$, the approximate periodic solution $x_{RHB}$ from [186] is given by:

$$x_{RHB}(t) = 0.8473604 \sin(0.6151734t) - 0.01168397 \sin(1.8455202t) + 0.000094103 \sin(3.075867t)$$

Applying FLSM we computed an approximate periodic solution of the problem (138) of the same order (containing terms up to $\sin(5 \cdot \omega \cdot t)$):

$$\tilde{x}_5(t) = 0.847966627859 \sin(0.6153509365t) + (2.7067525988410^{-8}) \sin(1.2307019873 \cdot t) - 0.0119562711384 \sin(1.84605298095t) - (9.5141883227910^{-11}) \sin(2.4614039746t) + (8.93050847544 \cdot 10^{-15}) \sin(3.07675496825 \cdot t)$$

The approximate frequency and period are

$$\omega_5^0 = 0.61535099365, \quad T_5^0 = 10.2107339908715,$$

with an error of $-0.000264516312410058$.

Table 34 presents the comparison of the absolute errors (computed as the difference in absolute value between the approximate solution and the corresponding numerical solution given by the Runge-Kutta method) corresponding to the approximate solutions $x_{RHB}$ from [186] and our approximate solutions $\tilde{x}_5$ for the case $B = 0.5$. In this case we omitted the
A graphical representation of the approximate solutions $x_{RHB}$ and $\tilde{x}_5$ since they are both very close to the numerical solution.

Table 34. *Comparison of the absolute errors corresponding to the solution* $x_{RHB}$ *from [186] and to our solution* $\tilde{x}_5$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$x_{RHB}$</th>
<th>$\tilde{x}_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t = 0$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$t = 1.0$</td>
<td>$2.07336138 \ 332 \cdot 10^{-4}$</td>
<td>$4.76705901 \ 847 \cdot 10^{-6}$</td>
</tr>
<tr>
<td>$t = 2.0$</td>
<td>$8.18802985 \ 278 \cdot 10^{-4}$</td>
<td>$7.01416299 \ 886 \cdot 10^{-6}$</td>
</tr>
<tr>
<td>$t = 3.0$</td>
<td>$6.29650984 \ 827 \cdot 10^{-4}$</td>
<td>$8.81003685 \ 005 \cdot 10^{-7}$</td>
</tr>
<tr>
<td>$t = 4.0$</td>
<td>$3.27159888 \ 572 \cdot 10^{-4}$</td>
<td>$1.19800236 \ 493 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>$t = 5.0$</td>
<td>$7.15072428 \ 261 \cdot 10^{-4}$</td>
<td>$2.05053864 \ 94 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>$t = 6.0$</td>
<td>$7.95019646 \ 471 \cdot 10^{-4}$</td>
<td>$2.11112229 \ 619 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>$t = 7.0$</td>
<td>$1.09803461 \ 819 \cdot 10^{-3}$</td>
<td>$1.31470477 \ 691 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>$t = 8.0$</td>
<td>$5.35184978 \ 947 \cdot 10^{-4}$</td>
<td>$5.37333608 \ 552 \cdot 10^{-6}$</td>
</tr>
<tr>
<td>$t = 9.0$</td>
<td>$8.01975762 \ 64 \cdot 10^{-4}$</td>
<td>$2.75784654 \ 626 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>$t = 10.0$</td>
<td>$1.42381620 \ 457 \cdot 10^{-3}$</td>
<td>$3.91402085 \ 78 \cdot 10^{-5}$</td>
</tr>
</tbody>
</table>

It is easy to see from the tables 5 and 6 that the solutions obtained by using FLSM are more accurate than the ones computed by using other methods. This fact is emphasized by table 35 which presents a comparison of approximate periods computed in several papers, as presented in [186]. In this table, $T_k$ denotes an approximate period obtained by means of an approximate solution containing terms up to $\sin(k \cdot \omega \cdot t)$. 
Table 35. Comparison of approximate periods

<table>
<thead>
<tr>
<th>B=0.5</th>
<th>Period</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Texact</td>
<td>10.210761</td>
<td>0</td>
</tr>
<tr>
<td>FLSM $T_1$</td>
<td>10.1976548730449</td>
<td>$-0.128356025129356$</td>
</tr>
<tr>
<td>FLSM $T_3$</td>
<td>10.2050839736674</td>
<td>$-0.0555984645279972$</td>
</tr>
<tr>
<td>FLSM $T_5$</td>
<td>10.210733908715</td>
<td>$-0.000264516312410058$</td>
</tr>
<tr>
<td>Wu et al.[2.202] $T_2$</td>
<td>10.1884</td>
<td>$-0.220$</td>
</tr>
<tr>
<td>Wu et al.[2.202] $T_3$</td>
<td>10.2107</td>
<td>$-0.001$</td>
</tr>
<tr>
<td>Leung et al.[2.186] $T_2$</td>
<td>10.213682</td>
<td>0.0286</td>
</tr>
<tr>
<td>Feng and Li [2.182] $T_2$</td>
<td>10.23452</td>
<td>0.232</td>
</tr>
<tr>
<td>Feng and Li [2.182] $T_3$</td>
<td>10.1935</td>
<td>$-0.169$</td>
</tr>
<tr>
<td>Feng and Li [2.182] $T_4$</td>
<td>10.2029</td>
<td>$-0.077$</td>
</tr>
<tr>
<td>Hu [2.200] $T_2$</td>
<td>10.247682</td>
<td>0.362</td>
</tr>
<tr>
<td>Hu [2.200] $T_3$</td>
<td>10.226464</td>
<td>0.154</td>
</tr>
<tr>
<td>Hu et al.[2.201] $T_2$</td>
<td>10.244058</td>
<td>0.326</td>
</tr>
<tr>
<td>Ramos [2.203]$T_2$</td>
<td>10.80674</td>
<td>5.84</td>
</tr>
<tr>
<td>Ramos [2.203]$T_3$</td>
<td>10.55034</td>
<td>3.33</td>
</tr>
<tr>
<td>Ramos [2.203] $T_2'$</td>
<td>10.24328</td>
<td>0.318</td>
</tr>
<tr>
<td>Ramos [2.203] $T_2''$</td>
<td>10.26079</td>
<td>0.49</td>
</tr>
<tr>
<td>Ma et al.[2.162] $T_2$</td>
<td>10.146603</td>
<td>$-2.27$</td>
</tr>
</tbody>
</table>
1.8.4 . Conclusions

The Fourier-Least Squares Method (FLSM) is a straightforward and efficient method to compute approximate periodic solutions for a very general class of nonlinear differential equations modeling oscillatory phenomena. Since the equation (124) is a very general one, being able to model a large class of oscillatory phenomena, FLSM can be considered a powerful and useful method.

The computation of approximate solutions by FLSM clearly illustrate the accuracy of the method by comparison with approximate solutions previously computed by using other methods.

2. Approximate analytical solutions of fractional-order differential equations

2.1. Approximate Analytical Solutions of the Fractional-Order Brusselator System Using the Polynomial Least Squares Method

2.1.1. Introduction

In recent years, in many practical applications in various fields such as physics, mechanics, chemistry, biology etc. (see for example [126]–[131]), the problem being studied are modeled using fractional nonlinear equations. For most of such fractional nonlinear equations, the exact solutions can not be found and, as a consequence, a numerical solution or, if possible, an analytical approximate solution of these equation is sought. Due to the complexity of this type of problems, a general approximation algorithm does not exists and thus, various approximation methods, each with its strong and weak points, were proposed, including among other:

- The Adomian Decomposition Method ([132]-[133], [142]-[143],[204]-[205])
- The Homotopy Analysis Method([206]-[208])
- The Homotopy Perturbation Method([209],[210])
- The Laplace Transform Method([211],[212])
- The Fourier Transform Method([213])
- The Variational Iteration Method([214]-[219])
2.1.2. Approximation method description

The Polynomial Least Squares Method (PLSM) allows us to compute approximate analytical solutions for the Brusselator system. The fractional order Brusselator system was recently studied by several authors ([220]-[222]).

We consider the following Brusselator system:

\[
\begin{align*}
D^\alpha_t x(t) &= a - (\mu + 1) \cdot x(t) + x(t)^2 \cdot y(t) \\
D^\alpha_t y(t) &= \mu \cdot x(t) - x(t)^2 \cdot y(t)
\end{align*}
\]  

(139)

together with the initial conditions:

\[x(0) = c_1, \ y(0) = c_2\]  

(140)

where \(a > 0\), \(\mu > 0\), \(0 < \alpha_1 \leq 1\), \(0 < \alpha_2 \leq 1\), \(c_1, c_2\) are real constants and \(D^\alpha_t\) denotes Caputo's fractional derivative ([2.211]):

\[
D^\alpha_t = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t - \zeta)^{-\alpha} \cdot x'(\zeta) d\zeta, \quad 0 < \alpha \leq 1
\]

We will introduce PLSM for the Brusselator System and then we will compare the approximate solutions obtained by using PLSM with the approximate solutions from [219]. The computations show that the approximations computed by using our method present an error smaller than the error of the corresponding solutions from [219].

For the problem (139,140) we consider the remainder operators:

\[
\begin{align*}
\mathcal{D}_1(x(t), y(t)) &= D^\alpha_t x(t) - [a - (\mu + 1) \cdot x(t) + x(t)^2 \cdot y(t)] \\
\mathcal{D}_2(x(t), y(t)) &= D^\alpha_t y(t) - [\mu \cdot x(t) - x(t)^2 \cdot y(t)]
\end{align*}
\]  

(141)

We will find approximate polynomial solutions \(\tilde{x}(t), \tilde{y}(t)\) of (139,140) on the \([0, b]\) interval, solutions which satisfy the following conditions:

\[
|D_j((\tilde{x}(\epsilon), \tilde{y}(\epsilon))| < \varepsilon, \ j = 1,2, \ \varepsilon > 0
\]

(142)

\[
\tilde{x}(0) = c_1, \ \tilde{y}(0) = c_2
\]  

(143)
We call an \( \varepsilon \)-approximate polynomial solution of the system (139,140) an approximate polynomial solution \((\tilde{x}(t), \tilde{y}(t))\) satisfying the relations (142,143).

We call a weak \( \delta \)-approximate polynomial solution of the system (139,140) an approximate polynomial solution \((\tilde{x}(t), \tilde{y}(t))\) satisfying the relations:

\[
\int_{0}^{b} \mathcal{D}_j^2(\tilde{x}(t), \tilde{y}(t))dt \leq \delta, \quad j=1,2
\]

together with the initial conditions (143).

We consider the sequence of polynomials

\[
P^i_m(t) = a^i_0 + a^i_1 t + \ldots + a^i_m t^m, \quad a^i_j \in \mathbb{R}, \quad i = 0,1,\ldots,m, \quad j = 1,2
\]
satisfying the conditions:

\[
P^1_m(0) = c_1, \quad P^2_m(0) = c_2, \quad m > 1, \quad m \in \mathbb{R}.
\]

We call the sequence of polynomials \(P^i_m(t)\) convergent to the solution of the system (139,140) if

\[
\lim_{m \to \infty} \mathcal{D}_j(P^i_m(t), P^j_m(t)) = 0.
\]

We will find weak \( \varepsilon \)-polynomial solutions of the type:

\[
\tilde{x}(t) = \sum_{k=0}^{m} d^1_k \cdot t^k, \quad \tilde{y}(t) = \sum_{k=0}^{m} d^2_k \cdot t^k, \quad m > 1
\]  

(144)

where the constants \(d^1_0,d^1_1,\ldots,d^1_m, \quad j = 1,2\) are calculated using the steps outlined in the following.

- We attach to the system (139,140) the following real functional:

\[
J(d^1_1,d^1_2,\ldots,d^1_m,d^2_1,d^2_2,\ldots,d^2_m) = \sum_{j=1}^{3} \int_{0}^{b} \mathcal{D}_j^2(\tilde{x}(t), \tilde{y}(t))dt
\]  

(145)

where \(d^1_0,d^2_0\) are computed as functions of \(d^1_1,d^1_2,\ldots,d^1_m,d^2_1,d^2_2,\ldots,d^2_m\) by using the initial conditions (143).
We compute the values of \( \overline{d}_1, \overline{d}_2, \ldots, \overline{d}_m, \overline{d}_2, \ldots, \overline{d}_m \) as the values which give the minimum of the functional (145) and the values of \( \overline{d}_0, \overline{d}_0 \) again as functions of \( \overline{d}_1, \overline{d}_1, \ldots, \overline{d}_m, \overline{d}_2, \overline{d}_2, \ldots, \overline{d}_m \) by using the initial conditions.

Using the constants \( \overline{d}_0, \overline{d}_1, \ldots, \overline{d}_m, \overline{d}_2, \overline{d}_2, \ldots, \overline{d}_m \) thus determined, we consider the polynomials:

\[
T_m^1(t) = \sum_{k=0}^{m} \overline{d}_1^k \cdot t^k, \quad T_m^2(t) = \sum_{k=0}^{m} \overline{d}_2^k \cdot t^k, \quad m > 1
\]  

(146)

The following convergence theorem holds:

**Theorem 1.** The necessary condition for the problem (139,140) to admit sequences of polynomials \( P_m^j(t) \) convergent to the solution of this problem is:

\[
\lim_{m \to \infty} \int_0^b \mathcal{D}_j(T_m^1(t), T_m^2(t)) dt = 0
\]

Moreover, \( \varepsilon > 0, \exists m_0 \in \mathbb{N} \) such that \( \forall m \in \mathbb{N}, m > m_0 \) it follows that

\[
T_m^j(t), \quad j = 1,2
\]

are weak \( \varepsilon \)-approximate polynomial solutions of the system (139,140).

**Remark 1.** Any \( \varepsilon \)-approximate polynomial solutions of the system (139,140) are also weak \( 2 \cdot b \)-approximate polynomial solutions, but the opposite is not always true. It follows that the set of weak approximate solutions of the system (139,140) also contains the approximate solutions of the system.

Taking into account the above remark, in order to find \( \varepsilon \)-approximate polynomial solutions of the system ((139,140)) by PLSM we will first determine weak approximate polynomial solutions, \( \bar{x}(t), \bar{y}(t) \).

If

\[
|D_j((\bar{x}(t), \bar{y}(t)))| < \varepsilon, \quad j = 1,2
\]

then

\( \bar{x}(t), \bar{y}(t) \)

are also \( \varepsilon \)-approximate polynomial solutions of the system.
2.1.3. Results and discussions

2.1.3.1. The fractional-order Brusselator system

We consider the following fractional-order Brusselator system ([219]):

\[
\begin{align*}
D^{\alpha_1}_t x(t) &= -2 \cdot x(t) + x(t)^2 \cdot y(t) \\
D^{\alpha_2}_t y(t) &= x(t) - x(t)^2 \cdot y(t)
\end{align*}
\] (147)

together with the initial conditions:

\[ x(0) = 1, \quad y(0) = 1 \] (148)

In [219] approximate solutions of (147,148) are computed using the Variational Iteration Method (VIM) for the case \( \alpha_1 = \alpha_2 = 0.98 \). Also, a comparison with numerical solutions is presented for the particular case \( \alpha_1 = \alpha_2 = 1 \), illustrating the applicability of the method.

1.1. The case \( \alpha_1 = \alpha_2 = 0.98 \)

For the case \( \alpha_1 = \alpha_2 = 0.98 \), using PLSM with \( m = 3 \) we obtain the following approximate polynomial solutions:

\[
x_{PLSM}(t) = 0.0243682 \cdot t^3 + 0.311138 \cdot t^2 - 1.08655 \cdot t + 1
\] (149)

\[
y_{PLSM}(t) = -0.184414 \cdot t^3 + 0.333424 \cdot t^2 + 0.0349127 \cdot t + 1
\]

The errors obtained by using PLSM are smaller than the ones obtained by using VIM.

1.2 The case \( \alpha_1 = \alpha_2 = 1 \)

For the case \( \alpha_1 = \alpha_2 = 0.98 \), using PLSM with \( m = 3 \) we obtain the following approximate polynomial solutions:

\[
x_{PLSM}(t) = 0.0750974 \cdot t^3 + 0.201028 \cdot t^2 - 1.02827 \cdot t + 1
\] (150)

\[
y_{PLSM}(t) = -0.180088 \cdot t^3 + 0.334087 \cdot t^2 + 0.0271107 \cdot t + 1
\]
In this case both approximations (VIM and PLSM) consist of third order polynomials.

Again, the errors obtained by using PLSM are smaller than the ones obtained by using VIM.

2.1.4. Conclusions

In this paper we present the Polynomial Least Squares Method, which is a relatively straightforward and efficient method to compute approximate solutions for the fractional order Brusselator system.

The comparison with previous results illustrates the accuracy of the method, since we were able to compute more precise approximations than the previously computed ones.

In closing we mention the fact that, due to the nature of the method, it is relatively easy to extend PLSM for the general case of fractional systems of $n \geq 3$ nonlinear differential equations.

3. Dynamical systems modeled by nonlinear partial differential equations

3.1. Approximate analytical solutions of the regularized long wave equation using the optimal homotopy perturbation method

3.1.1. Introduction

A significant part of the natural technological processes and phenomena are usually modelled by means of partial differential equations. Thus it is very important to find solutions of these equations. However, as in many cases the computation of exact solutions is not possible, numerical or approximate solutions must be computed.

We present a new approximation method named optimal homotopy perturbation method (OHPM). As the name suggests, the method is based on the homotopy perturbation method ([145],[146]) and its main feature is an
accelerated convergence compared to the regular homotopy perturbation method.

The applications presented show that the approximate solutions obtained by using OHPM requires less iterations in comparison with other iterative methods for approximate solutions of partial differential equations.

### 3.1.2. Approximation method description (The optimal homotopy perturbation method)

We consider the following problem:

\[
\mathcal{L}(u(x,t)) + \mathcal{N}(u(x,t)) - f(x,t) = 0, \quad B(u) = 0 \quad (151)
\]

Here \( \mathcal{L} \) is a linear operator, \( u(x,t) \) is the unknown function, \( \mathcal{N} \) is a nonlinear operator, \( f(x,t) \) is a known, given function and \( B \) is a boundary operator.

If \( \tilde{u} \) is an approximate solution of equation (121), we evaluate the error obtained by replacing the exact solution \( u \) with the approximate one \( \tilde{u} \) as the remainder:

\[
R(x,t,\tilde{u}) = \mathcal{L}(\tilde{u}(x,t)) + \mathcal{N}(\tilde{u}(x,t)) - f(x,t) \quad (152)
\]

The first step in applying OHPM is to attach to the problem (151) the family of equations (see [145],[146]):

\[
(1 - p) (\mathcal{L}(\Phi(x,t,p)) - f(x,t)) + p (\mathcal{L}(\Phi(x,t,p)) + \mathcal{N}(\Phi(x,t,p)) - f(x,t)) = 0 \quad (153)
\]

where \( p \in [0,1] \) is an embedding parameter, \( \Phi(x,t,p) \) is an unknown function. When \( p = 0 \), \( \Phi(x,t,0) = u_0(x,t) \) and when \( p = 1 \), \( \Phi(x,t,1) = u(x,t) \). Thus, as \( p \) increases from 0 to 1, the solution \( \Phi(x,t,p) \) varies from \( u_0(x,t) \) to the solution \( u(x,t) \), where \( u_0(x,t) \) is obtained from the equation:

\[
\mathcal{L}(u_0(x,t)) - f(x,t) = 0, \quad B(u_0) = 0 \quad (154)
\]

We consider the following expansion of \( \Phi(x,t,p) \):

\[
\Phi(x,t,p) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) p^m \quad (155)
\]
Substituting the relation (155) in (153), collecting the same powers of \( p \) and equating each coefficient of the powers of \( p \) with zero we obtain:

\[
\mathcal{L}(u_m(x,t)) = -\mathcal{N}_{m-1}(u_0(x,t),u_1(x,t),...,u_{m-1}(x,t))
\]

\[
m \geq 1,\ldots, B(u_m) = 0,
\]

(156)

where \( \mathcal{N}_i, i \geq 0 \) are the coefficients of \( p^i \) in the nonlinear operator \( \mathcal{N} \):

\[
\mathcal{N}(u(x,t)) = \mathcal{N}_0(u_0(x,t)) + p\mathcal{N}_1(u_0(x,t),u_1(x,t)) + p^2\mathcal{N}_2(u_0(x,t),u_1(x,t),u_2(x,t)) + \ldots
\]

(157)

We remark that \( u_m, m \geq 1 \) are obtained from the linear equations (156), which are easily solved together with the boundary conditions.

We denote \( f_m = u_0 + u_1 + \ldots + u_m \).

We consider the set \( S_m \) \((m=0,1,2,\ldots)\) containing the functions \( \varphi_{m0}, \varphi_{m1}, \varphi_{m2}, \ldots, \varphi_{mn_m} \), chosen as linearly independent functions in the vector space of the continuous functions on the real domain \( \Omega \) such that \( S_{m-1} \subseteq S_m \) and \( u_0 + u_1 + \ldots + u_m \) is a real linear combination of these functions.

We remark that such a construction is always possible. For example we can choose \( S_m = \{u_0, u_1, \ldots, u_m\}, m = 0,1,2,\ldots \). In this case

\[
\varphi_{m0} = u_0, \varphi_{m1} = u_1, \varphi_{m2} = u_2, \ldots, \varphi_{mn_m} = u_m.
\]

We call an **HP-sequence** of the problem (121) a sequence of functions \( \{s_m(x,t)\}_{m\in\mathbb{N}} \) of the form \( s_m(x,t) = \sum_{k=0}^{n_m} \alpha_m^k \varphi_{mk} \), where \( m \in \mathbb{N}, \alpha_m^k \in \mathbb{R} \).

A function of the sequence is called an **HP-function** of the problem (151).

We call the HP-sequence \( \{s_m(x,t)\}_{m\in\mathbb{N}} \), **convergent to the solution of the problem** (151) if

\[
\lim_{m \to \infty} R(x,t,s_m(x,t)) = 0.
\]

We call an **\( \varepsilon \)**-**approximate HP-solution** of the problem (151) on the real domain \( \Omega \) a HP-function \( \tilde{u} \) which satisfy the following condition:

\[
|R(x,t,\tilde{u})| < \varepsilon
\]

(158)

together with the boundary conditions from (151).
We call an \textit{weak} $\delta$-approximate HP-solution of the problem (151) on the real domain $\Omega$ a HP-function $\tilde{u}$ satisfying the relation
\[ \int_{\Omega} R^2(x,t,\tilde{u}) dx dt \leq \delta, \]
and together with the boundary conditions from (151).

We will find a weak $\varepsilon$-approximate HP-solution of the type $\tilde{u} = \sum_{k=0}^{n_m} c_m^k \varphi_{mk}$ where $m \geq 0$ and the constants $c_m^k$ are calculated using the following steps:

\begin{itemize}
  \item We substitute the approximate solution $\tilde{u}$ in the equation (151) and obtain the expression:
  \[ \mathcal{R}(x,t,c_m^k) = R(x,t,\tilde{u}) \quad (159) \]
  \item We attach to the problem (121) the following real functional:
  \[ J(c_m^k) = \int_{\Omega} \mathcal{R}^2(x,t,c_m^k) dx dt \quad (160) \]
  where, by imposing the boundary conditions we can determine $l \in N$, $l \leq m$ such that $c_0^m, c_1^m, ..., c_l^m$ are computed as functions of $c_0^m, c_1^m, ..., c_n^m$.
  \item We compute the values of $\tilde{c}_0^m, \tilde{c}_1^m, ..., \tilde{c}_l^m$ as the values which give the minimum of the functional (160) and the values of $\tilde{c}_0^m, \tilde{c}_1^m, ..., \tilde{c}_l^m$ again as functions of $\tilde{c}_0^m, \tilde{c}_1^m, ..., \tilde{c}_l^m$ by using the boundary conditions.
  \item Using the constants $\tilde{c}_0^m, ..., \tilde{c}_l^m$ thus determined, we consider the HP-sequence:
  \[ s_m(x,t) = \sum_{k=0}^{n_m} \tilde{c}_m^k \varphi_{mk} \quad (161) \]
  The following convergence theorem holds:
  \[ \text{Theorem 1. The HP-sequence } s_m(x,t) \text{ from (151) satisfies the property:} \]
  \[ \lim_{m \to \infty} \int_{\Omega} R^2(x,t,s_m(x,t)) dx dt = 0 \]
  Moreover, $\forall \varepsilon > 0, \exists m_0 \in \mathbb{N}$ such that $\forall m \in \mathbb{N}, m > m_0$ it follows that $s_m(t)$ is a weak $\varepsilon$-approximate HP-solution of the problem (151).
Based on the way the HP-function \( s_m(x,t) \) is computed, the following inequality holds:

\[
0 \leq \int_{\Omega} R^2(x,t,s_m(x,t))dxdt \leq \int_{\Omega} R^2(t,f_m(x,t))dxdt, \quad \forall m \in \mathbb{N}.
\]

It follows that:

\[
0 \leq \lim_{m \to \infty} \int_{\Omega} R^2(x,t,s_m(x,t))dxdt \leq \lim_{m \to \infty} \int_{\Omega} R^2(x,t,f_m(x,t))dxdt = 0,
\]

\( \forall m \in \mathbb{N} \).

We obtain:

\[
\lim_{m \to \infty} \int_{\Omega} R^2(x,t,s_m(x,t))dxdt = 0.
\]

From this limit we obtain that \( \forall \varepsilon > 0, \exists m_0 \in \mathbb{N} \) such that \( \forall m \in \mathbb{N}, m > m_0 \) it follows that \( s_m(x,t) \) is a weak \( \varepsilon \)-approximate HP-solution of the problem (151) q.e.d.

Any \( \varepsilon \)-approximate HP-solution of the problem (151) is also a weak approximate HP-solution, but the opposite is not always true. It follows that the set of weak approximate HP-solutions of the problem (151) also contains the approximate HP-solutions of the problem.

Taking into account the above remark, in order to find \( \varepsilon \)-approximate HP-solutions of the problem (151) by the OHPM method we will first determine weak approximate HP-solutions, \( \tilde{u} \). If \( |R(x,t,\tilde{u})| < \varepsilon \) then \( \tilde{u} \) is also an \( \varepsilon \)-approximate HP-solution of the problem.

### 3.1.3. Results and discussion

In this section we apply OHPM to find approximate analytical solutions for the regularized long wave (RLW) equation.

The RLW equation is a nonlinear evolution equations which are frequently used to model a variety of physical phenomena such as: ion-acoustic waves in plasma, magnetohydrodynamics waves in plasma, longitudinal dispersive waves in elastic rods, pressure waves in liquid-gas bubble mixtures, rotating flow down a tube etc.

The RLW equation was introduced in [147] where it was used to describe the behaviour of the undular bore.

For some restricted initial and boundary conditions, exact analytical solutions for the RLW equation were computed (see for example [148]). However, in most cases it is not possible to find such exact analytical solutions
and usually numerical methods are used. Among the numerical methods recently employed for RLW-type equations we mention finite difference methods ([149], [150], [151], [152]), multistep mixed finite element methods ([153]), the method of lines ([154]) and meshless finite-point methods ([155]).

Taking into account the usefulness of analytical solutions versus numerical ones, various approximation methods were also employed to find approximate analytical solutions for various RLW-type equations, such as the homotopy parametric method ([156]), the variational iteration method ([156]), the homotopy asymptotic method ([157], [158]) and the Riccati expansion method ([159]).

In the following, for two test problems presented in [156], we compare solutions obtained by using OHPM with previous results obtained by using the homotopy parametric method and the variational iteration method.

**Application 1**

Our first application is the following RLW problem ([156]):

$$
\begin{align*}
\left\{ \begin{array}{l}
\frac{\partial u}{\partial t} - u_{xx} + \left( \frac{u^2}{2} \right)_x = 0 \\
u(x,0) = x
\end{array} \right.
\end{align*}
$$

(162)

In [156] approximate solutions of (152) are computed using the homotopy perturbation method (HPM) and the variational iteration method (VIM). The exact solution of this problem is $u_e(x,t) = \frac{x}{t+1}$.

Using OHPM, the following steps are performed:

- Choosing the same homotopy (153) as used in [156] we obtain the same solutions:
  $$
  \begin{align*}
  u_0(x,t) &= x(t+1) \\
u_1(x,t) &= -x \cdot t \cdot (2 + t + t^2 / 3) \\
u_2(x,t) &= 2 \cdot x \cdot t^2 \cdot (15 + 15 \cdot t + 5 \cdot t^2 + t^3) / 15
  \end{align*}
  $$

  It follows that we obtain the sets $S_0 = \{ x, x \cdot t \}$, $S_1 = \{ x \cdot t, x \cdot t^2, x \cdot t^3 \}$, $S_2 = \{ x \cdot t^2, x \cdot t^3, x \cdot t^4, x \cdot t^5 \}$.
We will compute a second order approximate solution, by taking into account the terms from $S_0$, $S_1$ and $S_2$ and we will compare this solution with the fifth order solutions from [156]. Our second order approximate solution will have the expression:

$$u_{OHPM}(x,t) = c_0 \cdot x + c_1 \cdot x \cdot t + c_2 \cdot x \cdot t^2 + c_3 \cdot x \cdot t^3 + c_4 \cdot x \cdot t^4 + c_5 \cdot x \cdot t^5.$$ 

- Imposing the boundary condition $u_{OHPM}(x,0) = x$ we obtain $c_0 = 1$.

Replacing this expression of $c_0$ in the expression of $u_{OHPM}$ we obtain:

$$u_{OHPM}(x,t) = x + c_1 \cdot x \cdot t + c_2 \cdot x \cdot t^2 + c_3 \cdot x \cdot t^3 + c_4 \cdot x \cdot t^4 + c_5 \cdot x \cdot t^5.$$ 

We introduce $u_{OHPM}$ in the remainder $\mathcal{R}$ given by (152,159) and we compute the functional $J(c_1,c_2,c_3,c_4,c_5)$ of (160).

- We compute the minimum of the functional $J$ and, by replacing the corresponding values of the parameters $c_1,c_2,c_3,c_4,c_5$, we obtain the following second order approximation:

$$\tilde{u}_{OHPM}(x,t) = -0.109895^5 x + 0.434798^4 x - 0.789112^3 x + 0.961938^2 x - 0.997729 x + x.$$ 

Figure 24 presents the comparison of the absolute errors (computed as the absolute values of the differences between the exact solutions and the approximate solutions) corresponding to the fifth order approximation obtained by using HPM (red surface), to the fifth order approximation obtained by using VIM (blue surface) and to the second order approximation obtained by OHPM (green surface).

Table 27 presents the same comparison for several values of $x$ and $t$. 
Table 27: The absolute differences corresponding to the HPM solution (red surface), VIM solution (blue surface) and OHPM solution (green surface) for problem (162)

<table>
<thead>
<tr>
<th></th>
<th>HPM</th>
<th>VIM</th>
<th>OHPM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = t = 0$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x = t = 0.2$</td>
<td>$7.894 \times 10^{-5}$</td>
<td>$3.256 \times 10^{-7}$</td>
<td>$1.081 \times 10^{-5}$</td>
</tr>
<tr>
<td>$x = t = 0.4$</td>
<td>$1.171 \times 10^{-2}$</td>
<td>$2.555 \times 10^{-5}$</td>
<td>$1.398 \times 10^{-5}$</td>
</tr>
<tr>
<td>$x = t = 0.6$</td>
<td>$2.346 \times 10^{-1}$</td>
<td>$2.819 \times 10^{-4}$</td>
<td>$9.787 \times 10^{-6}$</td>
</tr>
<tr>
<td>$x = t = 0.8$</td>
<td>$2.075$</td>
<td>$1.420 \times 10^{-3}$</td>
<td>$3.280 \times 10^{-5}$</td>
</tr>
<tr>
<td>$x = t = 1$</td>
<td>$1.172 \times 10^{1}$</td>
<td>$4.700 \times 10^{-5}$</td>
<td>$2.408 \times 10^{-7}$</td>
</tr>
</tbody>
</table>

Figure 24: The absolute differences corresponding to the HPM solution (red surface), VIM solution (blue surface) and OHPM solution (green surface) for problem (162)
It is easy to see that, overall, the approximations obtained by using OHPM are much more accurate than the ones previously computed by using HPM and VIM. Moreover, our approximate solutions are not only more accurate but, at the same time, they present a much simpler expression since they are second order approximate solutions while the previous ones are fifth order approximate solutions.

**Application 2**

Our second application is the RLW problem (also from [156]):

\[
\begin{aligned}
    u_t - u_{xxx} &= 0 \\
    u(x,0) &= \sin(x)
\end{aligned}
\]  

(163)

Again in [156] approximate solutions of (163) are computed using the homotopy parametric method (HPM) and the variational iteration method (VIM).

The exact solution of this problem is \( u_e(x,t) = e^{-t} \sin(x) \).

The fourth order solution computed in [156] by using the variational iteration method is:

\[
u_{vIM}(x,t) = \frac{1}{24} \left( t^4 - 4t^3 + 12t^2 - 24t + 24 \right) \sin(x)
\]

The third order solution computed in [156] by using the homotopy perturbation method is of the form:

\[
u_{HPM}(x,t) = -\frac{1}{24} \left( t^4 + 4t^3 - 12t^2 + 24t - 24 \right) \sin(x)
\]

Using OHPM, the following steps are performed:

- Choosing the same homotopy (153) as used in [156] we obtain the same solutions:

\[
\begin{aligned}
u_0(x,t) &= (t+1) \cdot \sin(x) \\
u_1(x,t) &= \frac{1}{2} \cdot t \cdot (t+4) \cdot (-\sin(x)) \\
u_2(x,t) &= \frac{1}{6} \cdot t^2 \cdot (t+6) \cdot \sin(x)
\end{aligned}
\]

It follows that we obtain the sets

\[
S_0 = \{ \sin(x), \sin(x) \cdot t \}, \quad S_1 = \{ \sin(x) \cdot t, \sin(x) \cdot t^2 \}, \\
S_2 = \{ \sin(x) \cdot t^2, \sin(x) \cdot t^3 \}.
\]
Hence we will compute a second order approximate solution of the form:

\[ u_{OHPM}(x,t) = c_0 \cdot \sin(x) + c_1 \cdot \sin(x) \cdot t + c_2 \cdot \sin(x) \cdot t^2 + c_3 \cdot \sin(x) \cdot t^3. \]

- Imposing the boundary condition \( u_{OHPM}(x,0) = x \) we obtain \( c_0 = 1 \).

Replacing this expression of \( c_0 \) in the expression of \( u_{OHPM} \) we obtain:

\[ u_{OHPM}(x,t) = \sin(x) + c_1 \cdot \sin(x) \cdot t + c_2 \cdot \sin(x) \cdot t^2 + c_3 \cdot \sin(x) \cdot t^3. \]

We introduce \( u_{OHPM} \) in the remainder \( \mathcal{R} \) given by (152,159) and we compute the functional \( J(c_1,c_2,c_3) \) of (160).

- We compute the minimum of the functional \( J \) and, by replacing the corresponding values of the parameters \( c_1,c_2,c_3 \), we obtain the following second order approximation:

\[
\bar{u}_{OHPM}(x,t) = -0.102902t^3 \sin(x) + 0.465235t^2 \sin(x) - 0.994455t \sin(x) + \sin(x)
\]

Figure 25 presents the comparison of the absolute errors corresponding to the third order approximation obtained by using HPM (red surface), to the fourth order approximation obtained by using VIM (blue surface) and to the second order approximation obtained by OHPM (green surface).

Table 28 presents the same comparison for several values of \( x \) and \( t \).

<table>
<thead>
<tr>
<th>( x = t = 0 )</th>
<th>HPM</th>
<th>VIM</th>
<th>OHPM</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( x = t = 0.2 )</td>
<td>2.598 ( 10^{-5} )</td>
<td>5.126 ( 10^{-7} )</td>
<td>3.268 ( 10^{-5} )</td>
</tr>
<tr>
<td>( x = t = 0.4 )</td>
<td>7.996 ( 10^{-4} )</td>
<td>3.113 ( 10^{-5} )</td>
<td>9.737 ( 10^{-5} )</td>
</tr>
<tr>
<td>( x = t = 0.6 )</td>
<td>5.766 ( 10^{-3} )</td>
<td>3.322 ( 10^{-4} )</td>
<td>1.279 ( 10^{-3} )</td>
</tr>
<tr>
<td>( x = t = 0.8 )</td>
<td>2.276 ( 10^{-2} )</td>
<td>1.725 ( 10^{-3} )</td>
<td>1.233 ( 10^{-2} )</td>
</tr>
<tr>
<td>( x = t = 1 )</td>
<td>6.413 ( 10^2 )</td>
<td>5.992 ( 10^{-3} )</td>
<td>9.511 ( 10^{-7} )</td>
</tr>
</tbody>
</table>

Table 28. The absolute differences corresponding to the HPM solution (red surface), VIM solution (blue surface) and OHPM solution (green surface) for problem (163)
3.1.4. Conclusions

Again, overall, the approximations obtained by using OHPM are more accurate than the ones previously computed by using HPM and VIM while, at the same time, they present a much simpler expression.

The optimal homotopy perturbation method has an accelerated convergence compared to the regular homotopy perturbation method, fact proved by the included applications. The method is a powerful one since not only were we capable to find more accurate approximations, but the approximations computed consist of fewer terms than the previous solutions.
4. Further research

In the following we will present two possible research directions related to the extension of the methods included in the present thesis.

The first one is to develop these methods in order to search for analytical approximate solutions for the dynamical systems modelling the following engineering problems:

* The heat transfer of a micropolar fluid through a porous medium
* The MHD boundary-layer equations
* The steady MHD convective and slip flow due to a rotating disk
* The unsteady boundary-layer flow and heat transfer due to a stretching sheet
* The steady three-dimensional problem of condensation film on inclined rotating disk
* The mixed convection about an inclined flat plate embedded in a porous medium.
* The electrically conducting fluid past a rotating disk in the presence of a uniform vertical magnetic field
* The MHD stagnation-point flow in porous media with heat transfer
* The non-Newtonian flow and heat transfer over a non-isothermal wedge
* The two-dimensional viscous flow in a rectangular domain bounded by two moving porous walls
* The three-dimensional Navier–Stokes equations for the flow near an infinite rotating disk
* The third grade non-Newtonian fluid flow between two parallel plates using the multi-step differential transform method
* The MHD flow in a laminar liquid film from a horizontal stretching surface.

In recent years fractional differential equations were extensively used to model various problems frequently encountered in fields such as physics, chemistry and engineering. Taking into account the fact that in many cases, exact solutions for these equations can not be found, it is necessary to find approximate solutions. Thus, the second research direction consists in the extension of the methods included in this thesis for various dynamical systems modelled by fractional differential equations.
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