HABILITATION THESIS

Approximation by certain positive linear operators

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Contents

Notations and symbols

$\mathbb{N}$	the set of natural numbers,
$\mathbb{N}_0$	the set of natural number including zero,
$\mathbb{R}$	the set of real numbers,
$\mathbb{R}_+$	the set of positive real numbers,
$[a, b]$	a closed interval,
(a, b)	an open interval,
$C[a, b]$	the set of all real-valued and continuous function defined $[a, b]$, on the compact interval
$C^r[a, b]$	the set of all real-valued, r-times continuously differentiable function ($r \in \mathbb{N}$),
(X, d)	metric space equipped with metric d,
$Lip_r M$	the set of all $C[a, b]-$ functions that verify the Lipschitz condition $\lvert f(x_2) - f(x_1)\rvert \leq M\lvert x_2 - x_1\rvert^r$, for all $x_1, x_2 \in [a, b]$, $0 < r \leq 1$, $M > 0$,
$B(X)$	the set of all real-valued and bounded functions defined on X,
$L_p(X)$	the class of the p-Lebesque integrable functions on X, $p \geq 1$,
Π_n	the linear space of all real polynomials with the degree at most n,
e_n	denotes the n-th monomials with $e_n : [a, b] \to \mathbb{R}$, $e_n(x) = x^n$, $n \in \mathbb{N}_0$,
$(x)_n$	the rising factorial $(x)_n := x(x + 1) \cdots (x + n - 1)$ and $(x)_0 := 1$,
B^A	$B^A := \{f : A \to B\}$,
$C[0, \infty)$	the set of all continuous functions defined on $[0, \infty)$,
$B_2[0, \infty)$	the set of all functions$f$ defined on $[0, \infty)$ satisfying the condition $\lvert f(x)\rvert \leq M(1 + x^2)$, M is a positive constant,
$C_2[0, \infty)$	the subspace of all continuous function in $B_2[0, \infty]$,
$C_2^*[0, \infty)$	the subspace of all function $f \in C_2[0, \infty)$ for which $\lim\limits_{x\to\infty} \frac{f(x)}{1 + x^2}$ is finite,
$\lVert \cdot \rVert_*$	$\lVert f\rVert_* = \sup\limits_{x\geq 0} \frac{\lvert f(x)\rvert}{1 + x^2}$,
$\Omega(f; \delta)$	the weighted modulus of continuity,
$t^{[n,h]}$	the n^{th} factorial power of t with increment h, $t^{[n,h]} = t(t - h) \dots (t - (n - 1)h)$

Abstract

In this habilitation thesis we have described the significant results achieved by the author after obtaining her PhD degree in Mathematics from Babeş-Bolyai University, in 2007. Approximation theory represents on old topic of mathematical analysis which still remains an attractive research area with many applications. The research results presented here are concerned with the approximation by certain classes of positive linear operators. In the first part of this thesis we were interested in how non-multiplicative can a linear functional be. In order to give an answer to this question, we considered the generalized Chebyshev functional

$$T_L(f,g) := L(f \cdot g) - L(f) \cdot L(g),$$

for a positive linear functional L and we obtained the estimates as follows

$$|T_L(f,g)| \leq \mathcal{E}(L,f,g).$$

These inequalities have been applied in the case of known operators. The estimates for the differences of positive linear operators is another topic which we presented. In the second part of this thesis, the basic results and direct estimates in both local and global approximation for certain classes of positive linear operators were considered. Also, some results concerning the asymptotic expansion of some linear positive operators were given. The rate of convergence for functions having derivatives of bounded variation was presented. In the last decades, the applications of q-calculus represent one of the most attractive research area in approximation theory. The first q-analogue of the well-known Bernstein operator was introduce by Lupaş in 1987. In 1997 Phillips considered another q-analogue of the classical Bernstein operator. Later several other researchers have obtained the q-extension of the well-known operators. In the last years we published some results related to this research area, but we did not included them in this thesis ([12], [13], [15], [17], [114], [115]).

I would like to highlight that some of these scientific results were obtained as joint work with researchers from Germany, Turkey, India and Romania, while collaborating in common research project and taking part as invited professor in scientific seminars. In 2008, I started a colaboration with H. Gonska from Diusburg-Essen University, Germany, by studying the Ostrowski-type inequality. This inequality was introduced by A.M. Ostrowski in [137] and can be given in a variety of forms. We showed that the conditions assuming differentiability properties of the functions are not necessary and gave a generalization of Ostrowski's inequality for an arbitrary continuous function $f \in C[a,b]$ and certain linear operators (see [25]).

Another renowned classical inequality was introduced by G. Grüss in [92]. In 2010, I continued to collaborate with H. Gonska and I. Raşa during my visit at Duisburg-Essen University in DAAD program dedicated to "Academic Reconstruction of South-Eastern Europe", by studying

Grüss type inequalities on spaces of continuous functions defined on a compact metric space (see [24]). Using the least concave majorant of the modulus of continuity we obtained a Grüss inequality for the functional $L(f) = H(f;x)$, where $H : C[a,b] \to C[a,b]$ is a positive linear operator and $x \in [a,b]$ is fixed. We applied this inequality in the case of known operators, for example the Bernstein, Hermite-Fejér interpolation, convolution-type operators. Moreover, we derived inequalities of the Grüss-type using Cauchy's mean value theorem. A Grüss inequality on a compact metric space for more than two functions was given. This important study was later continued by H. Gonska, I. Raşa, G. Tachev, B. Gavrea, M. Rusu, I. Gavrea, N. Minculete, L. Ciurdariu in [67], [70], [75], [76], [131], [169].

The collaboration with H. Gonska was continued in 2015 during his vist at "Lucian Blaga" University of Sibiu, by studying a sequence of composite bivariate Bernstein operators and the cubature formula associated with them. This contribution is a continuation of [74]. We described the upper-bounds for the remainder term of cubature formula in terms of moduli of continuity of order two (see [11]). Our study is motivated by a recent series of articles by Barbosu et al. (see [40]-[43]).

In 2013, during my visit at Duisburg-Essen University under Erasmus Program, I started a good collaboration with Maria Rusu concerned with bivariate Grüss-type inequalities via discrete oscillations and applied them to different tensor products of linear, (not necessarily) positive, well-known operators. Also, we obtained a Grüss-type inequality with discrete oscillations for more than two functions (see [18]).

In 2014, motivated by the recent results which give a solution to a problem proposed by A. Lupaş in [123], we introduced new inequalities for the differences of certain positive linear operators (see [9]).

In 2015, during my visit as Invited Professor at Kirikkale University, Turkey, I had scientific collaboration with T. Acar and G. Ulusoy. Our discussion about the Kantorovich modification of order k of Baskakov operators have been concretized in the paper [3]. Another direction in my papers refers to the study of approximation properties of some operators based on Pólya distribution. This topic was considered in some joint papers with mathematicians from India (V. Gupta, P.N. Agrawal, T. Neer).

The thesis consists of seven chapters, the last one being dedicated to description of some future plans regarding the scientific and professional career of the author. At the beginning of each chapter we mention the papers where the results are contained.

The **first chapter** comprises preliminary instruments that will be further used for deriving our results, as the moduli of smoothness, the K-functional and the connection to the moduli, the weighted space and corresponding modulus of continuity, positive linear operators.

Chapter 2 is dedicated to the contribution of the authors concerning Grüss type inequalities on space of continuous functions defined on a compact metric space and the applications of these inequalities in the cases of known operators. Our results are motivated by a theorem which can be found in the paper [34] by D. Andrica and C. Badea. It is the aim of this chapter to look again at Grüss' inequality from a somewhat different point of view. We are interested in how non-multiplicative can a linear functional be.

Let $H_n : C[a,b] \to C[a,b]$ be the positive linear operators which reproduce constant functions.

For $x \in [a, b]$ we consider $L = \epsilon_x \circ H_n$, so $L(f) = H_n(f; x)$. Denote

$$D(f, g) := H_n(fg; x) - H_n(f; x) \cdot H_n(g; x).$$

In this chapter we obtained a result that suggests how non-multiplicative the functional $L(f) = H_n(f; x)$ is for a given $x \in [a, b]$. Some applications of this result in the case of known positive linear operators were considered. Also, we studied non-multiplicativity for the functional L using Cauchy's mean value theorem. This study was motivated by B.G. Pachpatte's result obtained in [139].

In **chapter 3** we gave a new upper bound for the approximation error of cubature formula associated with the bivariate Bernstein operators. The bounds are described in terms of moduli of continuity of order two. The consideration of this cubature formula was motivated by Bărbosu and Pop's result [42]. Also, we constructed the bivariate composite Bernstein operators and the order of convergence is considered involving the second modulus of continuity. Some inequalities of Grüss type were proven. These results were obtained using some general inequalities published in [153]. Another study considered in this chapter concerns Grüss-type inequalities which involves oscillations of functions.

In **chapter 4** we introduced new inequalities for the differences of positive linear operators in terms of moduli of continuity. These results are based on some inequalities involving positive linear functionals. Firstly we established such inequalities for smooth functions and the norms of their derivatives. Then, using some deep results from [85] and [87], we gave inequalities for continuous functions in terms of moduli of smoothness. In the last section of this chapter we applied the general results to certain positive linear operators.

Chapter 5 contains approximation properties of some operators based on Pólya distribution. In the first section we construct a sequence of Lupaş operators based on Pólya distribution using a function τ. This function is any function on $[0, 1]$ continuously differentiable ∞ times, such that $\tau(0) = 0$, $\tau(1) = 1$ and $\tau'(x) > 0$ for $x \in [0, 1]$. Note that the Korovkin set $\{1, e_1, e_2\}$ is generalized to $\{1, \tau, \tau^2\}$ and these operators present a better degree of approximation then the original ones. We gave a direct approximation theorem by means of the Ditzian-Totik modulus of smoothness and a Voronovskaja type theorem. In the second section we introduced the Bézier variant of genuine-Durrmeyer type operators having Pólya basis functions. We gave a global approximation theorem in terms of second order modulus of continuity, a direct approximation theorem by means of the Ditzian-Totik modulus of smoothness and a Voronovskaja type theorem by using the Ditzian-Totik modulus of smoothness. The rate of convergence for functions whose derivatives are of bounded variation was obtained. Further, we showed the rate of convergence of these operators to certain functions by illustrative graphics using the Maple algorithms. The results presented in this chapter were published in [10] and [136].

Chapter 6 contains a study of the k-th order Kantorovich type modification of Baskakov operators. We first established explicit formulas giving the images of monomials and the central moments up to order six. Also, we gave a quantitatine Voronowskaya theorem for differentiated Baskakov operators in weighted spaces.

Chapter 7 includes a perspective plan for the present and future projects in scientific research and the teaching career. I will continue my research in the field of approximation by linear positive operators. At the same time, I will maintain focus on the certain type of inequalities and their applications in theory of linear positive operators. Also, I intend to write two scientific monographs related to my contributions in approximation formulas of definite integrals and a survey on Grüss and Ostrowski type inequalities and applications of these inequalities in context of approximation by linear positive operators.

In the approximation theory by linear positive operators, the Baskakov operators and Szász-Mirakjan operators are two of the most useful tools to approximate the functions on unbounded intervals. Then, I intend to start a new joint project with Purshottam Agrawal and Arun Kajla related to the Baskakov-Szász operators based on inverse Pólya-Eggenberger distribution. Inspired by the recently work of Gupta and Greubel [94], I would like to start a new joint project with Vijay Gupta related to certain summation-integral type operators such as Baskakov-Szász-Mirakjan operators.

In [112] was introduced the Durrmeyer type Jakimovski-Leviatan operators. Motivated by this work, I would like to start a new joint work with Purshottam Agrawal and Trapti Neer related to this subject. We intend to introduce the Chlodowsky variant of these operators and to give the rate of approximation in terms of first order modulus of continuity and the Ditzian-Totik modulus of smoothness. Also we will introduce a Voronovskaja-type asymptotic formula and we will give some approximation results for a weighted space.

In [9] we obtained some inequalities for a positive linear functional using Taylor's formula. These results led us to new estimates of the differences of certain positive linear operators. Applications for some known positive linear operators were given. I would like to continue this joint project with Ioan Raşa by giving estimates for such differences of certain positive linear operators involving their derivatives.

Also, I will focus my attention on the P_n-simple functionals. Some estimates for P_n-simple functionals using the least concave majorant of the modulus of continuity were considered by Gavrea (see [68], [69]) and Raşa [150]. These results motivate me to obtain some approximation properties for linear positive operators expressed in terms of modulus of smoothness using the estimates for P_n-simple functionals.

Rezumat

In aceasta teză descriem rezultatele semnificative obţinute de autor după obţinerea în 2007 a titlului de doctor în matematică la Universitatea Babeş-Bolyai din Cluj-Napoca. Teoria aproximării reprezintă o ramură veche a analizei matematice, care încă a rămas un domeniu de cercetare atractiv, cu multe aplicaţii. Rezultatele ştiinţiifice prezentate se referă la aproximarea prin anumite clase de operatori liniari şi pozitivi. In prima parte a tezei suntem interesaţi să studiem cât de non-multiplicativă poate fi o funcţională liniară. Cu scopul de a da un răspuns la această întrebare considerăm funcţionala Chebyshev generalizată

$$T_L(f, g) := L(f \cdot g) - L(f) \cdot L(g),$$

unde L este o funcţională liniară, şi obţinem estimari de forma

$$|T_L(f, g)| \leq \mathcal{E}(L, f, g).$$

Aceste inegalităţi au fost aplicate în cazul unor operatori cunoscuţi. Estimarea diferenţelor de operatori liniari şi pozitivi este un alt subiect abordat. În partea a doua a tezei am considerat rezultate de bază şi estimări directe în raport cu aproximarea locală şi globală pentru anumite clase de operatori liniari şi pozitivi. De asemenea, câteva rezultate privind comportarea asimptotică a unor operatori liniari şi pozitivi au fost obţinute. Rata de convergenţă în cazul funcţiilor a căror derivată este cu variaţie mărginita a fost studiată. In ultimul deceniu, aplicaţii ale q-calculus reprezintă unul dintre cele mai atractive domenii de cercetare. În 1987, Lupaş a introdus q-analog al operatorului Bernstein. Mai tarziu, in 1997, Phillips consideră o noua variantă a q-analog pentru operatorul Bernstein. Mai târziu mulţi matematicieni au obţinut extinderi în q-calculus pentru operatorii liniari si pozitivi clasici. În ultimii ani am publicat câteva rezultate ce abordează acest subiect, dar acestea nu au fost incluse in prezenta lucrare ([12], [13], [15], [17], [114], [115]).

As dori să menţionez că anumite rezultate au fost obţinute in colaborare cu cercetători din Germania, Turcia, India şi România, în anumite proiecte de cercetare şi ca profesor invitat la diferite seminarii ştiinţifice. In 2008 am început colaborarea cu H. Gonska de la Universitatea Duisburg-Essen din Germania, abordând ca şi studiu inegalităţile de tip Ostrowski. Această inegalitate a fost introdusă de A.M. Ostrowski în [137], ulterior obţinându-se o varietate de forme ale acesteia. Am arătat ca nu este necesară condiţia de derivabilitate şi am obţinut o generalizare a inegalităţii Ostrowski pentru funcţii continue oarecare $f \in C[a, b]$ şi anumiţi operatori liniari (vezi [25]).

În 2010 am continuat colaborarea cu H. Gonska şi I. Raşa în timpul vizitei mele la Universitatea Duisburg-Essen din Germania în cadrul programului DAAD dedicat " Reconstrucţiei Academice în sud-estul Europei", considerând inegalităţile de tip Grüss pe spaţiul funcţiilor continue definite pe un spaţiu metric compact (see [24]). Folosind cel mai mic majorant concav al modulului de continuitate am obţinut o inegalitate de tip Grüss pentru funcţionala $L(f) = H(f; x)$, unde $H : C[a, b] \to C[a, b]$ este un operator pozitiv şi liniar şi $x \in [a, b]$ este fixat. Am aplicat această inegalitate în cazul unor operatori clasici, ca de exemplu Bernstein, operatorii de interpolare Hermite-Fejér, operatorii de tip convoluţie. Mai mult, am obţinut inegalităţi de tip Grüss folosind teorema de media a lui Cauchy. De asemenea, am studiat inegalităţile de tip Grüss pe un spaţiu metric compact pentru mai mult de două funcţii. Mai târziu acest important studiu a fost continuat de H. Gonska, I. Raşa, G. Tachev, B. Gavrea, M. Rusu, I. Gavrea, N. Minculete, L. Ciurdariu în [67], [70], [75], [76], [131], [169].

Colaborarea cu H. Gonka a continuat în 2015 în timpul vizitei sale la Universitatea "Lucian Blaga" din Sibiu. În această perioadă ne-am ocupat de şirul operatorilor Bernstein bidimensionali compoziţi şi de formula de cuadratură asociată acestora. Aceste rezultate reprezintă o continuare a cercetarilor facute în [74]. Marginea superioara a termenului rest în formula de cuadratură este dată utilizând modulul de continuitate de ordinul doi (vezi [11]). Studiul nostru este motivat de seria recentă de articole publicate de Bărbosu, Miclăuş şi Pop (vezi [40]-[43]).

În 2013, în timpul unei mobilităţi Erasmus la Universitatea Duisburg-Essen din Germania am pus bazele unei bune colaborări cu Maria Rusu, obţinând astfel inegalităţi de tip Grüss pentru cazul bidimensional utilizând oscilaţii discrete şi am aplicat aceste rezultate în cazul operatorilor liniari bidimensionali clasici, (nu în mod necesar) pozitivi. De asemenea, am obţinut inegalităţi de tip Grüss pentru mai mult de două funcţii utilizând oscilaţii discrete (vezi [18]).

În 2014, motivaţi de recentele rezultate ce dau o soluţie a problemei propuse de A. Lupaş in [123], am introdus noi inegalităţi pentru diferenţele anumitor operatori pozitivi şi liniari (vezi [9]).

În 2015, în timpul visitei în calitate de profesor invitat la Universitatea Kirikkale din Turcia am colaborat cu T. Acar şi G. Ulusoy. Discuţia noastră cu privire la modificarea Kantorovich de ordin k a operatorilor Baskakov s-a concretizat în articolul [3]. O altă direcţie de cercetare în aricolele mele se referă la studiul proprietăţiilor de aproximare ale unor operatori bazaţi pe distribuţia Pólya. Acest subiect a fost tratat în câteva articole cu matematicieni din India (V. Gupta, P.N. Agrawal, T. Neer).

Teza conţine şapte capitole, ultimul fiind dedicat descrierii unor planuri de viitor privind cariera ştiinţifică şi profesională a autoarei. La începutul fiecărui capitol menţionăm articolele în care rezultatele prezentate au fost obţinute.

În **primul capitol** sunt prezentate noţiunile preliminare ce vor fi utilizate în obţinerea rezultatelor, ca modulul de continuitate, K- funcţionala şi legătura cu modulul de contimuitate, spaţiul ponderat şi modulul de continuitate corespunzător, operatorii pozitivi şi liniari.

Capitolul 2 este dedicat contribuţiei autoarei privind inegalităţile de tip Grüss pe spaţiul funcţiilor continue definite pe un spaţiu metric compact şi aplicaţiilor acestor inegalităţi în cazul operatorilor clasici. Rezultatele noastre sunt motivate de o teoremă ce poate fi găsită in articolul [34] al lui D. Andrica şi C. Badea. Dorinţa noastră în acest capitol este de a privi inegalitatea Grüss dintr-un alt punct de vedere. Suntem interesaţi în a studia cât de non-multiplicativă poate fi o funcţională liniară.

Fie $H_n : C[a, b] \to C[a, b]$ un şir de operatori ce reproduc funcţii constante. Pentru $x \in [a, b]$

considerăm $L = \epsilon_x \circ H_n$, so $L(f) = H_n(f; x)$. Notăm

$$D(f, g) := H_n(fg; x) - H_n(f; x) \cdot H_n(g; x).$$

În acest capitol am obţinut un rezultat ce sugerează cât de non-multiplicativă este funcţionala $L(f) = H_n(f; x)$ pentru $x \in [a, b]$. Aplicaţii ale acestui rezultat în cazul operatorilor clasici au fost considerate. De asemenea, am studiat non-multiplicativitatea funcţionalei L folosind teorema de medie a lui Cauchy. Acest studiu a fost motivat de rezultatul lui B.G. Pachpatte publicat în [139].

În **capitolul 3** am obţinut o nouă margine a erorii de aproximare în cazul formulei de cubatură asociată operatorilor Bernstein bidimensionali. Marginile au fost descrise utilizând modulul de continuitate de ordinul 2. Studiul acestei formule de cubatură a fost motivat de rezultatul lui Bărbosu şi Pop [42]. De asemenea, am construit operatorii Bernstein bidimensionali compoziţi şi am studiat ordinul de convergenţă utilizând modulul de continuitate de ordinul doi. Câteva tipuri de inegalităţi de tip Grüss au fost considerate. Aceste rezultate au fost obţinute aplicând inegalităţi generale publicate în [153]. Un alt studiu prezentat în acest capitol se referă la inegalităţile de tip Grüss obţinute cu ajutorul oscilaţiilor funcţiilor.

În **capitolul 4** am introdus noi inegalităţi pentru diferenţele de operatori liniari şi pozitivi în termenii modulului de continuitate. Aceste rezultate au la bază anumite inegalităti obţinute pentru funcţionale liniare. Mai întâi am stabilit astfel de inegalităţi pentru funcţii derivabile şi pentru normele derivatelor acestor funcţii. Utilizând apoi rezultatele din [85] şi [87] am obţinut inegalităţi pentru funcţii continue în termenii modulului de continuitate. În ultima secţiune a acestui capitol am aplicat aceste rezultate generale unor operatori pozitivi şi liniari.

În **capitolul 5** am studiat proprietăţi de aproximare ale unor operatori bazaţi pe distribuţia Pólya folosind funcţia τ. Această funcţie este definită pe intervalul $[0, 1]$ şi este indefinit continuu diferenţiabilă astfel încât $\tau(0) = 0$, $\tau(1) = 1$ şi $\tau'(x) > 0$ pentru $x \in [0, 1]$. Notăm că mulţimea Korovkin $\{1, e_1, e_2\}$ este generalizată prin $\{1, \tau, \tau^2\}$ şi aceşti operatori prezintă un grad de aproximare mai bun decât versiunea iniţială. Am obţinut teoreme de aproximare ce implică modulul de continuitate Ditzian-Totik şi o teoremă de tip Voronovskaja. În sectiunea a doua am introdus varianta Bézier a operatorilor de tip genuine-Durrmeyer bazaţi pe distribuţia Pólya. Am obţinut o teoremă de aproximare globală ce implică modulul de continuitate de ordinul 2, o teoremă de aproximare directă utilizând modulul de continuitate Ditzian-Totik şi o teoremă de tip Voronovskaja. Rata de convergenţă pentru funcţii ale căror derivate au variaţie mărginită a fost obţinută. Mai mult, utilizând grafice realizate în Maple am pus în evidenţă rata de convergenţă a acestor operatori pentru anumite funcţii. Rezultatele prezentate în acest capitol au fost publicate în [10] and [136].

Capitolul 6 conţine un studiu cu privire la modificarea Kantorovich de ordin k a operatorilor Baskakov. Pentru început am stabilit formulele explicite dând imaginile monoamelor şi momentele centrale până la ordinul 6. De asemenea am demonstrat o teoremă de tip Voronovskaja pentru aceşti operatori pe spaţii ponderate.

Capitolul 7 prezintă un plan de perspectivă cu privire atât la activitatea de cercetare, cât şi la cea didactică. Intenţionez să continui activitatea de cercetare în domeniul aproximării prin operatori liniari şi pozitivi. În egală masură îmi voi îndrepta atenţia şi către inegalităţile matematice şi aplicaţile acestora în domeniul operatorilor liniari şi pozitivi. De asemenea, mi-am propus să scriu două monografii ce vor cuprinde contribuţii personale cu privire la formulele de aproximare ale integralelor definite, precum şi un istoric al inegalităţilor de tip Grüss şi Ostrowski şi aplicaţii ale acestora în contextul aproximării prin operatori liniari şi pozitivi.

În teoria aproximării prin operatori liniari şi pozitivi, operatorii Baskakov şi Szász-Mirakjan sunt două dintre cele mai utilizate instrumente folosite în aproximarea funcţiilor pe intervale nemărginite. Acesta este motivul pentru care intenţionez să demarez un nou proiect de cercetare cu Purshottam Agrawal şi Arun Kajla privind operatorii Baskakov-Szász bazaţi pe distribuţia Pólya-Eggenberger. Inspiraţi de recentele rezultate ale lui Gupta şi Greubel [94], doresc să pun bazele unei viitoare colaborări cu Vijay Gupta privind studiul unor operatori de tip Baskakov-Szász-Mirakjan.

În [112] a fost introdus operatorul Jakimovski-Leviatan de tip Durrmeyer. Motivaţi de acest rezultat imi doresc să aprofundez acest subiect propunând împreună cu Purshottam Agrawal şi Trapti Neer noi rezultate. Intenţionăm să introducem varianta Chlodowsky a acestor operatori şi să obţinem rata de convergenţă utilizând modulul de continuitate. De asemenea vom studia comportarea asimptotică a acestor operatori şi vom da câteva proprietăţi de aproximare pe spaţii ponderate.

În [9] am obţinut câteva inegalităţi pentru funcţionale liniare utilizând formula lui Taylor. Aceste rezultate ne-au condus către noi estimări ale diferenţelor unor operatori liniari şi pozitivi. Aplicaţii pentru operatori clasici au fost de asemenea considerate. Imi doresc să continui cu Ioan Raşa acest proiect ocupându-ne de obţinerea estimărilor unor astfel de diferenţe pentru operatori pozitivi şi liniari ce implică derivatele acestora.

De asemenea, intenţionez să ma ocup de studiul P_n-funcţionalelor simple. Estimări ale acestor funcţionale utilizând cel mai mic majorant concav al modulului de continuitate au fost considerate de Gavrea (vezi [68], [69]) şi Raşa [150]. Aceste rezultate mă motivează să studiez proprietăţi de aproximare pentru operatori pozitivi şi liniari utilizând estimări ale P_n-funcţionale simple.

1 Preliminaries

1.1 Moduli of smoothness

In this section we recall definitions and properties of the moduli of smoothness, that will be of interest in the whole thesis. The definition of the first moduli of smoothness was given in the Ph.D. thesis of D. Jackson [108] as follows:

Definition 1.1. *For a function $f \in C[a,b]$ and $t \geq 0$, we have*

$$\omega(f;t) := \sup \left\{ |f(x+h) - f(x)| : \, x, x+h \in [a,b], 0 \leq h \leq t \right\}.$$

When we want to establish the degree of convergence of positive linear operators towards the identity operators, a very important tool that we use is the least concave majorant of the modulus of continuity $\tilde{\omega}(f;\cdot)$. This is given by

$$\tilde{\omega}(f;t) := \begin{cases} \sup\limits_{0 \leq x \leq t \leq y \leq b-a,\, x \neq y} \dfrac{(t-x)\omega(f;y) + (y-t)\omega(f;x)}{y-x}, & \text{if } 0 \leq t \leq b-a, \\ \omega(f;b-a), \ \text{if } t > b-a. \end{cases} \tag{1.1}$$

The following relationship between the different moduli holds:

$$\omega(f;\cdot) \leq \tilde{\omega}(f;\cdot) \leq 2\omega(f;\cdot).$$

Some of the error estimates in this work are given in terms of the moduli of higher order. Therefore we give the definition of ω_k, $k \in \mathbb{N}$, as given in 1981 by L.L. Schumaker [155]:

Definition 1.2. *For $k \in \mathbb{N}$, $t \in \mathbb{R}_+$ and $f \in C[a,b]$ the modulus of smoothness of order k is defined by*

$$\omega_k(f;t) := \sup \left\{ |\Delta_h^k f(x)| : \, 0 \leq h \leq t, \, x, x+kh \in [a,b] \right\}, \tag{1.2}$$

where

$$\Delta_h^k f(x) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} f(x+(k-i)h) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x+jh).$$

Proposition 1.1. *[162] The modulus of smoothness of order k verifies the following properties:*

1) $\omega_k(f;0) = 0$;

2) $\omega_k(f;\cdot)$ is a positive, continuous and non-decreasing function on $\mathbb{R}_+$;

3) $\omega_k(f;\cdot)$ is sub-additive, i.e., $\omega_k(f;t_1+t_2) \leq \omega_k(f;t_1) + \omega_k(f;t_2)$, $t_i \geq 0$, $i = 1,2$;

4) $\omega_{k+1}(f;t) \leq 2\omega_k(f;t)$, for all $t \geq 0$;

5) If $f \in C^1[a,b]$, then $\omega_{k+1}(f;t) \le t\omega_k(f';t)$, $t \ge 0$;

6) If $f \in C^k[a,b]$, then $\omega_k(f;t) \le t^k \sup\limits_{t\in[a,b]} |f^{(k)}(t)|$;

7) $\omega_k(f;nt) \le n^k \omega_k(f;t)$, for all $t > 0$ and $n \in \mathbb{N}$;

8) $\omega_k(f;rt) \le (1+[r])^k \omega_k(f;t)$, for all $t > 0$ and $r > 0$, where $[a]$ is the integer part of a;

In the following we shall recall more general statements using the moduli of smoothness that were introduced by Gonska and Kovacheva [85] and Gonska [87].

Lemma 1.1. *[85] If $f \in C^q[0,1]$, then for all $0 < h \le \dfrac{1}{2}$ there are functions $g \in C^{q+2}[0,1]$, such that*

i) $\|f^{(q)} - g^{(q)}\| \le \dfrac{3}{4}\omega_2(f^{(q)};h)$,

ii) $\|g^{(q+1)}\| \le \dfrac{5}{h}\omega_1(f^{(q)};h)$,

iii) $\|g^{(q+2)}\| \le \dfrac{3}{2h^2}\omega_2(f^{(q)};h)$.

Lemma 1.2. *[87] Let $I = [0,1]$ and $f \in C^r(I)$, $r \in \mathbb{N}_0$. For any $h \in (0,1]$ and $s \in \mathbb{N}$ there exists a function $f_{h,r+s} \in C^{2r+s}(I)$ with*

i) $\|f^{(j)} - f^{(j)}_{h,r+s}\| \le c\omega_{r+s}(f^{(j)};h)$, *for* $0 \le j \le r$,

ii) $\|f^{(j)}_{h,r+s}\| \le ch^{-j}\omega_j(f;h)$, *for* $0 \le j \le r+s$,

iii) $\|f^{(j)}_{h,r+s}\| \le ch^{-(r+s)}\omega_{r+s}(f^{(j-r-s)};h)$, *for* $r+s \le j \le 2r+s$.

Here the constant c depends only on r and s.

1.2 K-functionals and the connection to the moduli of smoothness

Another important instrument to measure the smoothness of a function is the so-called Peetre's K-functional. It was introduced by J. Peetre [144] in 1968 and can be defined in a very general context. For the applications in this thesis we need to consider the following definition:

Definition 1.3. *Let $f \in C[a,b]$, $\delta \ge 0$ and $s \in \mathbb{N}$, $s \ge 1$. We denote*

$$K_s(f;t) := K(f;t;C[a,b],C^s[a,b]) := \inf\left\{\|f-g\| + t\|g^{(s)}\|,\ g \in C^s[a,b]\right\} \tag{1.3}$$

to be Peetre's K-functional of order s.

P.L. Butzer and H. Berens [51] were proven the following properties of the Peetre's K-functionals of order s. For more details the reader should consult the recent work on approximation theory [59], [86], [155].

Lemma 1.3. *[51] Let $K_s(f;\cdot)$ be the K-functional of order s defined in (1.3).*

1) The mapping $K_s(f;\cdot):\mathbb{R}_+\to\mathbb{R}_+$ *is continuous at* $t=0$*, i.e.,*

$$\lim_{t\to 0^+} K_s(f;t)=0=K_s(f;0).$$

2) For each fixed $f\in C[a,b]$ *the application* $K_s(f;\cdot):\mathbb{R}_+\to\mathbb{R}_+$ *is monotonically increasing and concave function.*

3) For arbitrary $\lambda,t\geq 0$*, and fixed* $f\in C[a,b]$*, one has the inequality*

$$K_s(f;\lambda t)\leq \max\{1,\lambda\}K_s(f;t).$$

4) For arbitrary $f_1,f_2\in C[a,b]$ *we have* $K_s(f_1+f_2;t)\leq K_s(f_1;t)+K_s(f_2;t)$*,* $t\geq 0$*.*

5) For each $t\geq 0$ *fixed,* $K_s(\cdot;t)$ *is a seminorm on* $C[a,b]$*, such that*

$$K_s(f;t)\leq \|f\|,$$

holds for all $f\in C[a,b]$*.*

6) For a fixed $f\in C[a,b]$ *and* $t\geq 0$ *the identity* $K_s(|f|;t)=K_s(f;t)$ *is true.*

A important connection between the K-functional and the moduli is given in the following theorem (see [111]):

Theorem 1.1. *There exists constants* c_1 *and* c_2 *depending only on the integer* $s=1$ *and* $[a,b]$*, such that*

$$c_1\omega_1(f;t)\leq K_1(f;t)\leq c_2\omega_1(f;t),$$

for all $f\in C[a,b]$ *and* $t>0$*.*

In general there no sharp constants known that satisfy the above inequality. There are two exceptional cases, namely for $s=1,2$. We will present them in the below results. The following lemma known as Brudnyĭ's representation theorem establishes the connection between $K_1(f;t)$ and the least concave majorant defined in (1.1).

Lemma 1.4. *Every function* $f\in C[a,b]$ *satisfies the equality*

$$K(f;t;C[a,b],C^1[a,b])=\frac{1}{2}\tilde{\omega}(f;2t),\ t\geq 0.$$

For more details concerning this lemma can be consult R. Păltănea's paper [140], B.S. Mitjagin and E.M. Semenov's paper [133], the book of R.T.Rockafellar [152] and the monograph of R.A. DeVore and G.G. Lorentz [59].

For the case $s=2$, H. Gonska [88] proved the following result that gives the connection between $K_2(f;t)$ and the moduli of smoothness.

Lemma 1.5. *Let* $f\in C[a,b]$ *and* $t\geq 0$*. Then we have*

$$\frac{1}{4}\omega_2(f;t)\leq K_2\left(f;\frac{t^2}{2};C[a,b],C^2[a,b]\right)$$

and

$$K_2\left(f;t^2;C[a,b],C^2[a,b]\right)\leq\left(\frac{3}{2}+2\max\left\{1,\frac{t^2}{(b-a)^2}\right\}\right)\omega_2(f;t).$$

1.3 The Ditzian-Totik first order modulus of smoothness

To describe our next results, we recall the definitions of the Ditzian-Totik first order modulus of smoothness and the K-functional [60]. Let $\varphi(x) := \sqrt{x(1-x)}$ and $f \in C[0,1]$. The first order modulus of smoothness is given by

$$\omega_\varphi(f;t) = \sup_{0<h\le t}\left\{\left|f\left(x+\frac{h\varphi(x)}{2}\right) - f\left(x-\frac{h\varphi(x)}{2}\right)\right|, x \pm \frac{h\varphi(x)}{2} \in [0,1]\right\}.$$

Further, the corresponding K-functional to (1.3) is defined by

$$K_\varphi(f;t) = \inf_{g\in W_\varphi[0,1]}\{||f-g|| + t||\varphi g'||\},\ t>0,$$

where $W_\varphi[0,1] = \{g : g \in AC_{loc}[0,1], \|\varphi g'\| < \infty\}$ and $g \in AC_{loc}[0,1]$ means that g is absolutely continuous on every interval $[a,b] \subset (0,1)$. It is well known [60, p. 11] that there exists a constant $C>0$ such that

$$K_\varphi(f;t) \le C\omega_\varphi(f;t). \tag{1.4}$$

1.4 Weighted spaces and corresponding modulus of continuity

Let $B_\rho(I)$ be the space of all functions f defined on the interval $I \subseteq \mathbb{R}$ for which there exist a constant $M>0$ such that $|f(x)| \le M\rho(x)$, for every $x \in I$, where ρ is a positive continuous function called weight. In 1974, A.D. Gadjiev [65, 66] introduced the weighted space $C_\rho(I)$, which is the set of all continuous functions f on the interval $I \subseteq \mathbb{R}$ and $f \in B_\rho(I)$. This space is a Banach space, endowed with the norm

$$\|f\|_\rho = \sup_{x\in I}\frac{|f(x)|}{\rho(x)}$$

For $I = [0,\infty)$, the subspace $C^*_\rho[0,\infty)$ is defined as follows:

$$C^*_\rho[0,\infty) := \left\{f \in C_\rho[0,\infty),\ \lim_{x\to\infty}\frac{f(x)}{\rho(x)} = k < +\infty\right\}.$$

Remark 1.1. *The polynomial weighted space $C_N[0,\infty)$ are obtained using weight*

$$\rho(x) = 1 + x^N, x \ge 0, N > 0.$$

Lopez-Moreno [122] introduced the modulus of continuity for the approximation on weighted spaces as follows:

$$\Omega(f;\delta) := \sup_{0<h\le\delta}\sup_{x\ge 0}\frac{|f(x+h)-f(x)|}{\rho(x+h)},$$

for $\rho(x) = 1 + x^m$, $m \in \mathbb{N}$ and $f \in C_\rho[0,\infty)$.

In [64], Gadjiev and Aral use the following modulus

$$\Omega_\rho(f;\delta) := \sup_{x,y\ge 0,\ |\rho(x)-\rho(y)|\le\delta}\frac{|f(x)-f(y)|}{[|\rho(x)-\rho(y)|+1]\rho(x)},$$

where $\rho \in C^1[0,\infty)$, $\rho(0) = 1$, $\inf_{x\ge 0}\rho'(x) > 0$ and $f \in C_\rho[0,\infty)$.

Many authors (see [2], [104], [107]) use the following modulus of continuity for weighted space

$$\Omega(f;\delta) := \sup_{x\ge 0,\ |h|\le\delta}\frac{|f(x+h)-f(x)|}{(1+h^2)(1+x^2)}. \tag{1.5}$$

1.5 Moduli of continuity and K-functional defined on metric space

Let $C(X)$ be the space of all real-valued and continuous functions defined on the compact metric space (X, d), with diameter $d(X) > 0$. In the following we will recal the definition for the (metric) modulus of continuity and its least concave majorant (see [89]).

Definition 1.4. *Let $f \in C(X)$. If, for $t \in [0, \infty)$, the quantity*

$$\omega_d(f; t) := \sup \{|f(x) - f(y)| \,; x, y \in X,\ d(x, y) \leq t\}$$

is the (metric) modulus of continuity, then its least concave majorant is given by

$$\widetilde{\omega}_d(f; t) = \begin{cases} \sup\limits_{0 \leq x \leq t \leq y \leq d(X), x \neq y} \frac{(t-x)\omega_d(f;y)+(y-t)\omega_d(f;x)}{y-x} & \text{for } 0 \leq t \leq d(X)\ , \\ \omega_d(f; d(X)) & \text{if } t > d(X)\ . \end{cases}$$

Denote

$$Lip_r = \left\{ g \in C(X) \,\middle|\, |g|_{Lip_r} := \sup_{d(x,y)>0} \frac{|g(x) - g(y)|}{d^r(x, y)} < \infty \right\},\ 0 < r \leq 1.$$

Lip_r is a dense subspace of $C(X)$ equipped with the supremum norm $\| \cdot \|$ and $| \cdot |_{Lip_r}$ is a seminorm on Lip_r.

The K-functional with respect to $(Lip_r, |\cdot|_{Lip_r})$ is given by

$$K(t; f; C(X), Lip_r) := \inf_{g \in Lip_r} \{\|f - g\| + t \cdot |g|_{Lip_r}\},$$

for $f \in C(X)$ and $t \geq 0$.

In the next result we give the relationship between the K-functional and the least concave majorant of the (metric) modulus of continuity (see [133]) .

Lemma 1.6. *Every continuous function f on X satisfies*

$$K\left(\frac{t}{2}; f; C(X), Lip_1\right) = \frac{1}{2} \cdot \widetilde{\omega}_d(f; t),\ 0 \leq t \leq d(X).$$

1.6 Positive linear operators

In this section we will give some basic definitions and some elementary properties regarding positive linear operators that will be considered in this thesis. For more details on this topic the reader should consult [162].

Definition 1.5. *Let X, Y be two linear spaces of real functions. Then the mapping $L : X \to Y$ is a linear operator if*

$$L(\alpha f + \beta g) = \alpha L(f) + \beta L(g),$$

for all $f, g \in X$ and $\alpha, \beta \in \mathbb{R}$. If for all $f \in X$, $f \geq 0$, it follows $Lf \geq 0$, then L is a positive operator.

Proposition 1.2. *Let $L : X \to Y$ be a positive linear operator. Then we have the following inequalities:*

i) If $f, g \in X$ with $f \leq g$, then $Lf \leq Lg$.

ii) For all $f \in X$, we have $|Lf| \leq L|f|$.

Definition 1.6. *Let $L : X \to Y$, where $X \subseteq Y$ are two linear normed spaces of real functions. To each operator L we assign a non-negative number $\|L\|$, given by*

$$\|L\| := \sup_{f \in X, \|f\|=1} \|Lf\| = \sup_{f \in X, 0<\|f\|\leq 1} \frac{\|Lf\|}{\|f\|}.$$

$\|\cdot\|$ is called the operator norm.

Corollary 1.1. *For $L : C[a,b] \to C[a,b]$ being positive and linear, it follows that L is also continuous and it holds:*

$$\|L\| = \|Le_0\|.$$

1.6.1 The Bernstein operators

The Bernstein operator was introduced by S.N. Bernstein in 1912 (see [45]) and it was used to prove the fundamental theorem of Weierstrass (see [173]).

The Bernstein operators are given by

$$B_n : C[0,1] \to C[0,1], \quad B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x), \tag{1.6}$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k},\ x \in [0,1].$$

Proposition 1.3. *The Bernstein operators satisfy the following properties:*

i) are linear and positive;

ii) $\sum_{k=0}^{n} p_{n,k}(x) = 1$;

iii) $B_n f$ interpolates at the endpoints of the interval, meaning $B_n(f;0) = f(0)$, $B_n(f;1) = f(1)$;

iv) $B_n(e_0;x) = 1$, $B_n(e_1;x) = x$ and $B_n(e_2;x) = x^2 + \frac{x(1-x)}{n}$.

1.6.2 The Durrmeyer operators

The Durrmeyer operators were considered for the first time by J.L. Durrmeyer [61] in 1967 and are defined by

$$M_n : L_1[0,1] \to C[0,1], \quad M_n(f;x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n,k}(t) f(t) dt.$$

Proposition 1.4. *The Durrmeyer operators satisfy the following properties:*

i) are linear and positive;

ii) $M_n(e_0;x) = 1,\ M_n(e_1;x) = x + \dfrac{1-2x}{n+2}$ *and* $M_n(e_2;x) = \dfrac{n(n-1)x^2+4nx+2}{(n+2)(n+3)}$*;*

iii) $M_n(\Pi_p) \subset \Pi_p$*, where* $p \le n$*.*

1.6.3 The genuine Bernstein-Durrmeyer operators

The Beta-type operators $\overline{\mathbb{B}}_n$ were introduced by A. Lupaş in his German thesis [126]. For $n = 1,2,3,\ldots$ and $f \in C[0,1]$ they are given by

$$\overline{\mathbb{B}}_n(f;x) := \begin{cases} f(0),\ x=0, \\ \dfrac{1}{B(nx,n-nx)}\displaystyle\int_0^1 t^{nx-1}(1-t)^{n-1-nx}f(t)dt,\ 0<x<1, \\ f(1),\ x=1, \end{cases}$$

where $B(\cdot,\cdot)$ is the Euler's Beta function.

The genuine Bernstein-Durrmeyer operators are introduced as a composition of Bernstein operators and Beta operators, namely $U_n = B_n \circ \overline{\mathbb{B}}_n$ (see [47], [90]). These are given in explicit form by

$$U_n(f;x) = (1-x)^n f(0) + x^n f(1) + (n-1)\sum_{k=1}^{n-1}\left(\int_0^1 f(t)p_{n-2,k-1}(t)dt\right)p_{n,k}(x), \quad f \in C[0,1].$$

These operators are defined as Bernstein operators at the end points and have a Durrmeyer-like construction inside of $[0,1]$.

Proposition 1.5. *The operators* U_n *have the following properties*

i) are linear and positive;

ii) $U_n(e_0;x) = 1,\ \ U_n(e_1;x) = x,\ \ U_n(e_2;x) = x^2 + \dfrac{2x(1-x)}{n+1},\ x \in [0,1]$*;*

iii) $\lim\limits_{n\to\infty} U_n(f;x) = f(x),\ f \in C[0,1],\ x \in [0,1]$.

1.6.4 The class of operators U_n^ρ

The class of operators U_n^ρ was introduced in [143] by Păltănea and further investigated by Păltănea and Gonska in [79] and [80].

Let $\rho > 0$ and $n \in \mathbb{N}$. The operators $U_n^\rho : C[0,1] \to \prod_n$ are defined by

$$\begin{aligned} U_n^\rho(f;x) &:= \sum_{k=0}^{n} F_k^\rho(f)p_{n,k}(x) \\ &:= \sum_{k=1}^{n-1}\left(\int_0^1 \frac{t^{k\rho-1}(1-t)^{(n-k)\rho-1}}{B(k\rho,(n-k)\rho)}f(t)dt\right)p_{n,k}(x) + f(0)(1-x)^n + f(1)x^n, \end{aligned}$$

for $f \in C[0,1]$, $x \in [0,1]$.

Remark 1.2. *For $\rho = 1$ and $f \in C[0,1]$, we obtain the genuine Bernstein-Durrmeyer operators, while for $\rho \to \infty$, for each $f \in C[0,1]$ the sequence $U_n^\rho(f;x)$ converge uniformly to the Bernstein polynomials $B_n(f;x)$.*

In [167] the following result for the images of the monomials under U_n^ρ is proved.

Theorem 1.2. *The images of the monomials under U_n^ρ can be written as*

$$U_n^\rho(e_m) = \frac{1}{(n\rho)_m} \sum_{l=0}^{m} c_{m-l}^{(m)} (n\rho)^l B_n(e_l) \tag{1.7}$$

where the coefficients $c_j^{(m)}$, $j = 0,1,...,m$ are given by the elementary symmetric sums:

$$\begin{aligned}
&c_0^{(m)} := 1, \quad c_m^{(m)} := 0,\\
&c_1^{(m)} = 1 + 2 + \cdots + (m-1) = \frac{m(m-1)}{2},\\
&c_2^{(m)} = 1\cdot 2 + 1\cdot 3 + \cdots + 1\cdot(m-1) + 2\cdot 3 + \cdots + (m-2)\cdot(m-1),\\
&\cdots\\
&c_{m-1}^{(m)} = 1\cdot 2\cdot 3\cdot \cdots \cdot (m-1) = (m-1)!.
\end{aligned} \tag{1.8}$$

1.6.5 The Kantorovich operators

In 1930 L.V. Kantorovich [116] introduced and studied the linear positive operators $K_n : L_1[0,1] \to C[0,1]$, defined by

$$K_n(f;x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt. \tag{1.9}$$

Proposition 1.6. *For all $n \in \mathbb{N}$ the following properties hold:*

i) K_n *are linear and positive;*

ii) $K_n(e_0;x) = 1,\ K_n(e_1;x) = \dfrac{n}{n+1}x + \dfrac{1}{2(n+1)}$,
$K_n(e_2;x) = \dfrac{1}{(n+1)^2}\left(n(n-1)x^2 + 2nx + \dfrac{1}{3}\right)$;

iii) For all $f \in L_1[0,1]$, *we have the representation* $K_n(f;x) = \dfrac{d}{dx}B_{n+1}(F;x)$, *where* $F(x) = \displaystyle\int_0^x f(t)dt$ *and* B_{n+1} *is the Bernstein operator of degree* $n+1$.

1.6.6 The Szász-Mirakjan operator

In the 1940s G.M. Mirakjan [132], J. Favard [62] and O. Szász [168] independently studied a sequence $(S_n)_{n\geq 1}$ of positive linear operators, that nowadays are called Szász-Mirakjan operators. These operators are defined by

$$S_n(f;x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \tag{1.10}$$

for $f \in C_2^*[0,\infty)$, $x \in [0,\infty) \subset \mathbb{R}$ and $n \in \mathbb{N}$, where $C_2^*[0,\infty)$ is the weighted space

$$C_2^*[0,\infty) := \left\{ f \in C[0,\infty) : \frac{f(x)}{1+x^2} \text{ is convergent as } x \to \infty \right\},$$

endowed with the norm $\|f\|_* := \sup\limits_{x \geq 0} \dfrac{|f(x)|}{1+x^2}$.

The series on the right-hand side of (1.10) is absolutely convergent (see [33]).

The moments of the operators S_n were obtained by Becker [44].

Lemma 1.7. *For $m \geq 1$ there holds*

$$S_n(e_m; x) = \sum_{j=1}^{m} a_{m,j} x^j n^{j-m} = x^m + \frac{m(m-1)}{2n} x^{m-1} + \cdots + n^{1-m} x$$

with positive coefficient $a_{m,j}$. In particular $S_n(e_m; x)$ is a polynomial of degree m without a constant term.

The coefficients $a_{m,j}$, $1 \leq j \leq m$ verify the following recurrence relations:

$$\begin{aligned}
&a_{m+1,m+1} = a_{m,m} = \cdots = a_{1,1} = 1,\\
&a_{m+1,1} = a_{m,1} = \cdots = a_{1,1} = 1,\\
&a_{m,m-1} = m(m-1)/2,\\
&a_{m+1,j} = j a_{m,j} + a_{m,j-1},\ m \geq 0,\ j \geq 1.
\end{aligned}$$

1.6.7 The Baskakov operator

The classical positive, linear Baskakov operators $(V_n)_{n \in \mathbb{N}}$ were introduced by Baskakov [38] and are defined as follows:

$$V_n(f; x) := \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}},$$

for every $f \in C_2^*[0,\infty)$ and $x \in [0,\infty)$.

Lemma 1.8. *The Baskakov operators satisfy the following properties:*

$$\begin{aligned}
V_n(e_1; x) &= x;\\
V_n(e_2; x) &= \frac{x}{n} + x^2 \frac{n+1}{n};\\
V_n(e_3; x) &= \frac{x}{n^2} + 3x^2 \frac{n+1}{n^2} + x^3 \frac{(n+1)(n+2)}{n^2};\\
V_n(e_4; x) &= \frac{x}{n^3} + 7x^2 \frac{n+1}{n^3} + 6x^3 \frac{(n+1)(n+2)}{n^3} + x^4 \frac{(n+1)(n+2)(n+3)}{n^3}.
\end{aligned}$$

For more details the reader should consult the book of F. Altomare and M. Campiti [33] and the paper of Becker [44].

2 Univariate Grüss-type inequalities for positive linear operators

In this chapter we discuss Grüss inequalities on spaces of continuous functions defined on a compact metric space. Using the least concave majorant of the modulus of continuity we will obtain a Grüss inequality for the functional $L(f) = H(f;x)$, where $H : C[a,b] \to C[a,b]$ is a positive linear operator and $x \in [a,b]$ is fixed. We will apply this inequality in the case of known operators, for example the Bernstein, Hermite-Fejér interpolation, convolution-type operators. Moreover, we derive inequalities of the Grüss-type using Cauchy's mean value theorem. A Grüss inequality on a compact metric space for more than two functions is given. All the results presented in this chapter were published in [24]. Also, we have continued the research in this field and new results concerning mathematical inequality were obtained ([14], [16], [37], [20], [25]).

2.1 Grüss-type inequalities for a positive linear functional

The original form of Grüss' inequality estimates the difference between the integral of a product of two functions and the product of integrals of the two functions and was published by G. Grüss in 1935 (see [92]):

Theorem A. *Let f and g be two functions defined and integrable on $[a,b]$. If $m \le f(x) \le M$ and $p \le g(x) \le P$ for all $x \in [a,b]$, then*

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{b-a} \int_a^b f(x)dx \cdot \frac{1}{b-a} \int_a^b g(x)dx \right| \le \frac{1}{4}(M-m)(P-p).$$

The constant 1/4 is the best possible.

Grüss' inequality attracted considerable interest after its publication. Here we mention only papers by E. Landau [120], J. Karamata [117], and a particularly useful one by A.M. Ostrowski [138]. We also note that a whole chapter in a book by D.S. Mitrinović et al. [134] is devoted to the inequality we discuss here.

Our present work is to a large extent motivated by a theorem which can be found in the paper [34] by D. Andrica and C. Badea. Here we cite a special form of it.

Theorem B. *Let $I = [a,b]$ be a compact interval of the real axis, $B(I)$ be the space of real-valued and bounded functions defined on I and $L : B(I) \to \mathbb{R}$ be a positive linear functional satisfying $L(e_0) = 1$ where $e_0 : I \ni x \mapsto 1$. Assuming that for $f, g \in B(I)$, i.e., $m \le f(x) \le M$, $p \le g(x) \le P$ for all $x \in I$, we have*

$$|L(fg) - L(f)L(g)| \le \frac{1}{4}(M-m)(P-p).$$

It is the aim of this section to look again at Grüss' inequality from a somewhat different point of view. In doing so we will be guided by the contribution of Andrica and Badea. That is: how non-multiplicative is a linear functional in the worst case? This is quite an intriguing question from the point of view of approximation theory.

In 2004, A. Mc. D. Mercer and Peter R. Mercer [129] give the following inequality for a positive, linear functional, $L : B(I) \to \mathbb{R}$, which $L(1) = 1$:

$$|L(fg) - L(f)L(g)| \leq \frac{1}{2} \min \left\{ (M-m)L(|g-G|), (P-p)L(|f-F|) \right\}, \tag{2.1}$$

where $m \leq f(x) \leq M$, $p \leq g(x) \leq P$ for all $x \in I$, $F := Lf$ and $G := Lg$.

In this section we will prove this kind of inequality on a compact metric space. Let $L : C(X) \to \mathbb{R}$ be a linear bounded functional, $L(1) = 1$, where $C(X)$ is a compact metric space with metric d. Then there are positive linear functionals L_+, L_-, $|L|$ such that $L = L_+ - L_-$ and $|L| = L_+ + L_-$. If L is a positive functional we have $|L| = L_+ = L$.

Since $M - m = \omega(f; d(X))$, $P - p = \omega(g; d(X))$, where $m = \inf f(x)$, $M = \sup f(x)$, $p = \inf g(x)$, $P = \sup g(x)$, we can prove using the idea of A. Mercer and P. Mercer's proof, the following inequality:

Theorem 2.1. *Let $L : C(X) \to \mathbb{R}$ be a linear, bounded functional, $L(1) = 1$, defined on the compact metric space $C(X)$. Then the inequality*

$$|L(fg) - L(f)L(g)| \leq \frac{1}{2} \min \left\{ \omega(f; d(X)) \cdot |L|(|g-G|), \omega(g; d(X)) \cdot |L|(|f-F|) \right\} \tag{2.2}$$

holds.

Remark 2.1. *The inequality is sharp in the sense that a non-positive functional A with $A(1) = 1$ exist such that equality occurs.*

Example 2.1. *Let us consider the following non-positive functional*

$$A : C[0,1] \to \mathbb{R}, \; A(f) = 2f(0) - f(1).$$

For this functional we have $A(1) = 1$, $A_+(f) = 2f(0)$, $A_-(f) = f(1)$, $|A|(f) = 2f(0) + f(1)$ and $A(fg) - A(f)A(g) = 2(f(1) - f(0))(g(0) - g(1))$. If we choose $f(t) = g(t) = t$, then $F = G = -1$ and

$$|A(fg) - A(f)A(g)| = 2 = \frac{1}{2} \min \left\{ \omega(f; 1) \cdot |A|(g+1), \omega(g, 1) \cdot |A|(f+1) \right\}.$$

Corollary 2.1. *If $L : C(X) \to \mathbb{R}$ is a positive and linear (and thus bounded) functional, then for all $f, g \in C(X)$ we have*

$$|L(fg) - L(g)L(g)| \leq \frac{1}{2} \min \left\{ \omega(f; d(X)) \cdot L(|g-G|), \omega(g; d(X)) \cdot L(|f-F|) \right\}, \tag{2.3}$$

$$|L(fg) - L(f)L(g)| \leq \frac{1}{4} \omega(f; d(X)) \omega(g; d(X)). \tag{2.4}$$

Proof. Since L is a positive functional it follows $|L| = L$; so the first inequality is proved.

In [129] A. Mercer and P. Mercer show that the inequalities

$$L(|g-G|) \leq \frac{1}{2}(P-p) \text{ and } L(|f-F|) \leq \frac{1}{2}(M-m) \tag{2.5}$$

hold. The inequality (2.4) can be obtained by using in (2.3) the inequalities (2.5). □

In [71], B. Gavrea and I. Gavrea raised the following

Problem. *Let L be a linear positive functional defined on C[0,1] with $L(1) = 1$ and f, g be two continuous functions. Do positive numbers $\delta_1 = \delta_1(f) < 1$ and $\delta_2 = \delta_2(f) < 1$ exist such that*

$$|L(fg) - L(f)L(g)| \leq \frac{1}{4}\tilde{\omega}(f;\delta_1)\tilde{\omega}(f;\delta_2)?$$

We will show that the answer to Gavreas' question is negative. Let us consider

$$L(f) = B_1\left(f;\frac{1}{2}\right) = \frac{1}{2}\left(f(0)+f(1)\right),\ f \in C[0,1],$$

where B_1 is the first Bernstein operator on $C[0,1]$.

If we choose $f(t) = g(t) = t$ we have

$$|L(fg) - L(f)L(g)| = |L(e_2) - L(e_1)^2| = \left|B_1\left(e_2;\frac{1}{2}\right) - B_1\left(e_1;\frac{1}{2}\right)^2\right| = \frac{1}{4}.$$

Moreover, for $0 \leq t \leq 1$, $\omega_1(e_1;t) = \tilde{\omega}(e_1;t) = t$, which follows

$$\frac{1}{4}\tilde{\omega}(f;t)\cdot\tilde{\omega}(g;t) = \frac{1}{4}\cdot t^2 < \frac{1}{4} \text{ for } 0 \leq t < 1.$$

Hence the conjecture of the two Gavreas is not true. □

One more question is if the upper bound in (2.4) has a corresponding lower bound, i.e., if there is a constant $c > 0$ such that for all $f, g \in C(X)$ we also have

$$c \cdot \omega(f;d(X)) \cdot \omega(g;d(X)) \leq |L(f\cdot g) - L(f)\cdot L(g)|. \tag{2.6}$$

The following example shows that this is not the case.

Example. Suppose $A : C(X) \to \mathbb{R}$ is a positive linear functional satisfying $A(1) = 1$. Write $D(f,g) := A(f\cdot g) - A(f)\cdot A(g)$.

Case 1: supp $A = \{x\}$ for $x \in X$. Then $A = \delta_x$, the point evaluation functional at x. Hence $D(f,g) = 0$ for all $f, g \in C(X)$, and for appropriate choices of X, f and g the l.h.s of (2.6) is non-zero.

Case 2: supp $A = \{x, y\}$, meaning that $A = \alpha\cdot\delta_x + \beta\cdot\delta_y$, where $\alpha, \beta > 0$ and $\alpha+\beta = 1$. Hence

$$D(f,g) = \alpha\cdot\beta(f(y)-f(x))(g(y)-g(x)) = 0$$

if and only if f or/and g is/are constant on supp A. Again for suitable choices of X, f and g the l.h.s. of (2.6) is non-zero.

Case 3: |supp $A| \geq 3$. Then there is an $h \in C(X)$ taking at least 3 distinct values on supp A. Let $a := A(h)$, $b := A(h^2)$, $c := A(h^3)$.

For all $t \in \mathbb{R}$ we have $(h - t \cdot 1)^2 \geq 0$, implying $A(h^2) - 2tA(h) + t^2 \geq 0$. Taking $t = A(h)$ shows that $A(h^2) \geq A^2(h)$. If $A(h^2) = A^2(h)$, then there is a $t_0 \in \mathbb{R}$ such that $A(h^2) - 2t_0 A(h) + t_0^2 = 0$, i.e., $A((h - t_0 1)^2) = 0$. This implies that $h - t_0 1$ is constant on supp A, which is a contradiction. Thus $A(h^2) - A^2(h) = b - a^2 > 0$. Let $f := h - a$, $g := h^2 + \frac{ab - c}{b - a^2} h$. Then $A(f) = 0$, $A(f \cdot g) = 0$, and so $D(f, g) = 0$. Clearly f is non-constant on supp A. Assuming that $g = d$ is constant on A, means $h^2 + \frac{ab - c}{b - a^2} h = d$, or $h^2 + \frac{ab - c}{b - a^2} h - d = 0$ on supp A. But this means that h attains at most two values on supp A, again a contradiction. Thus f and g are both non-constant on supp A and again the r.h.s. of (2.6) is non-zero.

□

In the following we will prove a Grüss inequality on a compact metric space for more than two functions.

Lemma 2.1. *Let $C(X)$ be a compact metric space and $f_k \in C(X)$, $1 \leq k \leq n$, $n \geq 1$. Then the following inequality*

$$\theta(f_1 f_2 \cdots f_n) \leq \sum_{i=1}^{n} \theta(f_i) \prod_{k=1, k \neq i}^{n} \|f_k\|, \tag{2.7}$$

where $\theta(f) := \max_X f - \min_X f$, $f \in C(X)$.

Proof. Inequality (2.7) can be proved using induction. □

Theorem 2.2. *Let $A : C(X) \to \mathbb{R}$ be a positive, linear functional, $A(1) = 1$, defined on the metric space $C(X)$. The inequality*

$$|A(f_1 f_2 \cdots f_n) - A(f_1)A(f_2) \cdots A(f_n)| \leq \frac{1}{4} \sum_{i,j=1, i<j}^{n} \theta(f_i)\theta(f_j) \prod_{k=1, k \neq i,j}^{n} \|f_k\| \tag{2.8}$$

holds.

Proof. The inequality (2.8) can be proved using induction and relation (2.7). □

Remark 2.2. *If $f_3, \ldots, f_n$ are constant, relation (2.8) reduces to*

$$|A(f_1 f_2) - A(f_1)A(f_2)| \leq \frac{1}{4}(M_1 - m_1)(M_2 - m_2),$$

where $M_i = \max_X f_i$, $m_i = \min_X f_i$, $i \in \{1, 2\}$.

2.2 Grüss-type inequalities for some positive linear operators

Let $H_n : C[a, b] \to C[a, b]$ be positive linear operators which reproduce constant functions. For $x \in [a, b]$ we consider $L = \varepsilon_x \circ H_n$, so $L(f) = H_n(f; x)$. Denote by

$$D(f, g) := H_n(fg; x) - H_n(f; x)H_n(g; x).$$

The following result suggests how non-multiplicative the functional $L(f) = H_n(f; x)$ is for a given $x \in [a, b]$.

Theorem 2.3. *If $f, g \in C[a,b]$ and $x \in [a,b]$ is fixed, then the inequality*

$$|D(f,g)| \leq \frac{1}{4}\tilde{\omega}\left(f; 2\sqrt{2H_n\left((e_1-x)^2;x\right)}\right) \cdot \tilde{\omega}\left(g; 2\sqrt{2H_n\left((e_1-x)^2;x\right)}\right)$$

holds.

Proof. Using the Cauchy-Schwarz inequality for positive linear functionals we can write

$$|H_n(f;x)| \leq H_n(|f|;x) \leq \sqrt{H_n(f^2;x) \cdot H_n(1;x)} = \sqrt{H_n(f^2;x)}, \text{ so}$$

$$D(f,f) = H_n(f^2;x) - H_n(f;x)^2 \geq 0.$$

Then D is a positive semidefinite bilinear form on $C[a,b]$. For $f, g \in C[a,b]$, using Cauchy-Schwarz for D, it follows that

$$|D(f,g)| \leq \sqrt{D(f,f)D(g,g)} \leq \|f\| \cdot \|g\|. \tag{2.9}$$

Since $H_n : C[a,b] \to C[a,b]$ is a positive linear operator which reproduces constant functions, $H_n(f;x)$, with $x \in [a,b]$ fixed, is a positive linear functional and can be represented as $H_n(f;x) = \int_a^b f(t)d\mu(t)$, where μ is a probability measure on $[a,b]$, i.e., $\int_a^b d\mu(t) = 1$.

We have

$$\begin{aligned}
&H_n(f^2;x) - H_n(f;x)^2 = \int_a^b f^2(t)d\mu(t) - \left(\int_a^b f(s)d\mu(s)\right)^2 \\
&= \int_a^b \left(f(t) - \int_a^b f(s)d\mu(s)\right)^2 d\mu(t) = \int_a^b \left(\int_a^b (f(t)-f(s))d\mu(s)\right)^2 d\mu(t) \\
&\leq \int_a^b \left(\int_a^b (f(t)-f(s))^2 d\mu(s)\right) d\mu(t) \leq \|f'\|^2 \int_a^b \left(\int_a^b (t-s)^2 d\mu(s)\right) d\mu(t) \\
&= \|f'\|^2 \int_a^b \left(t^2 - 2t\int_a^b s d\mu(s) + \int_a^b s^2 d\mu(s)\right) d\mu(t) \\
&= \|f'\|^2 \left[\int_a^b t^2 d\mu(t) - 2\int_a^b s d\mu(s) \int_a^b t d\mu(t) + \int_a^b s^2 d\mu(s)\right] \\
&= 2\|f'\|^2 \left[H_n(e_2;x) - H_n(e_1;x)^2\right] \leq 2\|f'\|^2 H_n\left((e_1-x)^2;x\right), \; f \in C^1[a,b].
\end{aligned}$$

Therefore

$$D(f,f) = H_n(f^2;x) - H_n(f;x)^2 \leq 2\|f'\|^2 \cdot H_n\left((e_1-x)^2;x\right). \tag{2.10}$$

Using relation (2.10) for differentiable functions $r, s \in C^1[a,b]$, we obtain the following estimate

$$|D(r,s)| \leq \sqrt{D(r,r)D(s,s)} \leq 2\|r'\|\|s'\| \cdot H_n\left((e_1-x)^2;x\right). \tag{2.11}$$

Moreover, if $f \in C[a,b]$, $s \in C^1[a,b]$, then

$$|D(f,s)| \leq \sqrt{D(f,f)D(s,s)} \leq \|f\| \cdot \sqrt{2} \cdot \|s'\| \cdot \sqrt{H_n\left((e_1-x)^2;x\right)}. \tag{2.12}$$

Likewise, for $r \in C^1[a,b]$, $g \in C[a,b]$, we have

$$|D(r,g)| \leq \|g\| \cdot \sqrt{2} \cdot \|r'\| \cdot \sqrt{H_n\left((e_1-x)^2;x\right)}. \tag{2.13}$$

Now let $f, g \in C[a,b]$ be fixed, $r, s \in C^1[a,b]$ arbitrary. Then

$$\begin{aligned}
&|D(f,g)| = |D(f-r+r, g-s+s)| \\
&\leq |D(f-r, g-s)| + |D(f-r, s)| + |D(r, g-s)| + |D(r,s)| \\
&\leq \|f-r\| \cdot \|g-s\| + \sqrt{2}\|f-r\| \cdot \|s'\| \sqrt{H_n\left((e_1-x)^2;x\right)} \\
&+\sqrt{2}\|g-s\| \cdot \|r'\| \sqrt{H_n\left((e_1-x)^2;x\right)} + 2 \cdot \|r'\| \cdot \|s'\| \cdot H_n\left((e_1-x)^2;x\right) \\
&= \left\{\|f-r\| + \|r'\|\sqrt{2H_n\left((e_1-x)^2;x\right)}\right\} \cdot \left\{\|g-s\| + \|s'\|\sqrt{2H_n\left((e_1-x)^2;x\right)}\right\}.
\end{aligned}$$

Passing to the infimum over r and $s \in C^1[a,b]$, respectively, shows

$$\begin{aligned}
&|D(f,g)| \leq K\left(\sqrt{2H_n\left((e_1-x)^2;x\right)}, f; C^0, C^1\right) \cdot K\left(\sqrt{2H_n\left((e_1-x)^2;x\right)}, g; C^0, C^1\right) \\
&= \frac{1}{2}\tilde{\omega}\left(f; \sqrt{8 \cdot H_n\left((e_1-x)^2;x\right)}\right) \cdot \frac{1}{2}\tilde{\omega}\left(g; \sqrt{8 \cdot H_n\left((e_1-x)^2;x\right)}\right) \\
&= \frac{1}{4}\tilde{\omega}\left(f; 2\sqrt{2 \cdot H_n\left((e_1-x)^2;x\right)}\right) \cdot \tilde{\omega}\left(g; 2\sqrt{2 \cdot H_n\left((e_1-x)^2;x\right)}\right),
\end{aligned}$$

which concludes the proof. $\square$

Remark 2.3. *If we choose $H_n = B_n$, the Bernstein operator, then this gives*

$$\begin{aligned}
&|B_n(fg;x) - B_n(f;x) \cdot B_n(g;x)| \\
&\leq \frac{1}{4}\tilde{\omega}\left(f; 2\sqrt{2B_n\left((e_1-x)^2;x\right)}\right) \cdot \tilde{\omega}\left(g; 2\sqrt{2B_n\left((e_1-x)^2;x\right)}\right) \\
&= \frac{1}{4}\tilde{\omega}\left(f; 2\sqrt{\frac{2x(1-x)}{n}}\right)\tilde{\omega}\left(g; 2\sqrt{\frac{2x(1-x)}{n}}\right) \\
&\leq \tilde{\omega}\left(f; \frac{1}{\sqrt{2n}}\right)\tilde{\omega}\left(g; \frac{1}{\sqrt{2n}}\right), \ f, g \in C[0,1].
\end{aligned}$$

In [75], Gonska et al. replaced the second moments $H((e_1-x)^2;x)$ by the smaller quantity $H(e_2;x) - H(e_1;x)^2$, proving that the above approch is not exactly the ideal choice.

Theorem 2.4. *(see [75, Theorem 3.1]) If $L : C[a,b] \to \mathbb{R}$ is a positive linear functional with $L(e_0) = 1$, then for $f, g \in C[a,b]$ we have*

$$|T(f,g)| \leq \frac{1}{4}\tilde{\omega}\left(f; 2\sqrt{T(e_1,e_2)}\right)\tilde{\omega}\left(g; 2\sqrt{T(e_1,e_2)}\right),$$

where

$$T(f,g) := L(f \cdot g) - L(f)L(g).$$

Moreover,

$$T\left(\frac{e_1-a}{b-a}, \frac{e_1-a}{b-a}\right) \leq \frac{1}{4},$$

with equality holding if and only if $L = \frac{1}{2}(\epsilon_a + \epsilon_b)$, where $\epsilon_x(f) = f(x)$, $x \in \{a,b\}$.

Corollary 2.2. *(see [75, Corollary 5.1]) If $H_n : C[a,b] \to C[a,b]$ is a positive linear operator which reproduces constant functions, then for $f, g \in C[a,b]$ and $x \in [a,b]$ fixed we have the inequalities:*

$$\begin{aligned}|D(f,g)| &= |H_n(f \cdot g; x) - H_n(f;x)H_n(g;x)| \\ &\le \frac{1}{4}\tilde{\omega}\left(f; 2\sqrt{H_n(e_2;x) - H_n(e_1;x)^2}\right)\tilde{\omega}\left(g; 2\sqrt{H_n(e_2;x) - H_n(e_1;x)^2}\right) \\ &\le \frac{1}{4}\tilde{\omega}\left(f; 2\sqrt{H_n\left((e_1-x)^2;x\right)}\right)\cdot\tilde{\omega}\left(g; 2\sqrt{H_n\left((e_1-x)^2;x\right)}\right).\end{aligned}$$

In [78], Gonska and Tachev used second order moduli of smoothness instead of the least concave majorant of the first order modulus of continuity and showed in the case of the classical Bernstein operators that in certain cases this leads to better results than those obtained earlier.

The result concerning the non-multiplicativity of positive linear operators reproducing linear function was remarkably generalized by Rusu [154] replacing $([a,b], |\cdot|)$ by a compact metric space (X,d), $H_n((e_1-x)^2;x)$ by $H_n(d^2(\cdot,x);x)$, and $K(\cdot, f; C[a,b], C^1[a,b])$ by $K(\cdot, f; C(X), Lip1)$.

Theorem 2.5. *(see [154, Theorem 3.1]) If $f, g \in C(X)$, where (X,d) is a compact metric space, and $x \in X$, then the inequality*

$$|D(f,g)| \le \frac{1}{4}\widetilde{\omega}_d\left(f; 4\sqrt{H_n(d^2(\cdot,x);x)}\right)\cdot\widetilde{\omega}_d\left(g; 4\sqrt{H_n(d^2(\cdot,x);x)}\right) \tag{2.14}$$

holds, where $H_n(d^2(\cdot,x);x)$ is the second moment of the operator H_n.

2.3 Applications

2.3.1 The Hermite-Fejér interpolation operator

The classical Hermite-Fejér interpolation operator is a positive linear operator and can be written as

$$L_n(f;x) = \sum_{k=1}^{n} f(x_k)(1 - x\cdot x_k)\cdot\left(\frac{T_n(x)}{n(x-x_k)}\right)^2, \tag{2.15}$$

where $f \in C[-1,1]$ and $x_k = \cos\frac{2k-1}{2n}\pi,\ 1 \le k \le n$ are the zeros of $T_n(x) = \cos(n \cdot \arccos)$, the n-th Chebyshev polynomial of the first kind.

For this operator we have $L_n\left((e_1-x)^2;x\right) = \frac{1}{n}\cdot T_n^2(x)$.

Remark 2.4. *If we choose in Theorem 2.3 $H_n = L_n$, the classical Hermite-Fejér interpolation operator, then this gives*

$$|L_n(fg;x) - L_n(f;x)\cdot L_n(g;x)| \le \frac{1}{4}\tilde{\omega}\left(f; \frac{2\sqrt{2}}{\sqrt{n}}|T_n(x)|\right)\cdot\tilde{\omega}\left(g; \frac{2\sqrt{2}}{\sqrt{n}}|T_n(x)|\right). \tag{2.16}$$

This is disappointing in view of the fact that L_n approximates much faster than B_n. Indeed, in [89] the following pointwise inequality was proved:

$$|L_n(f;x) - f(x)| \le 5\cdot\omega_1\left(f; \frac{|T_n(x)|}{n}\left\{\sqrt{1-x^2}\cdot\ln n + 1\right\}\right).$$

In this section we will give a different approach adapted to the Hermite-Fejér case. Denote by

$$D(f,g) := L_n(fg;x) - L_n(f;x)L_n(g;x).$$

Theorem 2.6. *If $f,g \in C[-1,1]$ and $x \in [-1,1]$ is fixed, then the following inequality is verified*

$$|D(f,g)| \leq \frac{1}{2}\min\left\{\|f\|\tilde{\omega}\left(g;\frac{40\cdot\ln n}{n}\right),\|g\|\tilde{\omega}\left(f;\frac{40\cdot\ln n}{n}\right)\right\}. \tag{2.17}$$

Proof. For $f \in C[-1,1]$, $s \in C^1[-1,1]$ proceed as follows:

$$\begin{aligned}
|D(f,s)| &= |L_n(f\cdot s;x) - L_n(f;x)\cdot L_n(s;x)| = |L_n(f(s-L_n(s;x));x)| \\
&= |L_n^t(f(t)(s(t)-s(x)+s(x)-L_n(s;x));x)| \\
&\leq \|f\|L_n^t(|s(t)-s(x)|+|s(x)-L_n(s;x)|;x) \\
&\leq \|f\|\cdot L_n(\|s'\|\cdot|e_1-x|+\|s'\|\cdot L_n(|e_1-x|;x);x) \\
&= 2\cdot\|f\|\cdot\|s'\|\cdot L_n(|e_1-x|;x).
\end{aligned}$$

Now, for $f,g \in C[-1,1]$ fixed and $s \in C^1[-1,1]$ arbitrary we get

$$\begin{aligned}
|D(f,g)| &= |D(f,g-s+s)| \leq |D(f,g-s)| + |D(f,s)| \\
&\leq \|f\|\cdot\|g-s\| + 2\|f\|\cdot\|s'\|\cdot L_n(|e_1-x|;x) \\
&= \|f\|\cdot\left\{\|g-s\| + 2\cdot L_n(|e_1-x|;x)\cdot\|s'\|\right\}.
\end{aligned}$$

Passing to the infimum over $s \in C^1[-1,1]$ yields

$$\begin{aligned}
|D(f,g)| &\leq \|f\|\cdot K(2\cdot L_n(|e_1-x|;x),g;C^0,C^1) \\
&= \|f\|\cdot\frac{1}{2}\cdot\tilde{\omega}(g,4\cdot L_n(|e_1-x|;x)).
\end{aligned}$$

By symmetry the same holds with f and g interchanged. Hence

$$|D(f,g)| \leq \frac{1}{2}\cdot\min\left\{\|f\|\cdot\tilde{\omega}(g,4\cdot L_n(|e_1-x|;x));\|g\|\cdot\tilde{\omega}(f,4\cdot L_n(|e_1-x|;x))\right\}.$$

But in [89] it was proved that

$$L_n(|e_1-x|;x) \leq \frac{4}{n}\cdot|T_n(x)|\cdot(\sqrt{1-x^2}\cdot\ln n+1) \leq 10\cdot\frac{\ln n}{n},\ n \geq 2 \tag{2.18}$$

and so

$$|D(f,g)| \leq \frac{1}{2}\min\left\{\|f\|\tilde{\omega}\left(g;\frac{40\cdot\ln n}{n}\right),\|g\|\tilde{\omega}\left(f;\frac{40\cdot\ln n}{n}\right)\right\}.$$

Remark 2.5. *If one of the function f or g is in $Lip1$, we have $|D(f,g)| = \mathcal{O}\left(\frac{\ln n}{n}\right)$, $n \to \infty$. The relation (2.16) implies in this case only $|D(f,g)| = o\left(\frac{1}{\sqrt{n}}\right)$. Also, the relation (2.16) implies $|D(f,g)| = \mathcal{O}\left(\frac{1}{n}\right)$ for $f,g \in Lip1$. This cannot be concluded from (2.17).*

2.3.2 The convolution-type operator

Definition 2.1. *For every function $f \in C[-1,1]$, and any natural number n, the operator $G_{m(n)}$ is defined by*

$$G_{m(n)}(f,t) := \pi^{-1} \cdot \int_{-\pi}^{\pi} f(\cos(\arccos t + v)) \cdot K_{m(n)}(v)dv,$$

where the kernel $K_{m(n)}$ is a trigonometric polynomial of degree $m(n)$ with the following properties:
(i) $K_{m(n)}$ is positive and even;
(ii) $\int_{-\pi}^{\pi} K_{m(n)}(v)dv = \pi$, i.e., $G_{m(n)}(1,t) = 1$ for $t \in [-1,1]$.

For each $f \in C[-1,1]$ the integral $G_{m(n)}(f,\cdot)$ from Definition 2.1 is an algebraic polynomial of degree $m(n)$. Moreover, in view of (i) and (ii) one has

$$K_{m(n)}(v) = \frac{1}{2} + \sum_{k=1}^{m(n)} \rho_{k,m(n)} \cdot \cos kv, \ v \in [-\pi,\pi].$$

Lemma 2.2. *[121] For $x \in I$ the inequality*

$$G_{m(n)}\left((e_1 - x)^2, x\right) = x^2 \left\{ \frac{3}{2} - 2\rho_{1,m(n)} + \frac{1}{2}\rho_{2,m(n)} \right\} + (1-x^2)\left\{ \frac{1}{2} - \frac{1}{2}\rho_{2,m(n)} \right\}$$

holds. Here e_1 denotes the first monomial given by $e_1(t) = t$ for $|t| \le 1$.

If $K_{m(n)}$ is the Fejér-Korovkin kernel with $m(n) = n-1$, then it is known that (see [128])

$$\rho_{1,n-1} = \cos\frac{\pi}{n+1}, \ \rho_{2,n-1} = \frac{n}{n+1}\cos\frac{2\pi}{n+1} + \frac{1}{n+1}. \tag{2.19}$$

Using the relations (2.19) we get

$$\begin{aligned} G_{n-1}\left((e_1-x)^2;x)\right) &\le \left| \frac{3}{2} - 2\rho_{1,n-1} + \frac{1}{2}\rho_{2,n-1} \right| + \frac{1}{2}\left|1 - \rho_{2,n-1}\right| \\ &\le 3\left(\frac{\pi}{n+1}\right)^2 + \left(\frac{\pi}{n+1}\right)^2 = 4 \cdot \left(\frac{\pi}{n+1}\right)^2. \end{aligned}$$

Remark 2.6. *If we consider in Theorem 2.3 the convolution-type operators with the Fejér-Korovkin kernel we have*

$$\begin{aligned} |D(f;g)| &= |G_{n-1}(fg;x) - G_{n-1}(f;x) \cdot G_{n-1}(g;x)| \\ &\le \frac{1}{4} \cdot \tilde{\omega}\left(f; 4\sqrt{2}\frac{\pi}{n+1}\right) \cdot \tilde{\omega}\left(g; 4\sqrt{2}\frac{\pi}{n+1}\right) = \mathcal{O}\left(\tilde{\omega}\left(f;\frac{1}{n}\right) \cdot \tilde{\omega}\left(g;\frac{1}{n}\right)\right). \end{aligned}$$

This is an improvement of what we obtained for the Bernstein and Hermite-Fejér operators.

2.4 Estimates via Cauchy's mean value theorem

Let $L : C[a,b] \to \mathbb{R}$ be a linear positive functional. We denote by

$$T(f,g) = L(fg) - L(f) \cdot L(g), \ f,g \in C[a,b].$$

In this section our aim is to study non-multiplicativity for the functional L using Cauchy's mean value theorem. Our work is motivated by B.G. Pachpatte's result obtained in [139] for the functional $L(f) = \dfrac{1}{\int_a^b w(x)dx}\displaystyle\int_a^b w(x)f(x)dx$, where $w : [a, b] \to [0, \infty)$ is an integrable function such that $\displaystyle\int_a^b w(x)dx > 0$.

Theorem 2.7. *If $L : C[a, b] \to \mathbb{R}$ is a linear positive functional, with $L(1) = 1$, then*
i) there is $(\eta, \theta) \in [a, b] \times [a, b]$ such that $T(f, g) = \dfrac{f'(\eta)}{h'(\eta)} \cdot \dfrac{g'(\theta)}{h'(\theta)} \cdot T(h, h)$.
ii) $|T(f, g)| \le \left\| \dfrac{f'}{h'} \right\| \cdot \left\| \dfrac{g'}{h'} \right\| \cdot |T(h, h)|$, where $f, g, h \in C^1[a, b]$ and $h'(t) \neq 0$ for each $t \in [a, b]$.

Proof. Let $x, y \in [a, b]$ with $y \neq x$. Applying Cauchy's mean value theorem, there exist points ξ_1 and ξ_2 between y and x such that

$$f(x) - f(y) = \frac{f'(\xi_1)}{h'(\xi_1)}(h(x) - h(y)), \tag{2.20}$$

$$g(x) - g(y) = \frac{g'(\xi_2)}{h'(\xi_2)}(h(x) - h(y)). \tag{2.21}$$

Multiplying the left sides and right sides of (2.20) and (2.21), we get

$$(f(x) - f(y))(g(x) - g(y)) = \frac{f'(\xi_1)}{h'(\xi_1)} \cdot \frac{g'(\xi_2)}{h'(\xi_2)} \cdot (h(x) - h(y))^2.$$

If we apply the functional L with respect to x and y it follows

$$2T(f, g) = L_y L_x \left(\frac{f'(\xi_1)}{h'(\xi_1)} \cdot \frac{g'(\xi_2)}{h'(\xi_2)} \cdot (h(x) - h(y))^2 \right). \tag{2.22}$$

If denote by

$$m = \min_{(x,y)\in[a,b]\times[a,b]} \frac{f'(x)}{h'(x)} \cdot \frac{g'(y)}{h'(y)},$$
$$M = \max_{(x,y)\in[a,b]\times[a,b]} \frac{f'(x)}{h'(x)} \cdot \frac{g'(y)}{h'(y)},$$

then we can write $m \le \dfrac{f'(\xi_1)}{h'(\xi_1)} \cdot \dfrac{g'(\xi_2)}{h'(\xi_2)} \le M$, namely

$$m(h(x) - h(y))^2 \le \frac{f'(\xi_1)}{h'(\xi_1)} \cdot \frac{g'(\xi_2)}{h'(\xi_2)}(h(x) - h(y))^2 \le M(h(x) - h(y))^2.$$

If apply the functional L with respect to x and y, we get

$$2mT(h, h) \le L_y L_x \left(\frac{f'(\xi_1)}{h'(\xi_1)} \cdot \frac{g'(\xi_2)}{h'(\xi_2)}(h(x) - h(y))^2 \right) \le 2MT(h, h).$$

Since

$$m \le \frac{L_y L_x \left(\frac{f'(\xi_1)}{h'(\xi_1)} \cdot \frac{g'(\xi_2)}{h'(\xi_2)}(h(x) - h(y))^2 \right)}{2T(h, h)} \le M,$$

it follows that there is $(\eta,\theta)\in[a,b]\times[a,b]$ such that

$$\frac{L_yL_x\left(\frac{f'(\xi_1)}{h'(\xi_1)}\cdot\frac{g'(\xi_2)}{h'(\xi_2)}(h(x)-h(y))^2\right)}{2T(h,h)}=\frac{f'(\eta)}{h'(\eta)}\cdot\frac{g'(\theta)}{h'(\theta)}.$$

Using the above relation in (2.22), it follows

$$T(f,g)=\frac{f'(\eta)}{h'(\eta)}\cdot\frac{g'(\theta)}{h'(\theta)}\cdot T(h,h). \tag{2.23}$$

From (2.23) we have

$$|T(f,g)|\leq\left\|\frac{f'}{h'}\right\|\cdot\left\|\frac{g'}{h'}\right\|\cdot|T(h,h)|.$$

□

Remark 2.7. *If in Theorem 2.7 we consider $h(x)=x,\ x\in[a,b]$, and $L(f)=\frac{1}{b-a}\int_a^b f(x)dx$, then*
i) there is $(\eta,\theta)\in[a,b]\times[a,b]$ such that

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{(b-a)^2}\int_a^b f(x)dx\cdot\int_a^b g(x)dx=\frac{(b-a)^2}{12}f'(\eta)\cdot g'(\theta).$$

This identity was found by Ostrowski [138] in 1970.
ii) $\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx\right|\leq\frac{(b-a)^2}{12}\sup_{x\in[a,b]}|f'(x)|\sup_{x\in[a,b]}|g'(x)|.$
This inequality was proved by Čebyšev [55] in 1882.

Theorem 2.8. *If $L:C[a,b]\to\mathbb{R}$ is a linear positive functional, with $L(1)=1$, then the following inequality is verified:*

$$|T(f,h)+T(g,h)|\leq|T(h,h)|\cdot\left(\left\|\frac{f'}{h'}\right\|+\left\|\frac{g'}{h'}\right\|\right),$$

where $f,g,h\in C^1[a,b]$ and $h'(t)\neq0$ for each $t\in[a,b]$.

Proof. Multiplying both sides of (2.20) and (2.21) by $h(x)-h(y)$ and adding the resulting identities we get

$$\begin{aligned}&(f(x)-f(y))(h(x)-h(y))+(g(x)-g(y))(h(x)-h(y))\\&=\frac{f'(\xi_1)}{h'(\xi_1)}(h(x)-h(y))^2+\frac{g'(\xi_2)}{h'(\xi_2)}(h(x)-h(y))^2.\end{aligned}$$

If apply the functional L with respect to x and y, we get

$$\begin{aligned}2T(f,h)+2T(g,h)&=L_yL_x\left(\frac{f'(\xi_1)}{h'(\xi_1)}(h(x)-h(y))^2\right)\\&+L_yL_x\left(\frac{g'(\xi_2)}{h'(\xi_2)}(h(x)-h(y))^2\right).\end{aligned} \tag{2.24}$$

In a similar way with the proof of Theorem 2.7 it can be shown that there are $\eta, \theta \in [a,b]$ such that

$$L_y L_x \left(\frac{f'(\xi_1)}{h'(\xi_1)} (h(x)-h(y))^2 \right) = 2T(h,h) \cdot \frac{f'(\eta)}{h'(\eta)},$$
$$L_y L_x \left(\frac{g'(\xi_2)}{h'(\xi_2)} (h(x)-h(y))^2 \right) = 2T(h,h) \cdot \frac{g'(\theta)}{h'(\theta)}.$$

Using the above identities in (2.24), we get

$$T(f,h)+T(g,h) = \left(\frac{f'(\eta)}{h'(\eta)} + \frac{g'(\theta)}{h'(\theta)} \right) T(h,h).$$

Therefore

$$|T(f,h)+T(g,h)| \leq \left(\left\| \frac{f'}{h'} \right\| + \left\| \frac{g'}{h'} \right\| \right) |T(h,h)|.$$

□

In the paper [138] Ostrowski defined the concept of synchronous functions. The functions $f, g : [a,b] \to \mathbb{R}$ are called synchronous, if we have, for any couple of points x, y from $[a,b]$, $f(x) \geq f(y)$ if and only if $g(x) \geq g(y)$.

In the case that f, g are synchronous, we get $T(f,g) \geq 0$.

Theorem 2.9. *If $L : C[a,b] \to \mathbb{R}$ is a linear positive functional, with $L(1) = 1$, then the following inequality is verified:*

$$|T(f,g)| \leq \frac{1}{2} \left[\left\| \frac{f'}{h'} \right\| |T(g,h)| + \left\| \frac{g'}{h'} \right\| |T(f,h)| \right], \tag{2.25}$$

where $f, g, h \in C^1[a,b]$, $h'(t) \neq 0$ for each $t \in [a,b]$ and the functions f, g, respectively g, h are synchronous.

Proof. Multiplying both sides of (2.20) and (2.21) by $g(x)-g(y)$ and $f(x)-f(y)$, respectively, adding the resulting identities, and applying the functionals L with respect to x and y, we get

$$\begin{aligned} 4T(f,g) &= L_y L_x \left(\frac{f'(\xi_1)}{h'(\xi_1)} (h(x)-h(y))(g(x)-g(y)) \right) \\ &+ L_y L_x \left(\frac{g'(\xi_2)}{h'(\xi_2)} (h(x)-h(y))(f(x)-f(y)) \right) \end{aligned}$$

Using this identity and the reasoning from the proof of the above theorems inequality (2.25) follows. □

3 Bivariate Grüss-type inequalities for positive linear operators

3.1 The composite bivariate Bernstein operators

In this chapter we consider a sequence of composite bivariate Bernstein operators and the cubature formula associated with them. The upper-bounds for the remainder term of cubature formula are described in terms of moduli of continuity of order two. This is motivated by a recent series of articles by Barbosu et al. (see [40]-[43]).

In the last years we published some results related to quadrature formulas ([21], [22], [23], [26], [27], [28], [29], [37]). Our contribution in this chapter is a continuation of [74]. Historically the origin of the method discussed seems to be in the article [165] by D.D. Stancu and A. Vernescu. The results presented in this chapter were published in [11].

We first introduce some notation which will be needed to formulate the general result.

Definition 3.1. *Let I and J be compact intervals of the real axis and let $L : C(I) \to C(I)$ and $M : C(J) \to C(J)$ be discretely defined operators, i.e.,*

$$L(g; x) = \sum_{e \in E} g(x_e) A_e(x),\ g \in C(I), x \in I,$$

where E is a finite index set, the $x_e \in I$ are mutually distinct and $A_e \in C(I)$, $e \in E$.

Analogously,

$$M(h; y) = \sum_{f \in F} h(y_f) B_f(y),\ h \in C(J), y \in J.$$

If L is of the form above, then its parametric extension to $C(I \times J)$ is given by

$$_xL(F; x, y) = L(F_y; x) = \sum_{e \in E} F_y(x_e) A_e(x) = \sum_{e \in E} F(x_e, y) A_e(x).$$

Here $F_y, y \in J,$ denote the partial functions of F given by $F_y(x) = F(x, y), x \in I$.

Similarly,

$$_yM(F; x, y) = \sum_{f \in F} F(x, y_f) B_f(y).$$

The tensor product of L and M (or M and L) is given by

$$\left(_xL \circ_y M\right)(F; x, y) = \sum_{e \in E} \sum_{f \in F} F(x_e, y_f) A_e(x) B_f(y).$$

The theorem below is given in terms of so-called partial moduli of smoothness of order r, given for the compact intervals $I, J \subset \mathbb{R}$, for $F \in C(I \times J)$, $r \in \mathbb{N}_0$ and $\delta \in \mathbb{R}_+$ by

$$\omega_r(F;\delta,0) := \sup\left\{\left|\sum_{\nu=0}^{r}(-1)^{r-\nu}\binom{r}{\nu}F(x+\nu h,y)\right| : (x,y),(x+rh,y)\in I\times J, |h|\le\delta\right\}$$

and symmetrically by

$$\omega_r(F;0,\delta) := \sup\left\{\left|\sum_{\nu=0}^{r}(-1)^{r-\nu}\binom{r}{\nu}F(x,y+\nu h)\right| : (x,y),(x,y+rh)\in I\times J, |h|\le\delta\right\}.$$

The total modulus of smoothness of order r is defined by

$$\omega_r(F;\delta_1,\delta_2) := \sup\left\{\left|\sum_{\nu=0}^{r}(-1)^{r-\nu}\binom{r}{\nu}F(x+\nu h_1,y+\nu h_2)\right| : \right.$$
$$\left.(x,y),(x+rh_1,y+rh_2)\in I\times J, |h_1|\le\delta_1,\ |h_2|\le\delta_2\right\}.$$

We now formulate and prove a simplified form of Theorem 37 in [47].

Theorem 3.1. *Let L and M be discretely defined operators as given above such that*

$$|(g-Lg)(x)| \le \sum_{\rho=0}^{r}\Gamma_{\rho,L}(x)\omega_\rho(g;\Lambda_{\rho,L}(x)),\ g\in C(I), x\in I,$$

and

$$|(h-Mh)(y)| \le \sum_{\sigma=0}^{s}\Gamma_{\sigma,M}(y)\omega_\sigma(h;\Lambda_{\sigma,M}(y)),\ h\in C(J), y\in J.$$

Here $\omega_\rho, \rho = 0,\ldots,r$, denote the moduli of order ρ, and Γ and Λ are bounded functions. Analogously for M. Then for $(x,y)\in I\times J$ and $F\in C(I\times J)$ the following hold:

$$\begin{aligned}|[F-({}_xL\circ_y M)F](x,y)| &\le \sum_{\rho=0}^{r}\Gamma_{\rho,L}(x)\omega_\rho(F;\Lambda_{\rho,L}(x),0)\\ &\quad + \|L\|\sum_{\sigma=0}^{s}\Gamma_{\sigma,M}(y)\omega_\sigma(F;0,\Lambda_{\sigma,M}(y)),\end{aligned}$$

where $\|L\|$ denotes the operator norm of L, which is finite due to the form of L.

Proof. We have

$$\begin{aligned}|\,[F-({}_xL\circ_y M)F]\,(x,y)| &= |\,[(Id -_x L) +_x L\circ(Id -_y M)]\,(F;x,y)|\\ &\le |(Id -_x L)(F;x,y)| + |_xL\circ(Id -_y M)(F;x,y)|\\ &=: E_1(x,y)+E_2(x,y).\end{aligned}$$

Now, for $x\in I$,

$$\begin{aligned}E_1(x,y) = |(Id-L)(F_y;x)| &\le \sum_{\rho=0}^{r}\Gamma_{\rho,L}(x)\cdot\omega_\rho\left(F_y;\Lambda_{\rho,L}(x)\right)\\ &\le \sum_{\rho=0}^{r}\Gamma_{\rho,L}(x)\cdot\omega_\rho\left(F;\Lambda_{\rho,L}(x),0\right).\end{aligned}$$

Furthermore, with $G := (Id -_y M)F$, we have

$$E_2(x,y) = |_x L(G;x,y)| = |L(G_y;x)| \le \|L(G_y)\|_{x\in I}.$$

Here again $G_y \in C(I)$ for all $y \in J$. By our assumption on L we have for any $g \in C(I)$ that

$$\|Lg\| \le \left(1 + \sum_{\rho=0}^{r} 2^\rho \cdot \|\Gamma_{\rho,L}\|\right) \cdot \|g\|.$$

Hence $\|L\| < \infty$.

In the situation at hand we have

$$\begin{aligned}
\|G_y\| &= \| \left[(Id -_y M)F\right]_y (\cdot)\| = \|(Id -_y M)F(\cdot,y)\| = \|(Id -_y M)F_x(y)\|_{x\in I} \\
&\le \| \sum_{\sigma=0}^{s} \Gamma_{\sigma,M}(y) \cdot \omega_\sigma\left(F_x; \Lambda_{\sigma,M}(y)\right) \| \le \sum_{\sigma=0}^{s} \Gamma_{\sigma,M}(y) \cdot \sup_{x\in I} \omega_\sigma\left(F_x; \Lambda_{\sigma,M}(y)\right) \\
&= \sum_{\sigma=0}^{s} \Gamma_{\sigma,M}(y) \cdot \omega_\sigma\left(F; 0, \Lambda_{\sigma,M}(y)\right).
\end{aligned}$$

Hence

$$\begin{aligned}
E_1(x,y) + E_2(x,y) &\le \sum_{\rho=0}^{r} \Gamma_{\rho,L}(x) \cdot \omega_\rho(F; \Lambda_{\rho,L}(x), 0) \\
&\quad + \|L\| \cdot \sum_{\sigma=0}^{s} \Gamma_{\sigma,M}(y) \cdot \omega_\sigma(F; 0, \Lambda_{\sigma,M}(y)).
\end{aligned}$$

□

3.1.1 The bivariate Bernstein operators

In this section we will apply the above results in the case of known bivariate Bernstein operators.

Example 3.1. *If we take $L = B_{n_1}$ and $M = B_{n_2}$ with two classical Bernstein operators mapping $C[0,1]$ into $C[0,1]$, then for $F \in C([0,1]\times[0,1])$ and $(x,y) \in [0,1]\times[0,1]$*

$$\left({}_x B_{n_1} \circ_y B_{n_2}\right)(F;x,y) = \sum_{i_1=0}^{n_1}\sum_{i_2=0}^{n_2} F\left(\frac{i_1}{n_1}, \frac{i_2}{n_2}\right) p_{n_1,i_1}(x) p_{n_2,i_2}(y),$$

where $p_{n,i}(x) = \binom{i}{n} x^i (1-x)^{n-i}, x \in [0,1],$ and

$$\begin{aligned}
&|[F - \left({}_x B_{n_1} \circ_y B_{n_2}\right) F](x,y)| \le \frac{3}{2}\left[\omega_2\left(F; \sqrt{\frac{x(1-x)}{n_1}}, 0\right) + \omega_2\left(F; 0, \sqrt{\frac{y(1-y)}{n_2}}\right)\right] \\
&\le \frac{3}{2}\left[\|F^{(2,0)}\| \frac{x(1-x)}{n_1} + \|F^{(0,2)}\| \frac{y(1-y)}{n_2}\right], F \in C^{2,2}([0,1]\times[0,1]).
\end{aligned}$$

Proof. We apply Theorem 3.1 with $r = s = 2$, $\Gamma_{0,B_n} = \Gamma_{1,B_n} = 0$, $\Gamma_{2,B_n} = \frac{3}{2}$, $\Lambda_{2,B_n}(z) = \sqrt{\frac{z(1-z)}{n}}$, for $n \in \{n_1, n_2\}$. The latter two choices are possible due to a well-known result of Păltănea (see [141]) showing that for the univariate Bernstein operators one has

$$|f(x) - B_n(f,x)| \le \frac{3}{2}\omega_2\left(f, \sqrt{\frac{x(1-x)}{n}}\right).$$

□

Remark 3.1. *From the last inequality we get*

$$|f(x) - B_n(f;x)| \leq \frac{3}{2}\|f''\|\frac{x(1-x)}{n},\ f \in C^2[0,1].$$

This is worse than the known inequality

$$|f(x) - B_n(f;x)| \leq \frac{1}{2}\|f''\|\frac{x(1-x)}{n}.$$

Our inequality was obtained from the more general statement in terms of ω_2 and well-known properties of the modulus.

However, we can use instead Theorem 1 in [46] (take $p = q = 2$, $p' = q' = 0$, $r = s = 0$, $\Gamma_{0,0,B_{n_1}}(x) = \frac{1}{2} \cdot \frac{x(1-x)}{n_1}$ and $\Gamma_{0,0,B_{n_2}}(y) = \frac{1}{2} \cdot \frac{y(1-y)}{n_2}$) to arrive at

$$\begin{aligned} |\left[F - ({}_xB_{n_1} \circ_y B_{n_2})F\right](x,y)| &\leq \frac{1}{2}\frac{x(1-x)}{n_1}\|F^{(2,0)}\| + \frac{1}{2}\frac{y(1-y)}{n_2}\|F^{(0,2)}\| \\ &+ \frac{1}{4}\frac{x(1-x)y(1-y)}{n_1n_2}\|F^{(2,2)}\| \\ &\leq \frac{1}{8n_1}\|F^{(2,0)}\| + \frac{1}{8n_2}\|F^{(0,2)}\| + \frac{1}{64n_1n_2}\|F^{(2,2)}\|. \end{aligned}$$

An estimate of this kind can be found in Theorem 2.3 of [43].

Such three-term expressions typically appear if one writes (I denoting the identity)

$$I - A \circ B = I - A + I - B - (I - A) \circ (I - B) = (I - A) \oplus (I - B),$$

that is, if one uses the fact that the remainder of the tensor product is the Boolean sum of the errors of the parametric extension. The approach behind the above Theorem 3.1 invokes the decomposition

$$I - A \circ B = I - A + A \circ (I - B),$$

and therefore leads to the two-term bound.

3.1.2 The Bernstein type cubature formula revisited

In this section we give a new upper bound for the approximation error of cubature formula associated with the bivariate Bernstein operators. The bounds are described in terms of moduli of continuity of order two. The consideration of this cubature formula is motivated by Bărbosu and Pop's result [42]. It deems necessary to also correct some of the wrong statements made there, in particular those with respect to Boolean sums.

Integrating the bivariate Bernstein polynomials for $F \in C([0,1] \times [0,1])$ one arrives at the following cubature formula

$$\int_0^1 \int_0^1 F(x,y)dxdy = \frac{1}{(n_1+1)(n_2+1)} \sum_{i_1=0}^{n_1} \sum_{i_2=0}^{n_2} F\left(\frac{i_1}{n_1}, \frac{i_2}{n_2}\right) + R_{n_1,n_2}[F], \tag{3.1}$$

where the remainder is bounded as follows:

$$|R_{n_1,n_2}[F]| \leq \frac{1}{12n_1}\|F^{(2,0)}\| + \frac{1}{12n_2}\|F^{(0,2)}\| + \frac{1}{144n_1n_2}\|F^{2,2}\|,$$

if $F \in C^{2,2}([0,1] \times [0,1])$.

This follows from the three-term upper bound of Remark 3.1. See [42] where the same integration error bound can be found.

The two-term bound from Example 3.1 leads to the following

Theorem 3.2. *For the remainder term of the cubature formula (3.1), $n_1, n_2 \in \mathbb{N}$ and $F \in C([0,1] \times [0,1])$ there holds*

$$|R_{n_1,n_2}[F]| \leq \frac{3}{2}\left[\int_0^1 \omega_2\left(F; \sqrt{\frac{x(1-x)}{n_1}}, 0\right) dx + \int_0^1 \omega_2\left(F; 0, \sqrt{\frac{y(1-y)}{n_2}}\right) dy\right].$$

Moreover, if $F \in C^{2,2}([0,1] \times [0,1])$, then the above implies

$$|R_{n_1,n_2}[F]| \leq \frac{1}{4}\left(\frac{1}{n_1}\|F^{(2,0)}\| + \frac{1}{n_2}\|F^{(0,2)}\|\right).$$

Proof. All that needs to be observed is that a function of type $[0,1/2] \ni z \to \omega_2(F; z, 0)$ (with F fixed and continuous) is continuous, thus integrable. The mixed moduli of smoothness of order (k,l), with $k, l \in \mathbb{N}_0$, given for $\delta_1, \delta_2 \geq 0$ by

$$\omega_{k,l}(F; \delta_1, \delta_2) := \sup\left\{\left|\sum_{\nu=0}^{k}\sum_{r=0}^{l}(-1)^{\nu+\mu}\binom{k}{\nu}\binom{l}{\mu}F(x+\nu\cdot h_1, y+\mu\cdot h_2)\right| : \right.$$
$$\left.(x,y), (x+kh_1, y+lh_2) \in [0,1]^2, |h_i| \leq \delta_i, i = 1,2\right\},$$

is a positive, continuous and non-decreasing function with respect to both variables (see [89], [171]). For continuous F these moduli are continuous in δ_1 and δ_2 and satisfy

$$\omega_k(F; \delta_1, 0) = \omega_{k,0}(F; \delta_1, \delta_2) \text{ and } \omega_k(F; 0, \delta_2) = \omega_{0,k}(F; \delta_1, \delta_2).$$

The latter is only relevant to us for $k = 2$. $\square$

3.1.3 The composite bivariate Bernstein operators

In this section we construct the bivariate composite Bernstein operators and the order of convergence is considered involving the second modulus of continuity. Also, some inequalities of Grüss type will be proven. These results are obtained using some general inequalities published in [153]. In order to give the main results of this section, we recall the following facts:

1. For $a, b \in \mathbb{R}$, $a < b$, and $f \in \mathbb{R}^{[a,b]}$ the Bernstein polynomial of degree $n \in \mathbb{N}$ associated to f is given for $x \in [a,b]$, by

$$B_n^{[a,b]}(f; x) = \frac{1}{(b-a)^n}\sum_{i=0}^{n}\binom{n}{i}(x-a)^i(b-x)^{n-i}f\left(a + i\frac{b-a}{n}\right).$$

2. For $g \in C^2[a,b]$ one has

$$g(x) - B_n^{[a,b]}(g; x) = -\frac{(x-a)(b-x)}{2n}g''(\xi_x),\ \xi_x \in (a,b).$$

If we divide $[0,1]$ into subintervals $\left[\frac{k-1}{m}, \frac{k}{m}\right]$, $k = 1, \ldots, m \in \mathbb{N}$, then on $\left[\frac{k-1}{m}, \frac{k}{m}\right]$ we consider

$$\begin{aligned} B_{n,k}(f;x) &= B_n^{\left[\frac{k-1}{m},\frac{k}{m}\right]}(f;x) \\ &= m^n \sum_{i=0}^{n} \binom{n}{i} \left(x - \frac{k-1}{m}\right)^i \left(\frac{k}{m} - x\right)^{n-i} f\left(\frac{kn-n+i}{nm}\right). \end{aligned}$$

Now we combine the $B_{n,k}$ to obtain the positive linear operator $\overline{B}_{n,m} : \mathbb{R}^{[0,1]} \to C[0,1]$,

$$\overline{B}_{n,m}(f;x) = B_{n,k}(f;x), \text{ if } x \in \left[\frac{k-1}{m}, \frac{k}{m}\right], \, 1 \le k \le m.$$

From now on (subscripted) symbols $n...$ will refer to a polynomial degree. (Subscripted) numbers $m...$ will be related to grids. Each function $\overline{B}_{n,m}(f)$ is a Schoenberg spline of degree n with respect to the knot sequence given as follows:

$$\begin{aligned} 0 = \frac{0}{m} &\quad (n+1) - \text{fold} \\ \frac{1}{m} &\quad\quad\;\; n - \text{fold} \\ \vdots &\quad\quad\quad\quad\;\; \vdots \\ \frac{m-1}{m} &\quad\quad\;\; n - \text{fold} \\ 1 = \frac{m}{m} &\quad (n+1) - \text{fold} \end{aligned}$$

We renounce to give a precise numbering of the knots since this will not be needed below. Thus $\overline{B}_{n,m}$ reproduces linear functions, interpolates at $\frac{k}{m}$, $0 \le k \le m$ and has operator norm $\|\overline{B}_{n,m}\| = 1$.

For $n_1, n_2, m_1, m_2 \in \mathbb{N}$ we now consider the parametric extension ${}_x\overline{B}_{n_1,m_1}$ and ${}_y\overline{B}_{n_2,m_2}$ and their product ${}_x\overline{B}_{n_1,m_1} \circ_y \overline{B}_{n_2,m_2}$. For brevity the latter will be denoted by $\overline{\mathcal{B}}$.

For $(x,y) \in \left[\frac{k-1}{m_1}, \frac{k}{m_1}\right] \times \left[\frac{l-1}{m_2}, \frac{l}{m_2}\right]$, it follows

$$\begin{aligned} \overline{\mathcal{B}}(f;x,y) = m_1^{n_1} \cdot m_2^{n_2} \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \binom{n_1}{i} \binom{n_2}{j} \left(x - \frac{k-1}{m_1}\right)^i \left(\frac{k}{m_1} - x\right)^{n_1-i} \\ \cdot \left(y - \frac{l-1}{m_2}\right)^j \left(\frac{l}{m_2} - y\right)^{n_2-j} f\left(\frac{k-1}{m_1} + \frac{i}{m_1 n_1}, \frac{l-1}{m_2} + \frac{j}{n_2 m_2}\right) \end{aligned}$$

and

$$\begin{aligned} \left|f(x,y) - \overline{\mathcal{B}}(f;x,y)\right| = \frac{\left(x - \frac{k-1}{m}\right)\left(\frac{k}{m_1} - x\right)}{2n_1}\|f^{(2,0)}\| + \frac{\left(y - \frac{l-1}{m_2}\right)\left(\frac{l}{m_2} - y\right)}{2n_2}\|f^{(0,2)}\| \\ + \frac{\left(x - \frac{k-1}{m}\right)\left(\frac{k}{m_1} - x\right)\left(y - \frac{l-1}{m_2}\right)\left(\frac{l}{m_2} - y\right)}{4n_1 n_2}\|f^{(2,2)}\|, \end{aligned}$$

where $f \in C^{2,2}([0,1] \times [0,1])$.

Using Theorem 3.1 again we get

Theorem 3.3. *For $f \in C([0,1]\times[0,1])$, $n_1, n_2, m_1, m_2 \in \mathbb{N}$ and $(x,y) \in [0,1]\times[0,1]$ there holds*

$$\left|f(x,y) - \overline{\mathcal{B}}(f;x,y\right| \leq \frac{3}{2}\left\{\omega_2\left(f;\sqrt{\frac{\left(x-\frac{k-1}{m_1}\right)\left(\frac{k}{m_1}-x\right)}{n_1}},0\right)\right.$$
$$\left.+\,\omega_2\left(f;0,\sqrt{\frac{\left(y-\frac{l-1}{m_2}\right)\left(\frac{l}{m_2}-y\right)}{n_2}}\right)\right\},$$

if $(x,y) \in \left[\frac{k-1}{m_1}, \frac{k}{m_1}\right] \times \left[\frac{l-1}{m_2}, \frac{l}{m_2}\right]$, $1 \leq k \leq m_1$, $1 \leq l \leq m_2$.

Proof. For the univariate case we have

$$\left|\overline{B}_{n_1,m_1}(f;x) - f(x)\right| \leq \frac{3}{2}\omega_2\left(f;\sqrt{\frac{\left(x-\frac{k-1}{m_1}\right)\left(\frac{k}{m_1}-x\right)}{n_1}}\right),$$

for $x \in \left[\frac{k-1}{m_1}, \frac{k}{m_1}\right]$, $1 \leq k \leq m_1$. Here ω_2 is the second order modulus over $[0,1]$. An analogous inequality holds for $\overline{B}_{n_2,m_2}$.

The theorem mentioned implies, with $r = s = 2$, the inequality claimed. □

Remark 3.2. *As mentioned earlier, for $g \in C^2[a,b]$ one has*

$$|g(x) - B_n^{[a,b]}(g;x)| = \left|-\frac{(x-a)(b-x)}{2n}g''(\xi_x)\right| \leq \frac{(b-a)^2}{8n}\|g''\|_{[a,b]}.$$

For $[a,b] = \left[\frac{k-1}{m}, \frac{k}{m}\right]$, the last expression equals $\dfrac{1}{8m^2n}\|g''\|_{\left[\frac{k-1}{m},\frac{k}{m}\right]}$.

If $f \in C^{2,2}([0,1]\times[0,1])$ and $(x,y) \in [0,1]\times[0,1]$, using Theorem 1 in [46], this leads to

$$\left|f(x,y) - \overline{\mathcal{B}}(f;x,y\right| \leq \frac{1}{8m_1^2n_1}\|f^{(2,0)}\| + \frac{1}{8m_2^2n_2}\|f^{(0,2)}\| + \frac{1}{64m_1^2n_1m_2^2n_2}\|f^{(2,2)}\|.$$

For $m_1 = m_2 = 1$ this is exactely the inequality in Remark 3.1.

3.1.4 A Grüss-type inequality

In what follows we present an inequality for the bivariate composite Bernstein operators, expressed in terms of the least concave majorant of a modulus of continuity. Let $C(X)$ be the Banach lattice of real valued continuous functions defined on the compact metric space (X,d).

Let $H : C(X^2) \to C(X^2)$ be a positive linear operator reproducing constant function and define

$$T(f,g;x,y) = H(fg;x,y) - H(f;x,y)\cdot H(g;x,y).$$

In order to give an inequality of Chebyshev-Grüss type we recall a general result given by M. Rusu in [153].

From now on we consider the Euclidian metric d_2 derived from d.

Theorem 3.4. *[153] If $f, g \in C(X^2)$ and $x, y \in X$ fixed, then the inequality*

$$|T(f,g;x,y)| \leq \frac{1}{4}\tilde{\omega}_{d_2}\left(f; 4\sqrt{H\left(d_2^2(\cdot,(x,y));x,y\right)}\right)\cdot\tilde{\omega}_{d_2}\left(g; 4\sqrt{H\left(d_2^2(\cdot,(x,y));x,y\right)}\right)$$

holds, where $H\left(d_2^2(\cdot,(x,y));x,y\right)$ is the second moment of the bivariate operator H.

Proposition 3.1. *For $f, g \in C(X^2)$ and $x, y \in X$ fixed, the following Grüss-type inequality holds*

$$|\overline{\mathcal{B}}(fg;x,y) - \overline{\mathcal{B}}(f;x,y)\cdot\overline{\mathcal{B}}(g;x,y)| \leq \frac{1}{4}\tilde{\omega}_{d_2}\left(f;4\sqrt{\Psi(x,y)}\right)\cdot\tilde{\omega}_{d_2}\left(g;4\sqrt{\Psi(x,y)}\right)$$
$$\leq \frac{1}{4}\tilde{\omega}_{d_2}\left(f;2\sqrt{\frac{1}{n_1m_1^2}+\frac{1}{n_2m_2^2}}\right)\cdot\tilde{\omega}_{d_2}\left(g;2\sqrt{\frac{1}{n_1m_1^2}+\frac{1}{n_2m_2^2}}\right)$$

where $\Psi(x,y) = \dfrac{\left(x-\frac{k-1}{m_1}\right)\left(\frac{k}{m_1}-x\right)}{n_1} + \dfrac{\left(y-\frac{l-1}{m_2}\right)\left(\frac{l}{m_2}-y\right)}{n_2}$ *and* $(x,y) \in \left[\frac{k-1}{m_1},\frac{k}{m_1}\right]\times\left[\frac{l-1}{m_2},\frac{l}{m_2}\right]$.

3.1.5 A cubature formula based on $\overline{\mathcal{B}}$

In this section some upper-bounds of the error of cubature formula associated with the bivariate Bernstein operators are given . In [41] D. Bărbosu and D. Miclăuş introduced the following cubature formula:

$$\int_0^1\int_0^1 f(x,y)dxdy = \sum_{k=1}^{m_1}\sum_{l=1}^{m_2}\int_{\frac{k-1}{m_1}}^{\frac{k}{m}}\int_{\frac{l-1}{m_2}}^{\frac{l}{m_2}} f(x,y)dxdy$$
$$\approx \sum_{k=1}^{m_1}\sum_{l=1}^{m_2}\int_{\frac{k-1}{m_1}}^{\frac{k}{m}}\int_{\frac{l-1}{m_2}}^{\frac{l}{m_2}} \overline{\mathcal{B}}(f;x,y)dxdy = \int_0^1\int_0^1 \overline{\mathcal{B}}(f;x,y)dxdy := \overline{\mathcal{I}}(f).$$

It follows

$$\int_{\frac{k-1}{m_1}}^{\frac{k}{m}}\int_{\frac{l-1}{m_2}}^{\frac{l}{m_2}} \overline{\mathcal{B}}(f;x,y)dxdy$$
$$= m_1^{n_1}m_2^{n_2}\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\binom{n_1}{i}\binom{n_2}{j}\int_{\frac{k-1}{m_1}}^{\frac{k}{n_1}}\left(x-\frac{k-1}{m_1}\right)^i\left(\frac{k}{m_1}-x\right)^{n_1-i}dx$$
$$\cdot\int_{\frac{l-1}{m_2}}^{\frac{l}{m_2}}\left(y-\frac{l-1}{m_2}\right)^j\left(\frac{l}{m_2}-y\right)^{n_2-j}dyf\left(\frac{k-1}{m_1}+\frac{i}{m_1n_1},\frac{l-1}{m_2}+\frac{j}{n_2m_2}\right)$$
$$= \sum_{i=0}^{n_1}\sum_{j=0}^{n_2}A_{n_1,n_2,m_1,m_2}f\left(\frac{k-1}{m_1}+\frac{i}{m_1n_1},\frac{l-1}{m_2}+\frac{j}{n_2m_2}\right),$$

where $A_{n_1,n_2,m_1,m_2} = \dfrac{1}{m_1m_2(n_1+1)(n_2+1)}$.

Theorem 3.5. *For $f \in C^{2,2}([0,1]\times[0,1])$ it follows*

$$\left|\int_0^1\int_0^1 f(x,y)dxdy - \overline{\mathcal{I}}(f)\right| \leq \frac{1}{12n_1m_1^2}\|f^{(2,0)}\| + \frac{1}{12n_2m_2^2}\|f^{(0,2)}\| + \frac{1}{144n_1n_2m_1^2m_2^2}\|f^{(2,2)}\|.$$

Proof. We have

$$\begin{aligned}
&\left|\int_0^1\int_0^1 f(x,y)dxdy-\overline{\mathcal{I}}(f)\right|\\
&=\left|\sum_{k=1}^{m_1}\sum_{l=1}^{m_2}\int_{\frac{k-1}{m_1}}^{\frac{k}{m_1}}\int_{\frac{l-1}{m_2}}^{\frac{l}{m_2}} f(x,y)dxdy-\sum_{k=1}^{m_1}\sum_{l=1}^{m_2}\int_{\frac{k-1}{m_1}}^{\frac{k}{m_1}}\int_{\frac{l-1}{m_2}}^{\frac{l}{m_2}}\overline{\mathcal{B}}(f;x,y)dxdy\right|\\
&\leq\sum_{k=1}^{m_1}\sum_{l=1}^{m_2}\int_{\frac{k-1}{m_1}}^{\frac{k}{m_1}}\int_{\frac{l-1}{m_2}}^{\frac{l}{m_2}}\left|f(x,y)-\overline{\mathcal{B}}(f;x,y)\right|dxdy\\
&=\sum_{k=1}^{m_1}\sum_{l=1}^{m_2}\int_{\frac{k-1}{m_1}}^{\frac{k}{m_1}}\int_{\frac{l-1}{m_2}}^{\frac{l}{m_2}}\left[\frac{\left(x-\frac{k-1}{m_1}\right)\left(\frac{k}{m_1}-x\right)}{2n_1}\|f^{(2,0)}\|+\frac{\left(y-\frac{l-1}{m_2}\right)\left(\frac{l}{m_2}-y\right)}{2n_2}\|f^{(0,2)}\|\right.\\
&\left.+\frac{\left(x-\frac{k-1}{m_1}\right)\left(\frac{k}{m_1}-x\right)\left(y-\frac{l-1}{m_2}\right)\left(\frac{l}{m_2}-y\right)}{4n_1n_2}\|f^{(2,2)}\|\right]dxdy\\
&\leq\sum_{k=1}^{m_1}\sum_{l=1}^{m_2}\left[\frac{1}{12n_1m_1^3m_2}\|f^{(2,0)}\|+\frac{1}{12n_2m_2^3m_1}\|f^{(0,2)}\|+\frac{1}{144n_1n_2m_1^3m_2^3}\|f^{(2,2)}\|\right]\\
&=\frac{1}{12n_1m_1^2}\|f^{(2,0)}\|+\frac{1}{12n_2m_2^2}\|f^{(0,2)}\|+\frac{1}{144n_1n_2m_1^2m_2^2}\|f^{(2,2)}\|.
\end{aligned}$$

□

One further estimate is given in

Theorem 3.6. *For $f\in C^{2,2}([0,1]\times[0,1])$ it follows*

$$\left|\int_0^1\int_0^1 f(x,y)dxdy-\overline{\mathcal{I}}(f)\right|\leq\frac{1}{4}\left\{\frac{1}{m_1^2n_1}\|f^{(2,0)}\|+\frac{1}{m_2^2n_2}\|f^{(0,2)}\|\right\}.$$

Proof. Integrating the error given in Theorem 3.3 leads to

$$\begin{aligned}
\left|\int_0^1\int_0^1 f(x,y)dxdy-\overline{\mathcal{I}}(f)\right|&\leq\frac{3}{2}\sum_{k=1}^{m_1}\sum_{l=1}^{m_2}\left\{\frac{1}{m_2}\int_{\frac{k-1}{m_1}}^{\frac{k}{m_1}}\omega_2\left(f;\sqrt{\frac{\left(x-\frac{k-1}{m_1}\right)\left(\frac{k}{m_1}-x\right)}{n_1}},0\right)dx\right.\\
&\left.+\frac{1}{m_1}\int_{\frac{l-1}{m_2}}^{\frac{l}{m_2}}\omega_2\left(f;0,\sqrt{\frac{\left(y-\frac{l-1}{m_2}\right)\left(\frac{l}{m_2}-y\right)}{n_2}}\right)dy\right\}.
\end{aligned}$$

Since $f\in C^{2,2}([0,1]\times[0,1])$ leads to

$$\begin{aligned}
\left|\int_0^1\int_0^1 f(x,y)dxdy-\overline{\mathcal{I}}(f)\right|&\leq\frac{3}{2}\sum_{k=1}^{m_1}\sum_{l=1}^{m_2}\left\{\frac{1}{m_2}\|f^{(2,0)}\|\int_{\frac{k-1}{m_1}}^{\frac{k}{m_1}}\frac{\left(x-\frac{k-1}{m_1}\right)\left(\frac{k}{m_1}-x\right)}{n_1}dx\right.\\
&\left.+\frac{1}{m_1}\|f^{(0,2)}\|\int_{\frac{l-1}{m_2}}^{\frac{l}{m_2}}\frac{\left(y-\frac{l-1}{m_2}\right)\left(\frac{l}{m_2}-y\right)}{n_1}dy\right\}\\
&=\frac{3}{2}\sum_{k=1}^{m_1}\sum_{l=1}^{m_2}\left\{\frac{1}{6m_1^3m_2n_1}\|f^{(2,0)}\|+\frac{1}{6m_1m_2^3n_2}\|f^{(0,2)}\|\right\}\\
&=\frac{1}{4}\left\{\frac{1}{m_1^2n_1}\|f^{(2,0)}\|+\frac{1}{m_2^2n_2}\|f^{(0,2)}\|\right\}.
\end{aligned}$$

□

3.1.6 Non-multiplicativity of the cubature formula

In this section we will give some results which suggest how non-multiplicative the functional $\overline{\mathcal{I}}(f) = \int_0^1 \int_0^1 \overline{\mathcal{B}}(f;(x,y))dxdy$ is.

Let (X,d) be a compact metric space and $L : C(X) \to \mathbb{R}$ be a positive linear functional reproducing constants. We consider the positive bilinear functional

$$D(f,g) := L(fg) - L(f)L(g).$$

Theorem 3.7. *If $f, g \in C(X)$, (X,d) a compact metric space, then the inequality*

$$|D(f,g)| \le \frac{1}{4}\tilde{\omega}_d\left(f; 2\sqrt{L^2(d^2(\cdot,\cdot))}\right)\tilde{\omega}_d\left(g; 2\sqrt{L^2(d^2(\cdot,\cdot))}\right)$$

holds.

Proof. Let $f, g \in C[a,b]$ and $r, s \in Lip_1$. Using the Cauchy-Schwarz inequality for positive linear functional gives

$$|L(f)| \le L(|f|) \le \sqrt{L(f^2)\cdot L(1)} = \sqrt{L(f^2)},$$

so we have

$$D(f,f) = L(f^2) - L(f)^2 \ge 0.$$

Therefore, D is a positive bilinear form on $C(X)$. Using the Cauchy-Schwarz inequality for D it follows

$$|D(f,g)| \le \sqrt{D(f,f)D(g,g)} \le \|f\|\|g\|.$$

Since L is a positive linear functional we can write

$$L(f) := \int_X f(t)d\mu(t),$$

where μ is a Borel probability measure on X, i.e., $\int_X d\mu(t) = 1$. For $r \in Lip_1$, it follows

$$\begin{aligned}
D(r,r) = L(r^2) - L(r)^2 &= \int_X r^2(t)d\mu(t) - \left(\int_X r(u)d\mu(u)\right)^2 \\
&= \int_X \left(r(t) - \int_X r(u)d\mu(u)\right)^2 d\mu(t) = \int_X \left(\int_X (r(t)-r(u))d\mu(u)\right)^2 d\mu(t) \\
&\le \int_X \left(\int_X (r(t)-r(u))^2 d\mu(u)\right) d\mu(t) \\
&\le |r|^2_{Lip_1} \int_X \left(\int_X d^2(t,u)d\mu(u)\right) d\mu(t) \\
&= |r|^2_{Lip_1} L^t\left[L\left(d^2(t,\cdot)\right)\right] = |r|^2_{Lip_1} L^2\left(d^2(\cdot,\cdot)\right).
\end{aligned}$$

For $r, s \in Lip_1$ we have

$$|D(r,s)| \le \sqrt{D(r,r)D(s,s)} \le |r|_{Lip_1}|s|_{Lip_1} L^2\left(d(\cdot,\cdot)\right).$$

Moreover, for $f \in C(X)$ and $s \in Lip_1$, we have the estimate

$$|D(f,s)| \leq \sqrt{D(f,f)D(s,s)} \leq \|f\||s|_{Lip_1}\sqrt{L^2\left(d(\cdot,\cdot)\right)}.$$

In a similar way, if $r \in Lip_1$ and $g \in C(X)$, we have

$$|D(r,g)| \leq \sqrt{D(r,r)D(g,g)} \leq \|g\||r|_{Lip_1}\sqrt{L^2\left(d(\cdot,\cdot)\right)}.$$

Let $f, g \in C(X)$ be fixed and $r, s \in Lip_1$ arbitrary, then

$$\begin{aligned}
|D(f,g)| &= |D(f-r+r, g-s+s)| \\
&\leq |D(f-r, g-s) + |D(f-r,s)| + |D(r,g-s)| + |D(r,s))| \\
&\leq \|f-r\| \cdot \|g-s\| + \|f-r\| \cdot |s|_{Lip_1}\sqrt{L^2\left(d^2(\cdot,\cdot)\right)} \\
&+ \|g-s\| \cdot |r|_{Lip_1}\sqrt{L^2\left(d^2(\cdot,\cdot)\right)} + |r|_{Lip_1}|s|_{Lip_1}L^2\left(d^2(\cdot,\cdot)\right) \\
&= \left\{\|f-r\| + |r|_{Lip_1}\sqrt{L^2\left(d^2(\cdot,\cdot)\right)}\right\}\left\{\|g-s\| + |s|_{Lip_1}\sqrt{L^2\left(d^2(\cdot,\cdot)\right)}\right\}.
\end{aligned}$$

Passing to the infimum over r and s, respectively, leads to

$$\begin{aligned}
|D(f,g)| &\leq K\left(\sqrt{L^2\left(d^2(\cdot,\cdot)\right)}, f; C(X), Lip_1\right) \cdot K\left(\sqrt{L^2\left(d^2(\cdot,\cdot)\right)}, g; C(X), Lip_1\right) \\
&\leq \frac{1}{4}\tilde{\omega}\left(f; 2\sqrt{L^2\left(d^2(\cdot,\cdot)\right)}\right)\tilde{\omega}\left(g; 2\sqrt{L^2\left(d^2(\cdot,\cdot)\right)}\right).
\end{aligned}$$

□

Applying Theorem 3.7 for $L(f) = \overline{\mathcal{I}}(f)$ we obtain the following result:

Corollary 3.1. *If $f, g \in C([0,1] \times [0,1])$, then*

$$\begin{aligned}
\left|\overline{\mathcal{I}}(fg) - \overline{\mathcal{I}}(f)\overline{\mathcal{I}}(g)\right| \leq &\frac{1}{4}\tilde{\omega}_{d_2}\left(f; 2\sqrt{\frac{1}{3}\left(1+\frac{1}{n_1m_1^2}+\frac{1}{n_2m_2^2}\right)}\right) \\
&\cdot \tilde{\omega}_{d_2}\left(g; 2\sqrt{\frac{1}{3}\left(1+\frac{1}{n_1m_1^2}+\frac{1}{n_2m_2^2}\right)}\right).
\end{aligned} \tag{3.2}$$

Proof. We have

$$\begin{aligned}
\overline{\mathcal{I}}\left(d_2^2(\cdot,\cdot)\right) &= \sum_{k,k_1=1}^{m_1}\sum_{l,l_1=1}^{m_2}\sum_{i,i_1=0}^{n_1}\sum_{j,j_1=0}^{n_2}\frac{1}{m_1^2m_2^2(n_1+1)^2(n_2+1)^2} \\
&\cdot\left[\left(\frac{k_1-1}{m_1}+\frac{i_1}{m_1n_1}-\frac{k-1}{m_1}-\frac{i}{m_1n_1}\right)^2+\left(\frac{l_1-1}{m_2}+\frac{j_1}{n_2m_2}-\frac{l-1}{m_2}-\frac{j}{n_2m_2}\right)^2\right] \\
&= \frac{1}{m_1^2(n_1+1)^2}\sum_{k,k_1=1}^{m_1}\sum_{i,i_1=0}^{n_1}\left(\frac{k_1-k}{m_1}+\frac{i_1-i}{m_1n_1}\right)^2 \\
&+ \frac{1}{m_2^2(n_2+1)^2}\sum_{l,l_1=1}^{m_2}\sum_{j,j_1=0}^{n_2}\left(\frac{l_1-l}{m_2}+\frac{j_1-j}{m_2n_2}\right)^2 = \frac{1}{3}\left(1+\frac{1}{m_1^2n_1}+\frac{1}{m_2^2n_2}\right).
\end{aligned}$$

Therefore, using Theorem 3.7 it follows

$$\left|\overline{\mathcal{I}}(fg) - \overline{\mathcal{I}}(f)\overline{\mathcal{I}}(g)\right| \leq \frac{1}{4}\tilde{\omega}_{d_2}\left(f; 2\sqrt{\frac{1}{3}\left(1+\frac{1}{n_1 m_1^2}+\frac{1}{n_2 m_2^2}\right)}\right) \cdot \tilde{\omega}_{d_2}\left(g; 2\sqrt{\frac{1}{3}\left(1+\frac{1}{n_1 m_1^2}+\frac{1}{n_2 m_2^2}\right)}\right).$$

□

3.2 Grüss-type inequalities via discrete oscillations

The aim of this section is to consider some new bivariate Grüss-type inequalities via discrete oscillations and to apply them to different tensor products of linear, (not necessarily) positive, well-known operators. Also, we compare the new inequalities with some older results. We give a Grüss-type inequality with discrete oscillations for more than two functions. The results presented in this section were published in [18].

3.2.1 Grüss-type inequalities via discrete oscillations for a linear functional

Gonska et al. [72] obtained a new Grüss-type inequality which involves oscillations of functions. This result is better than (2.14) in the sense that the oscillations of functions are relative only to certain points, while in (2.14) the oscillations, expressed in terms of $\tilde{\omega}$, are relative to the whole interval $[a,b]$. In this section we will give a generalization of the results obtained in [72], considering the bivariate discrete linear functional case.

Let X be an arbitrary set and $B(I)$ the set of all real-valued, bounded functions on $I = X^2$. Take $a_n, b_n \in \mathbb{R}$, $n \geq 0$, such that $\sum_{n=0}^{\infty} |a_n| < \infty$, $\sum_{n=0}^{\infty} a_n = 1$ and $\sum_{n=0}^{\infty} |b_n| < \infty$, $\sum_{n=0}^{\infty} b_n = 1$, respectively. Furthermore, let $x_n \in X, n \geq 0$ and $y_m \in X$, $m \geq 0$ be arbitrary mutually distinct points. For $f \in B(I)$ set $f_{n,m} := f(x_n, y_m)$. Now consider the functional $L : B(I) \to \mathbb{R}$, $Lf = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n b_m f_{n,m}$. The functional L is linear and reproduces constant functions.

Theorem 3.8. *The Grüss-type inequality for the above linear functional L is given by:*

$$|L(fg) - L(f)\cdot L(g)| \leq \frac{1}{2}\cdot osc_L(f)\cdot osc_L(g)\cdot \sum_{n,m,i,j=0,\ (n,m)\neq(i,j)}^{\infty} |a_n b_m a_i b_j|,$$

where $f, g \in B(I)$ and we define the oscillations to be:

$$osc_L(f) := \sup\{|f_{n,m} - f_{i,j}| : n,m,i,j \geq 0\},$$
$$osc_L(g) := \sup\{|g_{n,m} - g_{i,j}| : n,m,i,j \geq 0\}.$$

Proof. We have

$$L(fg) - L(f)\cdot L(g) = \sum_{n,m=0}^{\infty} a_n b_m f_{n,m} g_{n,m} - \sum_{n,m=0}^{\infty} a_n b_m f_{n,m}\cdot \sum_{i,j=0}^{\infty} a_i b_j g_{i,j}$$
$$= \sum_{n,m=0}^{\infty}\left(\sum_{i,j=0}^{\infty} a_i b_j\right) a_n b_m f_{n,m} g_{n,m} - \sum_{n,m=0}^{\infty} a_n b_m f_{n,m}\cdot \sum_{i,j=0}^{\infty} a_i b_j g_{i,j}$$

$$
\begin{aligned}
&= \sum_{n,m=0}^{\infty} a_n^2 b_m^2 f_{n,m} g_{n,m} + \sum_{n,m=0}^{\infty} \left(\sum_{i,j=0,(i,j)\neq(n,m)}^{\infty} a_i b_j \right) a_n b_m f_{n,m} g_{n,m} \\
&- \sum_{n,m=0}^{\infty} a_n^2 b_m^2 f_{n,m} g_{n,m} - \sum_{n,m=0}^{\infty} \left(\sum_{i,j=0,(i,j)\neq(n,m)}^{\infty} a_n b_m a_i b_j f_{n,m} g_{i,j} \right) \\
&= \sum_{n,m,i,j=0,(i,j)\neq(n,m)}^{\infty} a_i b_j a_n b_m f_{n,m} (g_{n,m} - g_{i,j}).
\end{aligned}
$$

The above identity can be written in the following way

$$
L(fg) - L(f) \cdot L(g) = \sum_{n,m,i,j=0,(i,j)\neq(n,m)}^{\infty} a_n b_m a_i b_j f_{i,j} (g_{i,j} - g_{n,m}).
$$

Therefore

$$
2\left(L(fg) - L(f) \cdot L(g)\right) = \sum_{n,m,i,j=0,(i,j)\neq(n,m)}^{\infty} a_i b_j a_n b_m \left(f_{n,m} - f_{i,j}\right)\left(g_{n,m} - g_{i,j}\right),
$$

and the theorem is proved. $\square$

Theorem 3.9. *In particular, if* $a_n \geq 0,\ b_m \geq 0,\ n, m \geq 0$, *then* L *is a positive linear functional and we have:*

$$
|L(fg) - L(f) \cdot L(g)| \leq \frac{1}{2} \cdot \left(1 - \sum_{n=0}^{\infty} a_n^2 \cdot \sum_{m=0}^{\infty} b_m^2 \right) \cdot osc_L(f) \cdot osc_L(g),
$$

for $f, g \in B(I)$ *and the oscillations given as above.*

Proof. In this case we have

$$
\begin{aligned}
\sum_{n,m,i,j=0,\ (n,m)\neq(i,j)}^{\infty} |a_n b_m a_i b_j| &= \sum_{n,m=0}^{\infty} a_n b_m \sum_{i,j=0,\ (i,j)\neq(n,m)}^{\infty} a_i b_j \\
&= \sum_{n,m=0}^{\infty} a_n b_m \left(\sum_{i,j=0}^{\infty} a_i b_j - a_n b_m \right) = \sum_{n,m=0}^{\infty} a_n b_m (1 - a_n b_m) \\
&= \sum_{n,m=0}^{\infty} a_n b_m - \sum_{n,m=0}^{\infty} a_n^2 b_m^2 = 1 - \left(\sum_{n=0}^{\infty} a_n^2 \right) \cdot \left(\sum_{m=0}^{\infty} b_m^2 \right),
\end{aligned}
$$

so the result follows as a consequence of Theorem 3.8. $\square$

In the following we will apply the Grüss-type inequality obtained in this section to known operators, for example Lagrange, Bernstein, Mirakjan-Favard-Szász and piecewise linear interpolation operators. Let

$$
T_H(f, g; x, y) = H(f \cdot g; x, y) - H(f; x, y) \cdot H(g; x, y),
$$

where (x, y) is fixed and H is a linear operator.

3.2.2 Application for the bivariate Lagrange operator

The bivariate Lagrange interpolation operator is constructed using the parametric extensions approach. We consider bivariate functions defined on $[-1,1]\times[-1,1]=I$. Let $(x_{k_1,n_1}, y_{k_2,n_2}) \in I$, $k_1=\overline{1,n_1}$, $k_2=\overline{1,n_2}$. Then the bidimensional Lagrange operator L_{n_1,n_2} is given by

$$L_{n_1,n_2}(f;x,y) := \sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2} f(x_{k_1,n_1}, y_{k_2,n_2}) l_{k_1,n_1}(x) l_{k_2,n_2}(y),$$

for $f \in \mathbb{R}^I$ and the Lagrange fundamental functions given as usual by

$$l_{k_1,n_1}(x) = \frac{\omega_{n_1}(x)}{\omega'_{n_1}(x_{k_1,n_1})(x - x_{k_1,n_1})},\ 1 \le k_1 \le n_1,$$

where $\omega_{n_1}(x) = \prod_{k_1=1}^{n_1}(x - x_{k_1,n_1})$. The fundamental functions $l_{k_2,n_2}(y)$ are defined analogously.

The corresponding Lebesgue functions are

$$\Lambda_{n_1}(x) := \sum_{k_1=1}^{n_1} |l_{k_1,n_1}(x)| \ \text{ and } \ \Lambda_{n_2}(y) := \sum_{k_2=1}^{n_2} |l_{k_2,n_2}(y)|.$$

Regarding the sums of the squared fundamental functions of a Lagrange interpolation based upon any infinite matrix X, we recall a result from [103]. It holds, for $\alpha = 2$ in the relation (3.1) in the cited article, that

$$\sum_{k_1=1}^{n_1} l_{k_1,n_1}^2(x) \ge \frac{1}{4},\ \text{for } -1 \le x \le 1,$$

and the same holds for the squares of the fundamental functions with respect to y.

We only apply this kind of inequality using these special oscillations to the Lagrange operators in the bivariate case. In the case of oscillations involving the least concave majorant of the modulus of continuity, the inequalities are more complicated.

Theorem 3.10. *The Grüss-type inequality for the bivariate Lagrange operator is given by*

$$\left|T_{L_{n_1,n_2}}(f,g;x,y)\right| \le \frac{1}{2} \cdot osc_{L_{n_1,n_2}}(f) \cdot osc_{L_{n_1,n_2}}(g) \cdot \left[\Lambda_{n_1}^2(x)\Lambda_{n_2}^2(y) - \frac{1}{16}\right],$$

where $f, g \in C(I)$. The oscillation for f is defined by

$$\begin{aligned} osc_{L_{n_1,n_2}}(f) :=& \max\{|f(x_{k_1,n_1}, y_{k_2,n_2}) - f(x_{m_1,n_1}, y_{m_2,n_2})| : \\ & 1 \le k_1, m_1 \le n_1, 1 \le k_2, m_2 \le n_2\}. \end{aligned}$$

A similar definition is given for the oscillation of the second function.

Proof. We have

$$\begin{aligned} &\sum_{k_1,m_1=1}^{n_1}\left(\sum_{k_2,m_2=1,(k_1,k_2)\neq(m_1,m_2)}^{n_2} |l_{k_1,n_1}(x)l_{k_2,n_2}(y)l_{m_1,n_1}(x)l_{m_2,n_2}(y)|\right) \\ &= \left(\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} |l_{i,n_1}(x)l_{j,n_2}(y)|\right)^2 - \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} l_{i,n_1}^2(x) l_{j,n_2}^2(y) \\ &= \Lambda_{n_1}^2(x)\Lambda_{n_2}^2(y) - \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} l_{i,n_1}^2(x) l_{j,n_2}^2(y) \le \Lambda_{n_1}^2(x)\Lambda_{n_2}^2(y) - \frac{1}{16}, \end{aligned}$$

and the theorem is proved. □

3.2.3 Application for the bivariate Bernstein operator

Let $X = [0,1]$ and $I = [0,1] \times [0,1]$ endowed with the Euclidean metric

$$d((s,t),(x,y)) := \sqrt{(s-x)^2 + (t-y)^2},$$

for $(s,t),(x,y) \in I$. Consider the bivariate Bernstein operators

$$B_{n_1,n_2}(f;x,y) := \sum_{i_1=0}^{n_1} \sum_{i_2=0}^{n_2} f\left(\frac{i_1}{n_1}, \frac{i_2}{n_2}\right) b_{n_1,i_1}(x) b_{n_2,i_2}(y), \;\; f \in \mathbb{R}^I, \;\; x,y \in X,$$

where $b_{n_1,i_1}(x) := \binom{n_1}{i_1} x^{i_1}(1-x)^{n_1-i_1}$ and $b_{n_2,i_2}(y) := \binom{n_2}{i_2} y^{i_2}(1-y)^{n_2-i_2}$.

According to Theorem 3.9, for each $x, y \in X, \; f,g \in B(I)$ we have

$$\left|T_{B_{n_1,n_2}}(f,g;x,y)\right| \leq \frac{1}{2}\left(1 - \sum_{i_1=0}^{n_1} b^2_{n_1,i_1}(x) \cdot \sum_{i_2=0}^{n_2} b^2_{n_2,i_2}(y)\right) \cdot osc_{B_{n_1,n_2}}(f) \cdot osc_{B_{n_1,n_2}}(g), \tag{3.3}$$

where

$$osc_{B_{n_1,n_2}}(f) := \max\{|f_{k,l} - f_{s,t}| : k,s = 0,\dots,n_1; l,t = 0,\dots,n_2\},$$

and $f_{k,l} := f\left(\frac{k}{n_1}, \frac{l}{n_2}\right)$; similar definition apply to g.

Let $\varphi_{n_1}(x) := \sum_{i_1=0}^{n_1} b^2_{n_1,i_1}(x)$, $x \in X$. Then we get immediately

$$\varphi_{n_1}(x) \geq \frac{1}{n_1+1}, \;\; x \in X, \tag{3.4}$$

and the same holds for $\varphi_{n_2}(y), \;\; y \in X$. Therefore it holds

$$\begin{aligned} \left|T_{B_{n_1,n_2}}(f,g;x,y)\right| &\leq \frac{1}{2}\left(1 - \frac{1}{n_1+1} \cdot \frac{1}{n_2+1}\right) \cdot osc_{B_{n_1,n_2}}(f) \cdot osc_{B_{n_1,n_2}}(g) \\ &= \frac{1}{2} \cdot \frac{n_2 n_1 + n_2 + n_1}{(n_1+1)(n_2+1)} \cdot osc_{B_{n_1,n_2}}(f) \cdot osc_{B_{n_1,n_2}}(g), \end{aligned} \tag{3.5}$$

for $x, y \in X$.

Remark 3.3. *It was proved in [72] that*

$$\varphi_{n_1}(x) \geq \frac{1}{4^{n_1}} \binom{2n_1}{n_1}, \;\; x \in X, \tag{3.6}$$

with equality if and only if $x = 1/2$; the same result holds for $\varphi_{n_2}(y)$, for $y \in X$.

Consequently we have:

Theorem 3.11. *The new Grüss-type inequality for the bivariate Bernstein operator is:*

$$\begin{aligned} &\left|T_{B_{n_1,n_2}}(f,g;x,y)\right| \\ &\leq \frac{1}{2}\left(1 - \binom{2n_1}{n_1}\binom{2n_2}{n_2} \frac{1}{4^{n_1}} \cdot \frac{1}{4^{n_2}}\right) \cdot osc_{B_{n_1,n_2}}(f) \cdot osc_{B_{n_1,n_2}}(g), \;\; x,y \in X. \end{aligned} \tag{3.7}$$

In what follows we present another inequality for the bivariate Bernstein operators, expressed in terms of the least concave majorant of the modulus of continuity. The second moment of the bivariate Bernstein operator is given by

$$\begin{aligned} B_{n_1,n_2}(d^2(\cdot,(x,y));x,y) &= \sum_{i_1=0}^{n_1}\sum_{i_2=0}^{n_2} d^2\left(\left(\frac{i_1}{n_1},\frac{i_2}{n_2}\right),(x,y)\right) b_{n_1,i_1}(x)b_{n_2,i_2}(y) \\ &= \frac{x(1-x)}{n_1}+\frac{y(1-y)}{n_2} \le \frac{1}{4}\left(\frac{1}{n_1}+\frac{1}{n_2}\right). \end{aligned}$$

The bivariate Grüss-type inequality (compare Theorem 3.1, in [154]) looks as follows:

$$\begin{aligned} &\left|T_{B_{n_1,n_2}}(f,g;x,y)\right| \qquad (3.8) \\ &\le \frac{1}{4}\widetilde{\omega}_d\left(f;4\sqrt{\frac{x(1-x)}{n_1}+\frac{y(1-y)}{n_2}}\right)\cdot\widetilde{\omega}_d\left(g;4\sqrt{\frac{x(1-x)}{n_1}+\frac{y(1-y)}{n_2}}\right), \end{aligned}$$

which implies

$$\left|T_{B_{n_1,n_2}}(f,g;x,y)\right| = \frac{1}{4}\widetilde{\omega}_d\left(f;2\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}\right)\cdot\widetilde{\omega}_d\left(g;2\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}\right), \qquad (3.9)$$

for two functions $f,g\in B(I)$ and $x,y\in X$ fixed.

Remark 3.4. *In (3.3) and (3.8), the right-hand side depends on (x,y) and vanishes when $(x,y)\to(i,j)$, $i,j\in\{0,1\}$. The maximum value of it, as a function of (x,y), is attained for $x=y=\frac{1}{2}$, and (3.5), (3.7), (3.9), illustrate this fact. On the other hand, in (3.3) the oscillations of f and g are relative only to the points $\left(\frac{k}{n_1},\frac{l}{n_2}\right)$, $0\le k\le n_1$, $0\le l\le n_2$ while in (3.8) the oscillations, expressed in terms of $\widetilde{\omega}$, are relative to the whole domain I.*

3.2.4 Application for the bivariate Szász-Mirakjan operators

The Szász-Mirakjan operators were first introduced in 1941, by G.M. Mirakjan [132]. Extension of these operators to the bivariate case was introduced in [164]:

$$M_{n_1,n_2}(f;x,y) := e^{-n_1\cdot x}e^{-n_2\cdot y}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{(n_1x)^{k_1}}{k_1!}\cdot\frac{(n_2y)^{k_2}}{k_2!}\cdot f\left(\frac{k_1}{n_1},\frac{k_2}{n_2}\right),$$

for every bounded function $f\in\mathbb{R}^I$, $I=[0,\infty)\times[0,\infty)$.

Denote

$$\sigma_{n_1}(x) := e^{-2n_1x}\sum_{k_1=0}^{\infty}\frac{(n_1x)^{2k_1}}{(k_1!)^2} \text{ and } \sigma_{n_2}(y) := e^{-2n_2y}\sum_{k_2=0}^{\infty}\frac{(n_2y)^{2k_2}}{(k_2!)^2}.$$

Theorem 3.12. *The Grüss-type inequality via discrete oscillations in the bivariate case for Szász-Mirakjan operators is given by*

$$\left|T_{M_{n_1,n_2}}(f,g;x,y)\right| \le \frac{1}{2}\cdot(1-\sigma_{n_1}(x)\cdot\sigma_{n_2}(y))\cdot osc_{M_{n_1,n_2}}(f)\cdot osc_{M_{n_1,n_2}}(g),$$

where $f,g\in B(I)$, $osc_{M_{n_1,n_2}}(f) := sup\{|f_{s,l}-f_{r,t}| : s,r,l,t\ge 0\}$, with $f_{s,l} := f\left(\frac{s}{n_1},\frac{l}{n_2}\right)$ and $B(I)$ is the set of all real-valued, bounded functions on I.

It was proved in [72] that $\inf_{x\geq 0}\sigma_{n_1}(x)=0$ and $\inf_{y\geq 0}\sigma_{n_2}(y)=0$ hold. Then the above inequality looks as follows.

Theorem 3.13. *The Grüss-type inequality for the Szász-Mirakjan operators is given by*

$$|T(f,g;x,y)|\leq \frac{1}{2}osc_{M_{n_1,n_2}}(f)\cdot osc_{M_{n_1,n_2}}(g),$$

where the functions f and g and the oscillations are given as above.

3.2.5 Application for $S_{\Delta_{n_1},\Delta_{n_2}}$

In [77] the authors considered the operator $S_{\Delta_n}: C[0,1]\to C[0,1]$,

$$S_{\Delta_n}(f;x)=\frac{1}{n}\sum_{k=0}^{n}\left[\frac{k-1}{n},\frac{k}{n},\frac{k+1}{n};|\alpha-x|\right]_\alpha f\left(\frac{k}{n}\right),$$

where $[a,b,c;f]=[a,b,c;f(\alpha)]_\alpha$ denotes the divided difference of a function $f:D\to R$ on the distinct knots $a,b,c\subset D$, w.r.t. α.

Let $I=[0,1]\times[0,1]$ be endowed with the Euclidean metric. In this section we will consider the bivariate case of this operator, $S_{\Delta_{n_1},\Delta_{n_2}}: C(I)\to C(I)$, which can be explicitly described as

$$S_{\Delta_{n_1},\Delta_{n_2}}(f;x,y)=$$

$$\frac{1}{n_1n_2}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\left[\frac{k_1-1}{n_1},\frac{k_1}{n_1},\frac{k_1+1}{n_1};|\alpha-x|\right]_\alpha\left[\frac{k_2-1}{n_2},\frac{k_2}{n_2},\frac{k_2+1}{n_2};|\alpha-y|\right]_\alpha f\left(\frac{k_1}{n_1},\frac{k_2}{n_2}\right).$$

Denote

$$u_{n_1,k_1}(x)=\frac{1}{n_1}\left[\frac{k_1-1}{n_1},\frac{k_1}{n_1},\frac{k_1+1}{n_1};|\alpha-x|\right]_\alpha, u_{n_1,k_1}\in C([0,1]),$$

a similar definition holding also for $u_{n_2,k_2}(y)$.

In order to apply Theorem 2.5 we will calculate the second moment of this operator. For $x\in\left[\frac{k_1-1}{n_1},\frac{k_1}{n_1}\right]$, $y=\left[\frac{k_2-1}{n_2},\frac{k_2}{n_2}\right]$, we get

$$S_{\Delta_1,\Delta_2}\left(d^2(\cdot,(x,y);x,y)\right)=\sum_{i_1=0}^{n_1}\sum_{i_2=0}^{n_2}u_{n_1,i_1}(x)u_{n_2,i_2}(y)d^2\left(\left(\frac{i_1}{n_1},\frac{i_2}{n_2}\right),(x,y)\right)$$

$$=\sum_{i_1=0}^{n_1}\sum_{i_2=0}^{n_2}u_{n_1,i_1}(x)u_{n_2,i_2}(y)\left\{\left(\frac{i_1}{n_1}-x\right)^2+\left(\frac{i_2}{n_2}-y\right)^2\right\}$$

$$=\sum_{i_1=0}^{n_1}\sum_{i_2=0}^{n_2}u_{n_1,i_1}(x)u_{n_2,i_2}(y)\left(\frac{i_1}{n_1}-x\right)^2+\sum_{i_1=0}^{n_1}\sum_{i_2=0}^{n_2}u_{n_1,i_1}(x)u_{n_2,i_2}(y)\left(\frac{i_2}{n_2}-y\right)^2$$

$$=\sum_{i_1=0}^{n_1}u_{n_1,i_1}(x)\left(\frac{i_1}{n_1}-x\right)^2+\sum_{i_2=0}^{n_2}u_{n_2,i_2}(y)\left(\frac{i_2}{n_2}-y\right)^2$$

$$=\sum_{i_1=0}^{n_1}\frac{n_1}{2}\left(\frac{i_1}{n_1}-x\right)^2\left\{\left|\frac{i_1+1}{n_1}-x\right|-2\left|\frac{i_1}{n_1}-x\right|+\left|\frac{i_1-1}{n_1}-x\right|\right\}$$

$$+\sum_{i_2=0}^{n_2}\frac{n_2}{2}\left(\frac{i_2}{n_2}-y\right)^2\left\{\left|\frac{i_2+1}{n_2}-y\right|-2\left|\frac{i_2}{n_2}-y\right|+\left|\frac{i_2-1}{n_2}-y\right|\right\}$$
$$=\left(x-\frac{k_1-1}{n_1}\right)\left(\frac{k_1}{n_1}-x\right)+\left(y-\frac{k_2-1}{n_2}\right)\left(\frac{k_2}{n_2}-y\right)\leq\frac{1}{4n_1^2}+\frac{1}{4n_2^2}.$$

By taking $H_n = S_{\Delta_{n_1},\Delta_{n_2}}$ in Theorem 2.5, the Grüss-type inequality for $S_{\Delta_{n_1},\Delta_{n_2}}$ is given by the following result:

Theorem 3.14. *If $f, g \in C(I)$ and $x, y \in [0,1]$, then the inequality*

$$\begin{aligned}&\left|T_{S_{\Delta_{n_1},\Delta_{n_2}}}(f,g;x,y)\right|\\ &\leq\frac{1}{4}\widetilde{\omega}_d\left(f;4\sqrt{S_{\Delta_{n_1},\Delta_{n_2}}(d^2(\cdot,(x,y));x,y)}\right)\widetilde{\omega}_d\left(g;4\sqrt{S_{\Delta_{n_1},\Delta_{n_2}}(d^2(\cdot,(x,y));x,y)}\right)\\ &\leq\frac{1}{4}\widetilde{\omega}_d\left(f;2\cdot\sqrt{\sum_{i=1}^{2}\frac{1}{n_i^2}}\right)\cdot\widetilde{\omega}_d\left(g;2\cdot\sqrt{\sum_{i=1}^{2}\frac{1}{n_i^2}}\right)\end{aligned}$$

holds.

Using Theorem 3.9, we get an inequality of the form

$$\left|T_{S_{\Delta_{n_1},\Delta_{n_2}}}(f,g;x,y)\right|\leq\frac{1}{2}\left(1-\sum_{k_1=0}^{n_1}u_{n_1,k_1}^2(x)\sum_{k_2=0}^{n_2}u_{n_2,k_2}^2(y)\right)osc_{S_{\Delta_{n_1},\Delta_{n_2}}}(f)\cdot osc_{S_{\Delta_{n_1},\Delta_{n_2}}}(g),$$

for each $x, y \in [0,1]$, $f, g \in B([0,1]\times[0,1])$, where

$$osc_{S_{\Delta_{n_1},\Delta_{n_2}}}(f) := \max\{|f_{s,l}-f_{r,t}| : 0\leq s,r\leq n_1, 0\leq l,t\leq n_2\}$$

and $f_{s,l} := f\left(\frac{s}{n_1},\frac{l}{n_2}\right)$.

In this case, we need to find the minimum of the sums $\tau_{n_1}(x) := \sum_{k_1=0}^{n_1}u_{n_1,k_1}^2(x)$ and $\tau_{n_2}(y) := \sum_{k_2=0}^{n_2}u_{n_2,k_2}^2(y)$. For particular intervals $x\in\left[\frac{k_1-1}{n_1},\frac{k_1}{n_1}\right]$ and $y\in\left[\frac{k_2-1}{n_2},\frac{k_2}{n_2}\right]$, we get that

$$\tau_{n_1}(x) := \sum_{k_1=0}^{n_1}u_{n_1,k_1}^2(x) = (n_1x-k_1+1)^2+(k_1-n_1x)^2,\ \text{for } k_1=1,\ldots,n_1$$

and the same for $\tau_{n_2}(y)$. The functions $\tau_{n_1}(x)$ and $\tau_{n_2}(y)$ are minimal if and only if $x=\frac{2k_1-1}{2n_1}$, $y=\frac{2k_2-1}{2n_2}$ and the minimum value for both $\tau_{n_1}(x)$ and $\tau_{n_2}(y)$ is $\frac{1}{2}$.

Theorem 3.15. *The new Grüss-type inequality for $S_{\Delta_{n_1},\Delta_{n_2}}$ is*

$$\begin{aligned}\left|T_{S_{\Delta_{n_1},\Delta_{n_2}}}(f,g;x,y)\right|&\leq\frac{1}{2}\left(1-\sum_{k_1=0}^{n_1}u_{n_1,k_1}^2(x)\cdot\sum_{k_2=0}^{n_2}u_{n_2,k_2}^2(y)\right)\cdot osc_{S_{\Delta_{n_1},\Delta_{n_2}}}(f)\cdot osc_{S_{\Delta_{n_1},\Delta_{n_2}}}(g)\\ &\leq\frac{1}{2}\left(1-\frac{1}{4}\right)osc_{S_{\Delta_{n_1},\Delta_{n_2}}}(f)\cdot osc_{S_{\Delta_{n_1},\Delta_{n_2}}}(g)\\ &\leq\frac{3}{8}osc_{S_{\Delta_{n_1},\Delta_{n_2}}}(f)\cdot osc_{S_{\Delta_{n_1},\Delta_{n_2}}}(g).\end{aligned}$$

3.2.6 Grüss-type inequalities via discrete oscillations for more than two functions

In [24], we introduce a Grüss-type inequality on a compact metric space for more than two functions. Now, a similar result using the new approach implying discrete oscillations is obtained. This result is better than what was obtained in [24] in the sense that the oscillations of functions are relative only to certain points, while in [24] the oscillations are relative to the whole compact metric space X. Moreover, in what follows X is an arbitrary set, $B(X)$ the set of all real-valued, bounded functions on X and $f^1, \ldots, f^p \in B(X)$. Take $a_n \in \mathbb{R}$, $a_n \geq 0$, $n \geq 0$, such that $\sum\limits_{n=0}^{\infty} a_n = 1$. Furthermore, let $x_n \in X$, $n \geq 0$ be arbitrary mutually distinct points of X. For $f^k \in B(X)$ set $f_n^k := f^k(x_n)$, $k = \overline{1,p}$. Consider a positive linear functional $L : B(X) \to \mathbb{R}$, such that $L(f) := \sum\limits_{n=0}^{\infty} a_n f_n$.

In this section we will use the following notation:

$$osc_L(f^k) := \sup \left\{ |f_n^k - f_m^k| : 0 \leq n < m < \infty \right\}.$$

The following result holds, concerning the oscillations.

Lemma 3.1. *Let $B(X)$ be the set of all real-valued and bounded functions on X and $f^i \in B(X)$, $i = \overline{1,p}$. Then the following inequality holds*

$$osc_L(\prod_{k=1}^{p} f^k) \leq \sum_{i=1}^{p} osc_L(f^i) \prod_{j=1, j\neq i}^{p} \sup_{0\leq n<\infty} \{|f_n^j|\}.$$

Proof. The above inequality can be proved by induction. If we consider two functions $f^1, f^2 \in B(X)$, we have

$$\begin{aligned} &\left|f^1(x_n)f^2(x_n) - f^1(x_m)f^2(x_m)\right| \\ &= \left|f^1(x_n)(f^2(x_n) - f^2(x_m)) + (f^1(x_n) - f^1(x_m))f^2(x_m)\right| \\ &\leq \sup_{0\leq k<\infty} \{|f_k^1|\} \left|f^2(x_n) - f^2(x_m)\right| + \sup_{0\leq k<\infty} \{|f_k^2|\} \left|f^1(x_n) - f^1(x_m)\right|. \end{aligned}$$

We take the supremum on both hand-sides and get

$$osc_L(f^1 f^2) \leq osc_L(f^2) \cdot \sup_{0\leq n<\infty} \{|f_n^1|\} + osc_L(f^1) \cdot \sup_{0\leq n<\infty} \{|f_n^2|\}.$$

Now consider the inequality to be true for p and prove it for $p+1$.

$$\begin{aligned} &osc_L(f^1 \cdot f^2 \ldots f^p \cdot f^{p+1}) \\ &\leq osc_L(f^1 \ldots f^p) \cdot \sup_{0\leq n<\infty} \{|f_n^{p+1}|\} + osc_L(f^{p+1}) \cdot \sup_{0\leq n<\infty} \{|f_n^1|\} \ldots \sup_{0\leq n<\infty} \{|f_n^p|\} \\ &\leq \sum_{i=1}^{p+1} osc_L(f^i) \prod_{j=1, j\neq i}^{p+1} \sup_{0\leq n<\infty} \{|f_n^j|\}. \end{aligned}$$

This ends our proof. □

The next result is a Grüss-type inequality via discrete oscillations for more than two functions.

Theorem 3.16. *For a positive linear functional, $L : B(X) \to \mathbb{R}$, $L(f) := \sum_{n=0}^{\infty} a_n f_n$, $a_n \in \mathbb{R}$, $a_n \geq 0$, $\sum_{n=0}^{\infty} a_n = 1$, the Grüss-type inequality via discrete oscillations, involving more than two functions is*

$$\begin{aligned}&\left|L(f^1 \cdot \ldots \cdot f^p) - L(f^1) \cdot \ldots \cdot L(f^p)\right| \\ &\leq \frac{1}{2}\left(1 - \sum_{n=0}^{\infty} a_n^2\right) \cdot \sum_{i,j=1, i<j}^{p} osc_L(f^i) \cdot osc_L(f^j) \cdot \prod_{k=1, k\neq i,j}^{p} \sup_{0\leq s<\infty} \{|f_s^k|\}.\end{aligned}$$

Proof. We prove by induction the following inequality:

$$\begin{aligned}&\left|L(f^1 \cdot \ldots \cdot f^p) - L(f^1) \cdot \ldots \cdot L(f^p)\right| \\ &\leq \left(\sum_{0\leq n<m<\infty} a_n a_m\right) \cdot \sum_{i,j=1, i<j}^{p} osc_L(f^i) \cdot osc_L(f^j) \cdot \prod_{k=1, k\neq i,j}^{p} \sup_{0\leq s<\infty} \{|f_s^k|\}\end{aligned} \tag{3.10}$$

It was proved in [72] that

$$\left|L(f^1 \cdot f^2) - L(f^1) \cdot L(f^2)\right| \leq \left(\sum_{0\leq n<m<\infty} a_n a_m\right) \cdot osc_L(f^1) osc_L(f^2),$$

therefore the inequality (3.10) is true for $p = 2$. We suppose that the inequality holds for p and we prove it for $p + 1$. We have

$$\begin{aligned}&\left|L(f^1 \cdots f^{p+1}) - L(f^1) \cdots L(f^p) L(f^{p+1})\right| = \\ &\left|L(f^1 \cdots f^{p+1}) - L(f^1 \cdots f^p) L(f^{p+1}) + L(f^1 \ldots f^p) L(f^{p+1}) - L(f^1) \cdots L(f^{p+1})\right| \\ &\leq \left(\sum_{0\leq n<m<\infty} a_n a_m\right) \cdot osc_L(f^1 \ldots f^p) \cdot osc_L(f^{p+1}) \\ &+ \left|L(f^1 f^2 \ldots f^p) - L(f^1) \ldots L(f^p)\right| \cdot \sup_{0\leq s<\infty} \{|f_s^{p+1}|\} \\ &\leq \left(\sum_{0\leq n<m<\infty} a_n a_m\right) \cdot \left(\sum_{i=1}^{p} osc_L(f^i) \prod_{j=1, j\neq i}^{p} \sup_{0\leq s<\infty} \{|f_s^j|\}\right) \cdot osc_L(f^{p+1}) \\ &+ \left(\sum_{0\leq n<m<\infty} a_n a_m\right) \left(\sum_{i,j=1; i<j}^{p} osc_L(f^i) osc_L(f^j) \prod_{k=1; k\neq i,j}^{p} \sup_{0\leq s<\infty} \{|f_s^k|\}\right) \sup_{0\leq s<\infty} \{|f_s^{p+1}|\} \\ &= \left(\sum_{0\leq n<m<\infty} a_n a_m\right) \sum_{i,j=1; i<j}^{p+1} osc_L(f^i) \cdot osc_L(f^j) \cdot \prod_{k=1; k\neq i,j}^{p+1} \sup_{0\leq s<\infty} \{|f_s^k|\}.\end{aligned}$$

Using in (3.10) the following identity

$$\sum_{0\leq n<m<\infty} a_n a_m = \frac{1}{2}\left(1 - \sum_{n=0}^{\infty} a_n^2\right),$$

the theorem is proved. □

4 Estimates for the differences of positive linear operators

The results obtained and presented in this chapter are motivated by the recent results which give a solution to a problem proposed by A. Lupaş in [123]. One of the questions raised by him was to give an estimate for

$$B_n \circ \overline{\mathbb{B}}_n - \overline{\mathbb{B}}_n \circ B_n =: U_n - S_n,$$

where B_n are the Bernstein operators and $\overline{\mathbb{B}}_n$ are the Beta operators. The results given in this chapter were published in [9]. Using the Taylor expansion with Peano remainder, Gonska et al. [84] obtained more general results in regard to Lupaş' problem. To present them, let ω_k be the k-th order modulus of smoothness, $\tilde{\omega}$ the least concave majorant of ω, and $e_i(t) = t^i$, $t \in [0,1]$, $i = 0, 1, \ldots$.

Theorem 4.1. *[84] Let $A, B : C[0,1] \to C[0,1]$ be positive operators such that*

$$(A - B)\left((e_1 - x)^i; x\right) = 0 \text{ for } i = 0,1,2,3 \text{ and } x \in [0,1].$$

Then for $f \in C^3[0,1]$ there holds

$$|(A-B)(f;x)| \le \frac{1}{6}(A+B)\left(|e_1 - x|^3; x\right)\tilde{\omega}\left(f'''; \frac{1}{4}\frac{(A+B)\left((e_1-x)^4;x\right)}{(A+B)\left(|e_1-x|^3;x\right)}\right).$$

Theorem 4.2. *[84] If A and B are given as in Theorem 4.1, also satisfying $Ae_0 = Be_0 = e_0$, then for all $f \in C[0,1]$, $x \in [0,1]$ we have*

$$|(A-B)(f;x)| \le c_1 \omega_4\left(f; \sqrt[4]{\frac{1}{2}(A+B)\left((e_1-x)^4;x\right)}\right).$$

Here c_1 is an absolute constant independent of f, x, A and B.

Using the above result the following solution to Lupaş' problem was given in [84]:

Proposition 4.1. *If S_n and U_n are given as above, then*

$$|(S_n - U_n)(f;x)| \le c_1 \omega_4\left(f; \sqrt[4]{\frac{3x(1-x)}{n(n+1)}}\right).$$

Here c_1 is an absolute constant independent of n, f and x.

Gonska et al. have continued their research on the differences of positive linear operators by giving estimates for such differences in [82]-[83]. In this chapter we introduce new inequalities for such differences in terms of moduli of continuity. These results are based on some inequalities involving positive linear functionals. Firstly we establish such inequalities for smooth functions and the norms of their derivatives. Then, using some deep results from [85] and [87], we get inequalities for continuous functions in terms of moduli of smoothness. Estimates in terms of the least concave majorant of the modulus of continuity can be found in [76]. In the last section of this chapter we apply the general results to certain positive linear operators.

The results from [76, 82, 83, 84] mentioned above are based on the fact that A and B have the same moments up to a certain order. Our approach involves operators constructed with the same "fundamental functions", but with different functionals in front of them. So the difference $A - B$ is controlled by the differences of these functionals. Of particular interest is the case when one functional is the point evaluation at the barycenter of the other one; a similar case was investigated in [48] and [149] . Concerning the differences of the operators from the family U_n^ρ, we improve a result from [148], [167].

4.1 Inequalities for positive linear functionals

Let $I \subset \mathbb{R}$ be an interval and $E(I)$ a space of real-valued continuous functions on I containing the polynomials. $E_b(I)$ will be the space of all $f \in E(I)$ with

$$\|f\| := \sup\{|f(x)| : x \in I\} < \infty.$$

For $i \in \mathbb{N}$ let $e_i(x) := x^i, x \in I$. Let $F : E(I) \to \mathbb{R}$ be a positive linear functional such that $F(e_0) = 1$. Set $b^F := F(e_1)$ and

$$\mu_i^F := \frac{1}{i!} F(e_1 - b^F e_0)^i, \ i \in \mathbb{N}.$$

Then $\mu_0^F = 1$, $\mu_1^F = 0$, $\mu_2^F = \dfrac{1}{2}\left(F(e_2) - (b^F)^2\right) \geq 0$.

Lemma 4.1. *Let $f \in E(I)$ with $f'' \in E_b(I)$. Then*

$$|F(f) - f(b^F)| \leq \mu_2^F \|f''\|. \tag{4.1}$$

Proof. By using Taylor's formula we get

$$\left|f - f(b^F)e_0 - f'(b^F)(e_1 - b^F e_0)\right| \leq \frac{1}{2}\|f''\|(e_1 - b^F e_0)^2,$$

and this leads immediately to (4.1). □

Lemma 4.2. *Let $f \in E(I)$ with $f^{IV} \in E_b(I)$. Then*

$$\left|F(f) - f(b^F) - \mu_2^F f''(b^F) - \mu_3^F f'''(b^F)\right| \leq \mu_4^F \|f^{IV}\|. \tag{4.2}$$

Proof. As before, we use Taylor's formula in order to get

$$\left|f - f(b^F)e_0 - f'(b^F)(e_1 - b^F e_0) - \frac{1}{2!}f''(b^F)(e_1 - b^F e_0)^2 - \frac{1}{3!}f'''(b^F)(e_1 - b^F e_0)^3\right|$$
$$\leq \frac{1}{4!}\|f^{IV}\|(e_1 - b^F e_0)^4,$$

which leads to (4.2). □

Proposition 4.2. *Let $I = [0,1]$, $f \in C[0,1]$, $\lambda \geq 2\sqrt{\mu_2^F} > 0$. Then*

$$\left|F(f) - f(b^F)\right| \leq \frac{3}{2}\omega_2\left(f, \frac{\sqrt{\mu_2^F}}{\lambda}\right)(1+\lambda^2). \tag{4.3}$$

Proof. Let $h := \dfrac{\sqrt{\mu_2^F}}{\lambda}$. Then $0 < h \leq \dfrac{1}{2}$. Consequently (see [85]), there exists $g \in C^2[0,1]$ such that $\|f-g\| \leq \dfrac{3}{4}\omega_2(f,h)$ and $\|g''\| \leq \dfrac{3}{2h^2}\omega_2(f,h)$. Now using (4.1) we get

$$\begin{aligned}|F(f) - f(b^F)| &\leq |F(f) - F(g)| + |F(g) - g(b^F)| + |g(b^F) - f(b^F)| \\ &\leq 2\|f-g\| + \mu_2^F\|g''\| \leq \frac{3}{2}\omega_2(f,h)\left(1 + \frac{\mu_2^F}{h^2}\right),\end{aligned}$$

and this yields (4.3). □

4.2 Differences of positive linear operators

Let K be a set of non-negative integers and $p_k \in C(I)$, $p_k \geq 0$, $k \in K$, such that $\sum_{k\in K} p_k = e_0$. For each $k \in K$ let $F_k : E(I) \to \mathbb{R}$ and $G_k : E(I) \to \mathbb{R}$ be positive linear functionals such that $F_k(e_0) = G_k(e_0) = 1$. Let $D(I)$ be the set of all $f \in E(I)$ for which $\sum_{k\in K} F_k(f)p_k \in C(I)$ and $\sum_{k\in K} G_k(f)p_k \in C(I)$.

Consider the positive linear operators $V : D(I) \to C(I)$ and $W : D(I) \to C(I)$ defined, for $f \in D(I)$, by

$$V(f;x) := \sum_{k\in K} F_k(f)p_k(x) \text{ and } W(f;x) := \sum_{k\in K} G_k(f)p_k(x).$$

Denote $\sigma(x) := \sum_{k\in K}\left(\mu_2^{F_k} + \mu_2^{G_k}\right)p_k(x)$ and $\delta := \sup_{k\in K}\left|b^{F_k} - b^{G_k}\right|$.

Theorem 4.3. *Let $f \in D(I)$ with $f'' \in E_b(I)$. Then*

$$|(V-W)(f;x)| \leq \|f''\|\sigma(x) + \omega_1(f,\delta). \tag{4.4}$$

Proof. Let $x \in I$. By using (4.1) we get

$$\begin{aligned}|(V-W)(f;x)| &\leq \sum_{k\in K}|F_k(f) - G_k(f)|p_k(x) \\ &\leq \sum_{k\in K} p_k(x)\left(|F_k(f) - f(b^{F_k})| + |G_k(f) - f(b^{G_k})| + |f(b^{F_k}) - f(b^{G_k})|\right) \\ &\leq \sum_{k\in K} p_k(x)\left[(\mu_2^{F_k} + \mu_2^{G_k})\|f''\| + \omega_1\left(f, |b^{F_k} - b^{G_k}|\right)\right] \\ &\leq \|f''\|\sigma(x) + \omega_1(f,\delta).\end{aligned}$$

□

Theorem 4.4. *Suppose that* $b^{F_k} = b^{G_k} = b_k$, $k \in K$. *Let* $f \in D(I)$ *with* f'', f''', $f^{IV} \in E_b(I)$. *Then for each* $x \in I$,

$$|(V-W)(f;x)| \le \|f''\|\alpha(x) + \|f'''\|\beta(x) + \|f^{IV}\|\gamma(x), \tag{4.5}$$

where

$$\alpha(x) := \sum_{k\in K} |\mu_2^{F_k} - \mu_2^{G_k}| p_k(x), \quad \beta(x) := \sum_{k\in K} |\mu_3^{F_k} - \mu_3^{G_k}| p_k(x), \quad \gamma(x) := \sum_{k\in K} (\mu_4^{F_k} + \mu_4^{G_k}) p_k(x).$$

Proof. Using (4.2) we have

$$\begin{aligned}
|F_k(f) - G_k(f)| &\le \left| F_k(f) - f(b_k) - \mu_2^{F_k} f''(b_k) - \mu_3^{F_k} f'''(b_k) \right| \\
&+ \left| G_k(f) - f(b_k) - \mu_2^{G_k} f''(b_k) - \mu_3^{G_k} f'''(b_k) \right| \\
&+ |\mu_2^{F_k} - \mu_2^{G_k}| \cdot |f''(b_k)| + |\mu_3^{F_k} - \mu_3^{G_k}| \cdot |f'''(b_k)| \\
&\le (\mu_4^{F_k} + \mu_4^{G_k}) \|f^{IV}\| + |\mu_2^{F_k} - \mu_2^{G_k}| \cdot |f''(b_k)| + |\mu_3^{F_k} - \mu_3^{G_k}| \cdot |f'''(b_k)|,
\end{aligned}$$

and this implies (4.5). □

Theorem 4.5. *Let* $I = [0,1]$, $f \in C[0,1]$, $0 < h \le \frac{1}{2}, x \in [0,1]$. *Then*

$$|(V-W)(f;x)| \le \frac{3}{2}\left(1 + \frac{\sigma(x)}{h^2}\right)\omega_2(f,h) + \frac{5\delta}{h}\omega_1(f,h). \tag{4.6}$$

Proof. According to (4.4), we obtain for all $g \in C^2[0,1]$,

$$\begin{aligned}
|(V-W)(f;x)| &\le |V(f;x) - V(g;x)| + |V(g;x) - W(g;x)| + |W(g;x) - W(f;x)| \\
&\le 2\|f-g\| + \sigma(x)\|g''\| + \delta\|g'\|.
\end{aligned}$$

For the given h there exists (see Lemma 1.1) a function $g \in C^2[0,1]$ such that

$$\|f-g\| \le \frac{3}{4}\omega_2(f,h); \quad \|g'\| \le \frac{5}{h}\omega_1(f,h); \quad \|g''\| \le \frac{3}{2h^2}\omega_2(f,h).$$

It follows that

$$|(V-W)(f;x)| \le \frac{3}{2}\omega_2(f,h) + \frac{3\sigma(x)}{2h^2}\omega_2(f,h) + \frac{5\delta}{h}\omega_1(f,h),$$

and this yields (4.6). □

Theorem 4.6. *Let* $I = [0,1]$, $f \in C[0,1]$, $0 < h < 1$, $x \in [0,1]$ *and* $b^{F_k} = b^{G_k}$, $k \in K$. *Then*

$$|(V-W)(f;x)| \le c\left[\left(2 + \frac{\gamma(x)}{h^4}\right)\omega_4(f,h) + \frac{\beta(x)}{h^3}\omega_3(f,h) + \frac{\alpha(x)}{h^2}\omega_2(f,h)\right], \tag{4.7}$$

where c *is an absolute constant.*

Proof. According to Lemma 1.2, there exist an absolute constant c and a function $g \in C^4[0,1]$ such that

$$\|f-g\| \le c\omega_4(f,h); \quad \|g''\| \le \frac{c}{h^2}\omega_2(f,h); \quad \|g'''\| \le \frac{c}{h^3}\omega_3(f,h); \quad \|g^{IV}\| \le \frac{c}{h^4}\omega_4(f,h).$$

Now we use (4.5) in order to get

$$\begin{aligned}|(V-W)(f;x)| &\le |V(f;x)-V(g;x)|+|V(g;x)-W(g;x)|+|W(g;x)-W(f;x)|\\ &\le 2\|f-g\|+\alpha(x)\|g''\|+\beta(x)\|g'''\|+\gamma(x)\|g^{IV}\|\\ &\le 2c\omega_4(f,h)+\alpha(x)\frac{c}{h^2}\omega_2(f,h)+\beta(x)\frac{c}{h^3}\omega_3(f,h)+\gamma(x)\frac{c}{h^4}\omega_4(f,h),\end{aligned}$$

and this entails (4.7). □

4.3 Applications

4.3.1 Differences of Bernstein operators and Durrmeyer operators

If we denote

$$F_k(f):=f\left(\frac{k}{n}\right),\quad G_k(f):=(n+1)\int_0^1 p_{n,k}(t)f(t)dt,$$

the Bernstein operators and the Durrmeyer operators can be written as

$$B_n(f;x)=\sum_{k=0}^{n}F_k(f)p_{n,k}(x),\quad M_n(f;x)=\sum_{k=0}^{n}G_k(f)p_{n,k}(x).$$

Proposition 4.3. *For Bernstein operators and Durrmeyer operators the following properties hold:*

i) $|(B_n-M_n)(f;x)|\le\sigma(x)\|f''\|+\omega_1\left(f,\frac{1}{n+2}\right)$, *for* $f''\in C[0,1]$;

ii) $|(B_n-M_n)(f;x)|\le 3\omega_2(f,\sqrt{\sigma(x)})+\frac{5}{(n+2)\sqrt{\sigma(x)}}\omega_1\left(f,\sqrt{\sigma(x)}\right)$, *for* $f\in C[0,1]$,
where $\sigma(x)=\frac{1}{2(n+3)(n+2)^2}\{x(1-x)n(n-1)+n+1\}\le\frac{1}{8(n+3)}$.

Proof. We have

$$b^{F_k}=F_k(e_1)=\frac{k}{n},\quad b^{G_k}=G_k(e_1)=\frac{k+1}{n+2}.$$

Since $|b^{F_k}-b^{G_k}|=\frac{|n-2k|}{n(n+2)}$, it follows $\delta:=\max|b^{F_k}-b^{G_k}|=\frac{1}{n+2}$.

Also,

$$\begin{aligned}\mu_2^{F_k}&:=\frac{1}{2!}F_k\left(e_1-b^{F_k}e_0\right)^2=0,\\ \mu_2^{G_k}&:=\frac{1}{2!}G_k\left(e_1-b^{G_k}e_0\right)^2=\frac{(k+1)(n-k+1)}{2(n+2)^2(n+3)}.\end{aligned}$$

Then $\sigma(x):=\sum_{k=0}^{n}\left(\mu_2^{F_k}+\mu_2^{G_k}\right)p_{n,k}(x)=\frac{1}{2(n+3)(n+2)^2}\{x(1-x)n(n-1)+n+1\}$.

Using Theorem 4.3 and Theorem 4.5 the properties i) and ii) are proved. □

4.3.2 Differences of Bernstein operators and the genuine Bernstein-Durrmeyer operators

Let U_n be the genuine Bernstein-Durrmeyer operators defined in Subsection 1.6.3.

Proposition 4.4. *The Bernstein operators and the genuine Bernstein-Durrmeyer operators verify the following properties*

i) $|(B_n - U_n)(f;x)| \le \sigma(x)\|f''\|,\ f'' \in C[0,1]$,

ii) $|(B_n - U_n)(f;x)| \le 3\omega_2(f, \sqrt{\sigma(x)}),\ f \in C[0,1]$,

where $\sigma(x) = \dfrac{x(1-x)(n-1)}{2n(n+1)} \le \dfrac{1}{8(n+1)}$.

Proof. If we denote

$$\begin{aligned} F_k(f) &= f\left(\frac{k}{n}\right), \\ G_k(f) &= (n-1)\int_0^1 p_{n-2,k-1}(t)f(t)dt, \quad 1 \le k \le n-1, \\ G_0(f) &= f(0), \quad G_n(f) = f(1), \end{aligned}$$

then $B_n(f;x) = \sum\limits_{k=0}^{n} F_k(f)p_{n,k}(x)$ and $U_n(f;x) = \sum\limits_{k=0}^{n} G_k(f)p_{n,k}(x)$. We have

$$\begin{aligned} b^{F_k} &:= F_k(e_1) = \frac{k}{n}, \quad \mu_2^{F_k} := \frac{1}{2!}F_k\left(e_1 - b^{F_k}e_0\right)^2 = 0, \\ b^{G_k} &:= G_k(e_1) = \frac{k}{n}, \quad \mu_2^{G_k} := \frac{1}{2!}G_k\left(e_1 - b^{G_k}e_0\right)^2 = \frac{k(n-k)}{2(n+1)n^2}, \\ \sigma(x) &= \frac{x(1-x)(n-1)}{2n(n+1)}. \end{aligned}$$

Using Theorem 4.3 and Theorem 4.5 this proposition is proved. □

4.3.3 Differences of the composition of two Bernstein operators and the genuine Bernstein-Durrmeyer operators

Let $D_n := B_n \circ B_{n+1}$ be the composition of two Bernstein operators. Since this mapping has some similarities with U_n, we propose to calculate the differences of these two operators.

Proposition 4.5. *The following properties are verified*

i) $|(D_n - U_n)(f;x)| \le \dfrac{n-1}{n(n+1)}x(1-x)\|f''\|,\ f'' \in C[0,1]$;

ii) $|(D_n - U_n)(f;x)| \le \dfrac{x(1-x)}{2(n+1)^2}\left(\dfrac{1}{3}\|f^{(3)}\| + \dfrac{1}{8}\|f^{(4)}\|\right),\ f^{(4)} \in C[0,1]$;

iii) $|(D_n - U_n)(f;x)| \le 3\omega_2\left(f, \sqrt{\dfrac{(n-1)x(1-x)}{n(n+1)}}\right),\ f \in C[0,1]$;

iv) $|(D_n-U_n)(f;x)|\leq c\left[\dfrac{33}{16}\omega_4\left(f,\sqrt[4]{\dfrac{x(1-x)}{(n+1)^2}}\right)+\dfrac{\sqrt[4]{x(1-x)}}{6\sqrt{n+1}}\omega_3\left(f,\sqrt[4]{\dfrac{x(1-x)}{(n+1)^2}}\right)\right]$,
$f\in C[0,1]$, *where* c *is an absolute constant and* $n\geq 6$.

Proof. Denote

$$F_k(f)=B_{n+1}\left(f;\frac{k}{n}\right),$$

$$G_k(f)=(n-1)\int_0^1 p_{n-2,k-1}(t)f(t)dt,\quad 1\leq k\leq n-1,\quad G_0(f)=f(0),\quad G_n(f)=f(1).$$

Then $D_n(f;x)=B_nB_{n+1}(f;x)=\sum_{k=0}^{n}F_k(f)p_{n,k}(x)$ and $U_n(f;x)=\sum_{k=0}^{n}G_k(f)p_{n,k}(x)$. We have

$$\begin{aligned}
b^{F_k}&:=F_k(e_1)=B_{n+1}\left(e_1;\frac{k}{n}\right)=\frac{k}{n},\\
\mu_2^{F_k}&:=\frac{1}{2!}F_k\left(e_1-b^{F_k}e_0\right)^2=\frac{k(n-k)}{2n^2(n+1)},\\
\mu_3^{F_k}&:=\frac{1}{3!}F_k(e_1-b^{F_k}e_0)^3=\frac{k(n^2+2k^2-3nk)}{6n^3(n+1)^2},\\
\mu_4^{F_k}&:=\frac{1}{4!}F_k(e_1-b^{F_k}e_0)^4=\frac{1}{24(n+1)^3}\frac{k}{n}\left(1-\frac{k}{n}\right)\left[3(n-1)\frac{k}{n}\left(1-\frac{k}{n}\right)+1\right],
\end{aligned}$$

and

$$\begin{aligned}
b^{G_k}&:=G_k(e_1)=\frac{k}{n},\\
\mu_2^{G_k}&:=\frac{1}{2!}G_k\left(e_1-b^{G_k}e_0\right)^2=\frac{k(n-k)}{2(n+1)n^2},\\
\mu_3^{G_k}&:=\frac{1}{3!}G_k(e_1-b^{G_k}e_0)^3=\frac{1}{3(n+2)(n+1)}\frac{k}{n}\left(1-\frac{k}{n}\right)\left(1-\frac{2k}{n}\right),\\
\mu_4^{G_k}&:=\frac{1}{4!}G_k(e_1-b^{G_k}e_0)^4=\frac{1}{8(n+3)(n+2)(n+1)}\frac{k}{n}\left(1-\frac{k}{n}\right)\left[(n-6)\frac{k}{n}\left(1-\frac{k}{n}\right)+2\right].
\end{aligned}$$

It follows

$$\sigma(x)=\frac{(n-1)x(1-x)}{n(n+1)},\alpha(x)=0,\delta=0.$$

Since

$$\begin{aligned}
|\mu_3^{F_k}-\mu_3^{G_k}|&=\frac{n}{6(n+1)^2(n+2)}\frac{k}{n}\left(1-\frac{k}{n}\right)\left|1-2\frac{k}{n}\right|\\
&\leq\frac{n}{6(n+1)^2(n+2)}\frac{k}{n}\left(1-\frac{k}{n}\right),
\end{aligned}$$

$$\begin{aligned}
\mu_4^{F_k}+\mu_4^{G_k}&=\frac{1}{24}\frac{k}{n}\left(1-\frac{k}{n}\right)\left[\frac{3(n-1)\frac{k}{n}\left(1-\frac{k}{n}\right)+1}{(n+1)^3}+\frac{3(n-6)\frac{k}{n}\left(1-\frac{k}{n}\right)+6}{(n+3)(n+2)(n+1)}\right]\\
&\leq\frac{1}{24}\frac{k}{n}\left(1-\frac{k}{n}\right)\left[\frac{\frac{3}{4}(n-1)+1}{(n+1)^3}+\frac{\frac{3}{4}(n-6)+6}{(n+3)(n+2)(n+1)}\right]\\
&=\frac{1}{96(n+1)}\frac{k}{n}\left(1-\frac{k}{n}\right)\left[\frac{3n+1}{(n+1)^2}+\frac{3n+6}{(n+3)(n+2)}\right]
\end{aligned}$$

$$\leq \frac{1}{32(n+1)}\frac{k}{n}\left(1-\frac{k}{n}\right)\left(\frac{1}{n+1}+\frac{1}{n+3}\right)=\frac{n+2}{16(n+1)^2(n+3)}\frac{k}{n}\left(1-\frac{k}{n}\right)$$
$$\leq \frac{1}{16(n+1)^2}\frac{k}{n}\left(1-\frac{k}{n}\right),$$

we have

$$\beta(x)\leq \frac{x(1-x)(n-1)}{6(n+2)(n+1)^2},\quad \gamma(x)\leq \frac{n-1}{16n(n+1)^2}x(1-x).$$

Using theorems from the previous section this proposition is proved. □

4.3.4 The difference $U_n^\rho - U_n^r$

Let us consider the class of operators U_n^ρ defined in Subsection 1.6.4. The following result was obtained with the method presented in [83].

Proposition 4.6. *([148], [167]) Let $f\in C[0,1]$, $n\geq 1$, $\rho, r>0$, $x\in[0,1]$. The following inequality is verified*

$$|(U_n^\rho - U_n^r)(f;x)|\leq c_1\omega_2\left(f;\sqrt{\frac{1}{2}(U_n^\rho+U_n^r)(|e_1-x|^2;x)}\right)$$
$$\leq c_1\omega_2\left(f;\sqrt{\frac{1}{2}\frac{2n\rho r+(n+1)(\rho+r)+2}{(n\rho+1)(nr+1)}x(1-x)}\right).$$

Here c_1 is an absolute constant independent of f,x,ρ and r.

Another result in this direction was obtained in [148] and [167]:

Theorem 4.7. *([148], [167]) Let $f\in C[0,1]$, $n\geq 1$, $\rho, r>0$, $x\in[0,1]$. Then*

$$|(U_n^\rho - U_n^r)(f;x)|\leq \frac{9}{4}\omega_2\left(f;\sqrt{\frac{(n-1)|\rho-r|}{(n\rho+1)(nr+1)}x(1-x)}\right).$$

In the next statement we give some estimates of the difference $U_n^\rho - U_n^r$ using the results proved in the previous section:

Proposition 4.7. *The following properties hold*

i) $|(U_n^\rho - U_n^r)(f;x)|\leq \dfrac{(n-1)[2+(\rho+r)n]}{2n(1+\rho n)(1+rn)}x(1-x)\|f''\|,\ f''\in C[0,1];$

ii) $$|(U_n^\rho - U_n^r)(f;x)|\leq \frac{(n-1)|r-\rho|}{2(1+\rho n)(1+rn)}x(1-x)\left\|f''\right\|$$
$$+\frac{1}{3}x(1-x)(n-1)\frac{|r-\rho|[3+n(r+\rho)]}{(1+\rho n)(2+\rho n)(1+rn)(2+rn)}\left\|f'''\right\|$$
$$+\frac{1}{32}x(1-x)\frac{n^2(\rho^2+r^2)+4n(\rho+r)+6}{(1+\rho n)(3+\rho n)(1+rn)(3+rn)}\|f^{IV}\|, f^{(4)}\in C[0,1],\quad n\rho\geq 6,\quad nr\geq 6;$$

iii) $|(U_n^\rho - U_n^r)(f;x)|\leq 3\omega_2\left(f,\sqrt{\dfrac{2+(\rho+r)n}{2(1+\rho n)(1+rn)}x(1-x)}\right),\ f\in C[0,1].$

Proof. We have

$$
\begin{aligned}
b^{F_k^\rho} &= F_k^\rho(e_1) = \frac{k}{n};\\
\mu_2^{F_k^\rho} &= \frac{k(n-k)}{2n^2(1+\rho n)};\\
\mu_3^{F_k^\rho} &= \frac{1}{3(1+\rho n)(2+\rho n)}\frac{k}{n}\left(1-\frac{k}{n}\right)\left(1-2\frac{k}{n}\right);\\
\mu_4^{F_k^\rho} &= \frac{1}{8(1+\rho n)(2+\rho n)(3+\rho n)}\frac{k}{n}\left(1-\frac{k}{n}\right)\left[\frac{k}{n}\left(1-\frac{k}{n}\right)(n\rho-6)+2\right].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sigma(x) &= \frac{(n-1)[2+(\rho+r)n]}{2n(1+\rho n)(1+rn)}x(1-x);\\
\mu_2^{F_k^\rho}-\mu_2^{F_k^r} &= \frac{k(n-k)}{2n}\cdot\frac{r-\rho}{(1+\rho n)(1+rn)};\\
\mu_3^{F_k^\rho}-\mu_3^{F_k^r} &= \frac{1}{3}\frac{k}{n}\left(1-\frac{k}{n}\right)\left(1-2\frac{k}{n}\right)\cdot\frac{n(r-\rho)[3+n(r+\rho)]}{(1+\rho n)(2+\rho n)(1+rn)(2+rn)};\\
\mu_4^{F_k^\rho}+\mu_4^{F_k^r} &\le \frac{1}{32}\frac{k}{n}\left(1-\frac{k}{n}\right)\left[\frac{1}{(1+\rho n)(3+\rho n)}+\frac{1}{(1+rn)(3+rn)}\right]\\
&= \frac{1}{32}\frac{k}{n}\left(1-\frac{k}{n}\right)\frac{n^2(\rho^2+r^2)+4n(\rho+r)+6}{(1+\rho n)(3+\rho n)(1+rn)(3+rn)};\\
\alpha(x) &= \frac{(n-1)|r-\rho|}{2(1+\rho n)(1+rn)}x(1-x);\\
\beta(x) &\le \frac{1}{3}x(1-x)(n-1)\frac{|r-\rho|[3+n(r+\rho)]}{(1+\rho n)(2+\rho n)(1+rn)(2+rn)};\\
\gamma(x) &\le \frac{1}{32}x(1-x)\frac{n^2(\rho^2+r^2)+4n(\rho+r)+6}{(1+\rho n)(3+\rho n)(1+rn)(3+rn)}.
\end{aligned}
$$

Now (i) follows from Theorem 4.3, (ii) from Theorem 4.4, and (iii) from Theorem 4.5. □

Remark 4.1. *The second argument of ω_2 in Proposition 4.7, (iii), is less than the corresponding argument in Proposition 4.6.*

4.3.5 Differences of Bernstein operators and Kantorovich operators

Let K_n be the Kantorovich operators defined in Subsection 1.6.5.

Proposition 4.8. *The Bernstein operators and the Kantorovich operators verify the following properties*

i) $|(K_n-B_n)(f;x)| \le \dfrac{1}{24(n+1)^2}\|f''\| + \omega_1\left(f, \dfrac{1}{2(n+1)}\right),\ f'' \in C[0,1];$

ii) $|(K_n-B_n)(f;x)| \le 3\omega_2\left(f, \dfrac{1}{2\sqrt{6}(n+1)}\right) + 5\sqrt{6}\,\omega_1\left(f, \dfrac{1}{2\sqrt{6}(n+1)}\right),\ f \in C[0,1].$

Proof. If we denote $F_k(f) := (n+1)\displaystyle\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt$, then the Kantorovich operators can be written as

$$K_n(f;x) = \sum_{k=0}^{n} F_k(f)p_{n,k}(x).$$

It follows that

$$b^{F_k} = F_k(e_1) = \frac{2k+1}{2(n+1)}, \quad \mu_2^{F_k} = \frac{1}{24(n+1)^2}.$$

Therefore,

$$\delta = \max_{0\le k\le n} \frac{|n-2k|}{2n(n+1)} = \frac{1}{2(n+1)}, \quad \sigma(x) = \frac{1}{24(n+1)^2}.$$

Using Theorems 4.3 and 4.5 this proposition is proved. □

In the following we will extend these results for a generalized class of Kantorovich-type operators. Let $C_b[0,\infty)$ be the space of all real-valued continuous function on $[0,\infty)$ with $\|f\| < \infty$ and $V_n : C_b[0,\infty) \to C_b[0,\infty)$, $V_n(f;x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\varphi_{n,k}(x)$ be a positive linear operator, where $(\varphi_{n,k})_{k\ge 0}$ is a sequence of real-valued functions which verify:

i) $\varphi_{n,k}(x) \ge 0,\ k\ge 0, \quad x\in[0,\infty)$,

ii) $\varphi_{n,k} \in C[0,\infty)$;

iii) $\sum_{k=0}^{\infty} \varphi_{n,k}(x) = 1$.

Let $W_n : C_b[0,\infty) \to C_b[0,\infty)$ be the Kantorovich generalized variant of the operator V_n. Therefore,

$$W_n(f;x) = n\sum_{k=0}^{\infty} \varphi_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt. \tag{4.8}$$

Some examples of the operators of the form (4.8) are the Kantorovich variants of the Szász-Mirakjan operators and the Baskakov operators. These operators are obtained choosing $\varphi_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}$, respectively $\varphi_{n,k}(x) = (1+x)^{-n}\binom{n+k-1}{k}\left(\frac{x}{1+x}\right)^k$.

Proposition 4.9. *With the above notation,*

$$|W_n(f;x) - V_n(f;x)| \le \frac{1}{24n^2}\|f''\| + \omega_1\left(f, \frac{1}{2n}\right), \ f^{(i)} \in C_b[0,\infty),\ i\in\{0,1,2\}.$$

The proof is similar to that of Proposition 4.8, i) and we omit it.

4.3.6 Bernstein operators and generalized Bernstein operators

Let $n \ge j \ge 2$. In [32] a generalized Bernstein operator $B_{n,j}$ is defined by

$$B_{n,j}(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\left(\frac{k(k-1)\dots(k-j+1)}{n(n-1)\dots(n-j+1)}\right)^{1/j}\right), \ x\in[0,1],\ f\in C[0,1],$$

where $p_{n,k}$ are the fundamental Bernstein polynomials. The operator $B_{n,j}$ preserves the functions e_0 and e_j. In particular, if $j=1$, this is the classical Bernstein operator.

It is easy to verify that

$$0 \le \frac{k}{n} - \left(\frac{k(k-1)\dots(k-j+1)}{n(n-1)\dots(n-j+1)}\right)^{1/j} \le \frac{j-1}{n},\ k=0,1,\dots,n.$$

It follows immediately that

$$|B_n(f;x) - K_n(f;x)| \leq \omega_1\left(f, \frac{j-1}{n}\right), \ f \in C[0,1].$$

5 Operators based on Pólya distribution

In this chapter we study approximation properties of some operators based on Pólya distribution. In the first section we construct a sequence of Lupaş operators based on Pólya distribution using a function τ. This function is any function on $[0,1]$ continuously differentiable ∞ times, such that $\tau(0)=0$, $\tau(1)=1$ and $\tau'(x)>0$ for $x\in[0,1]$. Note that the Korovkin set $\{1,e_1,e_2\}$ is generalized to $\{1,\tau,\tau^2\}$ and these operators present a better degree of approximation then the original ones. We give a direct approximation theorem by means of the Ditzian-Totik modulus of smoothness and a Voronovskaja type theorem. In the second section we introduce the Bézier variant of genuine-Durrmeyer type operators having Pólya basis functions. We give a global approximation theorem in terms of second order modulus of continuity, a direct approximation theorem by means of the Ditzian-Totik modulus of smoothness and a Voronovskaja type theorem by using the Ditzian-Totik modulus of smoothness. The rate of convergence for functions whose derivatives are of bounded variation is obtained. Further, we show the rate of convergence of these operators to certain functions by illustrative graphics using the Maple algorithms. The results presented in this chapter were published in [10] and [136].

5.1 Lupaş operators based on Pólya distribution

In 1968, Stancu [163] proposed the sequence of positive linear operators $P_n^{<\alpha>}:C[0,1]\to C[0,1]$, depending on a non-negative parameter α, defined as follows

$$P_n^{<\alpha>}(f;x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)p_{n,k}^{<\alpha>}(x),\ x\in[0,1], \tag{5.1}$$

where $p_{n,k}^{<\alpha>}(x)$ is the Pólya distribution with density function given by

$$p_{n,k}^{<\alpha>}(x)=\binom{n}{k}\frac{x^{[k,-\alpha]}(1-x)^{[n-k,-\alpha]}}{1^{[n,-\alpha]}}$$

and $t^{[n,h]}:=t(t-h)\cdots(t-\overline{n-1}h)$ is the n^{th} factorial power of t with increment h.

For $\alpha=0$ these operators reduce to the classical Bernstein operators. For $\alpha=\dfrac{1}{n}$ one obtains a special case of the operators (5.1), introduced first by L. Lupaş and A. Lupaş [124], and is given by

$$P_n^{<\frac{1}{n}>}(f;x)=\frac{2n!}{(2n)!}\sum_{k=0}^{n}\binom{n}{k}f\left(\frac{k}{n}\right)(nx)_k(n-nx)_{n-k}. \tag{5.2}$$

Recently Miclăuş [130] established some approximation results for the operators (5.1) and (5.2). Gupta and Rassias [99] introduced the Durrmeyer-type integral modification for the operators (5.2) and established some direct results that include an asymptotic formula, local and global approximation results for these operators in terms of modulus of continuity. Gupta [96] defined a genuine Durrmeyer-type modiffication of the operators defined in (5.2) and obtained a Voronovskaja-type asymptotic theorem and a local approximation theorem.

The values of the test function by Lupaş operators were given as follows

Lemma 5.1. *[124] If $x \in [0,1]$, then*

i) $P_n^{<\frac{1}{n}>}(e_0;x) = 1$,

ii) $P_n^{<\frac{1}{n}>}(e_1;x) = x$,

iii) $P_n^{<\frac{1}{n}>}(e_2;x) = x^2 + \dfrac{2x(1-x)}{n+1}$.

Recently, in [130] Miclăuş proposed a new technique to obtain the values of the test function, without using the properties of Bernstein operators. Also, the moments up to fourth order are established.

Lemma 5.2. *[130] The Lupaş operators verify*

i) $P_n^{<\frac{1}{n}>}\left((e_1-x)^2;x\right) = \dfrac{2x(1-x)}{n+1}$;

ii) $P_n^{<\frac{1}{n}>}\left((e_1-x)^3;x\right) = \dfrac{6x(1-x)(1-2x)}{(n+1)(n+2)}$;

iii) $P_n^{<\frac{1}{n}>}\left((e_1-x)^4;x\right) = \dfrac{12(n^2-7n)x^2(1-x)^2+(26n-2)x(1-x)}{n(n+1)(n+2)(n+3)}$.

5.1.1 Modified Lupaş operators

In the last year there is an increasing interest in modifying linear operators so that the new versions present a better degree of approximation than the original ones. In order to make the convergence faster, King [119] proposed for the classical Bernstein operators the following new version

$$((B_nf)\circ r_n)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}(r_n(x))^k(1-r_n(x))^{n-k},$$

where r_n is a sequence of continuous functions defined on $[0,1]$ with $0 \le r_n(x) \le 1$ for each $x \in [0,1]$ and $n \in \{1,2,\dots\}$. The modified Bernstein operators preserve e_0 and e_2 and present a degree of approximation at least as good. Using the same type of technique introduced by King or new methods many authors published new results dealing with this matter. Cárdenas-Morales et al. [54] extended this result considering a family of sequences of operators $B_{n,\alpha}$ that preserve e_0 and $e_2+\alpha e_1$ with $\alpha \in [0,\infty)$. Gonska et al [81] studied the sequence $V_n^\tau : C[0,1] \to C[0,1]$ defined by

$$V_n^\tau f := (B_nf)\circ(B_n\tau)^{-1}\circ\tau,$$

where τ is a continuous strictly increasing function defined on $[0,1]$ with $\tau(0)=0$ and $\tau(1)=1$. Note that if $\tau = \dfrac{e_2+\alpha e_1}{1+\alpha}$, then $V_n^\tau = B_{n,\alpha}$ and the operators V_n^τ preserve e_0 and τ. In [53], the authors consider the sequence of linear Bernstein-type operators defined for $f \in C[0,1]$ by

$B_n(f \circ \tau^{-1}) \circ \tau$, τ being any function that is continuously differentiable ∞ times on $[0, 1]$, such that $\tau(0) = 0$, $\tau(1) = 1$ and $\tau'(x) > 0$ for $x \in [0, 1]$. Inspired by the above ideas, Acar et al. [5] studied the modified Bernstein-Durrmeyer operators. Also, the modified Szasz operators were considered recently in [35].

In the following we introduce a modification of the Lupaş operators defined in (5.2). The main properties of this new approximation process are studied in this section.

Let τ be a continuously differentiable ∞ times on $[0, 1]$, such that $\tau(0) = 0$, $\tau(1) = 1$ and $\tau'(x) > 0$ for $x \in [0, 1]$. We introduce the sequence of linear Lupaş type operators for $f \in C[0, 1]$ by

$$P_n^{<\frac{1}{n},\tau>}(f; x) = \sum_{k=0}^{n} p_{n,k}^{<\frac{1}{n},\tau>}(x)(f \circ \tau^{-1})\left(\frac{k}{n}\right), \; x \in [0, 1], \tag{5.3}$$

where

$$p_{n,k}^{<\frac{1}{n},\tau>}(x) = \frac{2n!}{(2n)!}\binom{n}{k}(n\tau(x))_k\,(n - n\tau(x))_{n-k}\,.$$

Example 5.1. *Let* $\tau(x) = \dfrac{x^2 + x}{2}$ *and* $f(x) = \cos(10x)$*,* $x \in [0, 1]$*. For* $n = 20$*, the approximation to the function* f *by* $P_n^{<\frac{1}{n}>}$ *and* $P_n^{<\frac{1}{n},\tau>}$ *is illustrated in the Figure 5.1. The error of*

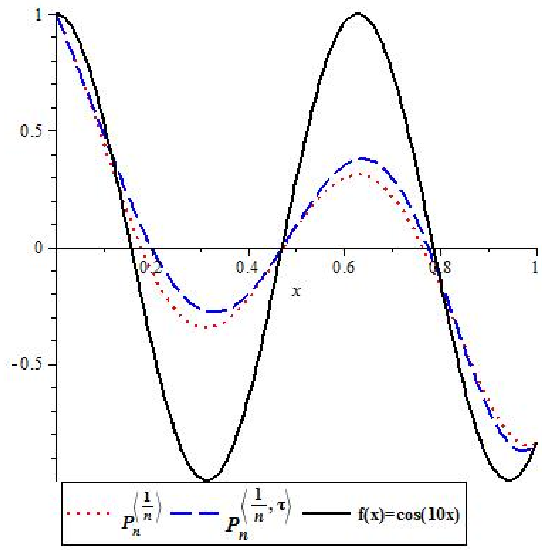

Figure 5.1: Approximation process by $P_n^{<\frac{1}{n}>}$ and $P_n^{<\frac{1}{n},\tau>}$

approximation for $P_n^{<\frac{1}{n}>}$ *and* $P_n^{<\frac{1}{n},\tau>}$ *at certain points from* $[0, 1]$ *is computed in the Table 5.1.*

Table 5.1. *Error of approximation for* $P_n^{<\frac{1}{n}>}$ *and* $P_n^{<\frac{1}{n},\tau>}$

x	$\lvert P_n^{<\frac{1}{n}>}(f;x)-f(x)\rvert$	$\lvert P_n^{<\frac{1}{n},\tau>}(f;x)-f(x)\rvert$
0.50	0.2044165867	0.1949861790
0.54	0.4438321941	0.4211072331
0.58	0.6139431834	0.5721366570
0.62	0.6884227223	0.6271754997
0.66	0.6584726760	0.5832604545
0.70	0.5345575707	0.4557703420
0.74	0.3445851032	0.2753358977
0.78	0.1287707481	0.0817297400
0.84	0.1461340572	0.1459312985
0.88	0.2418236388	0.2125904931
0.92	0.2431479318	0.1985866819
0.96	0.1534586699	0.1166998692
0.98	0.0820971154	0.0599585623

In order to prove our main results, we shall need some auxiliary results. The proofs are similar to the corresponding results for the Lupaş operators, therefore the details are omitted.

Lemma 5.3. *The modified Lupaş operator verify*

$$P_n^{<\frac{1}{n},\tau>} e_0 = e_0,\ P_n^{<\frac{1}{n},\tau>} \tau = \tau,\ P_n^{<\frac{1}{n},\tau>} \tau^2 = \tau^2 + \frac{2\tau(1-\tau)}{n+1}.$$

Let

$$\mu_{n,m}^{\tau}(x) = P_n^{<\frac{1}{n},\tau>}\left((\tau(t)-\tau(x))^m; x\right) = \sum_{k=0}^{n} p_{n,k}^{<\frac{1}{n},\tau>}(x)\left(\frac{k}{n}-\tau(x)\right)^m$$

be the central moment operator.

Lemma 5.4. *The central moment operator verifies:*

i) $\mu_{n,2}^{\tau}(x) = \dfrac{2}{n+1}\varphi_\tau^2(x);$

ii) $\mu_{n,4}^{\tau}(x) = \dfrac{12(n^2-7n)\varphi_\tau^2(x)+(26n-2)}{n(n+1)(n+2)(n+3)}\varphi_\tau^2(x),$

where $\varphi_\tau^2(x) := \tau(x)(1-\tau(x))$.

Lemma 5.5. *If* $f \in C[0,1]$, *then* $\|P_n^{<\frac{1}{n},\tau>} f\| \le \|f\|$, *where* $\|\cdot\|$ *is the uniform norm on* $C[0,1]$.

Proof. By the definition of the modified Lupaş operators (5.3) and using Lemma 5.3 we have

$$\left|P_n^{<\frac{1}{n},\tau>}(f;x)\right| \le \sum_{k=0}^{n} p_{n,k}^{<\frac{1}{n},\tau>}(x)\left|\left(f\circ\tau^{-1}\right)\left(\frac{k}{n}\right)\right| \le \|f\circ\tau^{-1}\| P_n^{<\frac{1}{n},\tau>}(e_0;x) = \|f\|.$$

□

Theorem 5.1. *If $f \in C[0,1]$, then $P_n^{<\frac{1}{n},\tau>} f$ converges to f as n tends to infinity, uniformly on $[0,1]$.*

Proof. Using Lemma 5.3, the Korovkin theorem and the fact that $\{e_0, \tau, \tau^2\}$ is an extended complete Tchebychev system on $[0,1]$, we obtain that the modified Lupaş operator $P_n^{<\frac{1}{n},\tau>} f$ converges uniformly to $f \in C[0,1]$. □

Example 5.2. *We consider $f : [0,1] \to \mathbb{R}$, $f(x) = \cos(10x)$ and $\tau(x) = \dfrac{x^2+x}{2}$. The convergence of the modified Lupaş operator to the function f is illustred in Figure 5.2. We remark that as the values of n increase, the error in the approximation of the function by the operator becomes smaller.*

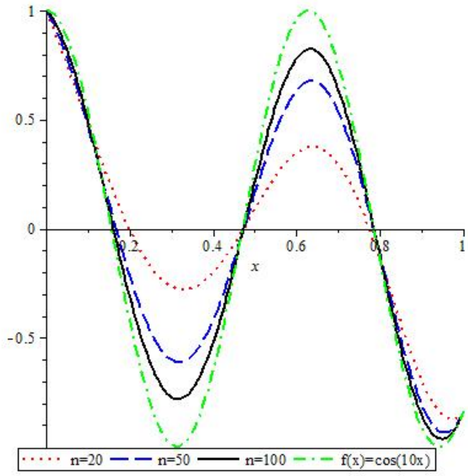

Figure 5.2: Approximation process by $P_n^{<\frac{1}{n},\tau>}$ for $n \in \{20, 50, 100\}$

Using the result of Shisha and Mond [159] we have

$$\left| P_n^{<\frac{1}{n},\tau>}(f;x) - f(x) \right| \leq \left(1 + \frac{\mu_{n,2}^{\tau}(x)}{\delta^2}\right) \omega(f,\delta), \text{ for } \delta > 0.$$

5.1.2 Approximation properties

Let $W^2 = \{g \in C[0,1] : g'' \in C[0,1]\}$, $f \in C[0,1]$ and $\delta > 0$. In order to prove the local approximation result we consider the Peetre's K-functional [144] defined by

$$K(f;\delta) = \inf_{g \in W^2} \{\|f-g\| + \delta \|g\|_{W^2}\},$$

where

$$\|f\|_{W^2} = \|f\| + \|f'\| + \|f''\|.$$

From [51, Proposition 3.4.1], there exists a constant $C_1 > 0$ independent of f and δ such that

$$K(f;\delta) \leq C_1 \left(\omega_2(f;\sqrt{\delta}) + \min\{1,\delta\}\|f\|\right), \tag{5.4}$$

where ω_2 is the second order modulus of smoothness of f defined in (1.2).

Throughout this section we assume that $\inf\limits_{x \in [0,1]} \tau'(x) \geq a$, $a \in \mathbb{R}^+$.

Theorem 5.2. *Let $f \in C[0,1]$. For the operator $P_n^{<\frac{1}{n},\tau>}$, there exists a constant $C_1 > 0$ independent of f and n such that*

$$\left| P_n^{<\frac{1}{n},\tau>}(f;x) - f(x) \right| \le C_1 \left\{ \omega_2 \left(f; \frac{\varphi_\tau(x)}{\sqrt{n+1}} \right) + \frac{\varphi_\tau^2(x)}{n+1} \|f\| \right\}.$$

Proof. Let $g \in W^2[0,1]$ and $t \in [0,1]$. Then by Taylor's expansion, we get

$$\begin{aligned} g(t) &= \left(g \circ \tau^{-1}\right)(\tau(t)) \\ &= \left(g \circ \tau^{-1}\right)(\tau(x)) + \left(g \circ \tau^{-1}\right)'(\tau(x))\,(\tau(t) - \tau(x)) + \int_{\tau(x)}^{\tau(t)} (\tau(t) - u) \left(g \circ \tau^{-1}\right)''(u)du. \end{aligned} \tag{5.5}$$

The quantity $\displaystyle\int_{\tau(x)}^{\tau(t)} (\tau(t) - u) \left(g \circ \tau^{-1}\right)''(u)du$ was estimated in [5, p. 35] as follows:

$$\begin{aligned} &\int_{\tau(x)}^{\tau(t)} (\tau(t) - u) \left(g \circ \tau^{-1}\right)''(u)du \\ &= \int_{\tau(x)}^{\tau(t)} (\tau(t) - u) \frac{g''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^2} du - \int_{\tau(x)}^{\tau(t)} (\tau(t) - u) \frac{g'(\tau^{-1}(u))\tau''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^3} du. \end{aligned} \tag{5.6}$$

From (5.5) and (5.6) we can write

$$\begin{aligned} g(t) = g(x) &+ \left(g \circ \tau^{-1}\right)'(\tau(x))\,(\tau(t) - \tau(x)) + \int_{\tau(x)}^{\tau(t)} (\tau(t) - u) \frac{g''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^2} du \\ &- \int_{\tau(x)}^{\tau(t)} (\tau(t) - u) \frac{g'(\tau^{-1}(u))\tau''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^3} du. \end{aligned} \tag{5.7}$$

Now applying $P_n^{<\frac{1}{n},\tau>}$ to both side of the relation (5.7) we can write

$$\begin{aligned} P_n^{<\frac{1}{n},\tau>}(g;x) = g(x) &+ P_n^{<\frac{1}{n},\tau>} \left(\int_{\tau(x)}^{\tau(t)} (\tau(t) - u) \frac{g''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^2} du; x \right) \\ &- P_n^{<\frac{1}{n},\tau>} \left(\int_{\tau(x)}^{\tau(t)} (\tau(t) - u) \frac{g'(\tau^{-1}(u))\tau''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^3} du; x \right). \end{aligned}$$

Therefore,

$$\left| P_n^{<\frac{1}{n},\tau>}(g;x) - g(x) \right| \le \frac{1}{2}\mu_{n,2}^\tau(x) \left(\frac{\|g''\|}{a^2} + \frac{\|g'\| \cdot \|\tau''\|}{a^3} \right) = \frac{\varphi_\tau^2(x)}{n+1} \left(\frac{\|g''\|}{a^2} + \frac{\|g'\| \cdot \|\tau''\|}{a^3} \right).$$

Let $C := \max\left\{2, \frac{1}{a^2}, \frac{\|\tau''\|}{a^3}\right\}$. By Lemma 5.5, it follows

$$\begin{aligned} \left| P_n^{<\frac{1}{n},\tau>}(f;x) - f(x) \right| &= \left| P_n^{<\frac{1}{n},\tau>}(f-g;x) \right| + \left| P_n^{<\frac{1}{n},\tau>}(g;x) - g(x) \right| + |g(x) - f(x)| \\ &\le 2\|f - g\| + \frac{\varphi_\tau^2(x)}{n+1} \left(\frac{\|g''\|}{a^2} + \frac{\|g'\| \cdot \|\tau''\|}{a^3} \right) \\ &\le C \left\{ \|f - g\| + \frac{\varphi_\tau^2(x)}{n+1} \|g\|_{W^2} \right\}. \end{aligned}$$

Taking the infimum on the right hand side over all $g \in W^2$ we obtain

$$\left| P_n^{<\frac{1}{n},\tau>}(f;x) - f(x) \right| \leq CK\left(f; \frac{\varphi_\tau^2(x)}{n+1}\right).$$

Using relation (5.4) the theorem is proved. □

Now, we establish a global approximation theorem by means of Ditzian-Totik modulus of smoothness.

Theorem 5.3. *Let $f \in C[0,1]$. Then for every $x \in (0,1)$, we have*

$$|C_n^\tau(f;x) - f(x)| \leq 2\omega_{\varphi_\tau}\left(f; \frac{2}{a\sqrt{n+1}}\right),$$

where $\varphi_\tau(x) = \sqrt{\tau(x)(1-\tau(x))}$ and ω_{φ_τ} is the Ditzian-Totik first modulus of smoothness.

Proof. Using the representation

$$g(t) = \left(g \circ \tau^{-1}\right)(\tau(t)) = \left(g \circ \tau^{-1}\right)(\tau(x)) + \int_{\tau(x)}^{\tau(t)} \left(g \circ \tau^{-1}\right)'(u)du$$

we get

$$\left| P_n^{<\frac{1}{n},\tau>}(g;x) - g(x) \right| = \left| P_n^{<\frac{1}{n},\tau>}\left(\int_{\tau(x)}^{\tau(t)} \left(g \circ \tau^{-1}\right)'(u)du \right) \right|. \tag{5.8}$$

But,

$$\begin{aligned} \left| \int_{\tau(x)}^{\tau(t)} \left(g \circ \tau^{-1}\right)'(u)du \right| &= \left| \int_x^t \frac{g'(y)}{\tau'(y)}\tau'(y)dy \right| = \left| \int_x^t \frac{\varphi_\tau(y)}{\varphi_\tau(y)} \cdot \frac{g'(y)}{\tau'(y)}\tau'(y)dy \right| \\ &\leq \frac{\|\varphi_\tau g'\|}{a} \left| \int_x^t \frac{\tau'(y)}{\varphi_\tau(y)}dy \right|, \end{aligned} \tag{5.9}$$

and

$$\begin{aligned} \left| \int_x^t \frac{\tau'(y)}{\varphi_\tau(y)}dy \right| &\leq \left| \int_x^t \left(\frac{1}{\sqrt{\tau(y)}} + \frac{1}{\sqrt{1-\tau(y)}} \right) \tau'(y)dy \right| \\ &\leq 2\left(\left| \sqrt{\tau(t)} - \sqrt{\tau(x)} \right| + \left| \sqrt{1-\tau(t)} - \sqrt{1-\tau(x)} \right| \right) \\ &= 2\left|\tau(t) - \tau(x)\right| \left(\frac{1}{\sqrt{\tau(t)} + \sqrt{\tau(x)}} + \frac{1}{\sqrt{1-\tau(t)} + \sqrt{1-\tau(x)}} \right) \\ &< 2|\tau(t) - \tau(x)| \left(\frac{1}{\sqrt{\tau(x)}} + \frac{1}{\sqrt{1-\tau(x)}} \right) \leq \frac{2\sqrt{2}|\tau(t) - \tau(x)|}{\varphi_\tau(x)}. \end{aligned} \tag{5.10}$$

From relations (5.8)-(5.10) and using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |P_n^{<\frac{1}{n},\tau>}(g;x) - g(x)| &\leq 2\sqrt{2}\frac{\|\varphi_\tau g'\|}{a\varphi_\tau(x)} P_n^{<\frac{1}{n},\tau>}\left(|\tau(t) - \tau(x)|; x\right) \\ &\leq 2\sqrt{2}\frac{\|\varphi_\tau g'\|}{a\varphi_\tau(x)} \left[P_n^{<\frac{1}{n},\tau>}\left((\tau(t) - \tau(x))^2; x\right) \right]^{1/2} \leq \frac{4}{a\sqrt{n+1}}\|\varphi_\tau g'\|. \end{aligned} \tag{5.11}$$

Using Lemma 5.5 and (5.11) it follows

$$\begin{aligned}\left|P_n^{<\frac{1}{n},\tau>}(f;x)-f(x)\right| &\leq \left|P_n^{<\frac{1}{n},\tau>}(f-g;x)\right|+|f(x)-g(x)|+\left|P_n^{<\frac{1}{n},\tau>}(g;x)-g(x)\right| \\ &\leq 2\left\{\|f-g\|+\frac{2}{a\sqrt{n+1}}\|\varphi_\tau g'\|\right\}.\end{aligned}$$

Taking infimum on the right hand side of the above inequality over all $g \in W_{\varphi_\tau}[0,1]$, we get

$$\left|P_n^{<\frac{1}{n},\tau>}(f;x)-f(x)\right| \leq 2K_{\varphi_\tau}\left(f;\frac{2}{a\sqrt{n+1}}\right).$$

Using the relation (1.4) this theorem is proven. □

5.1.3 Voronovskaja type theorem

In this section we prove a quantitative Voronovskaja type theorem for the operator $P_n^{<\frac{1}{n},\tau>}$. This result is given using the first order Ditzian-Totik modulus of smoothness. In the recent years, several researchers have made significant contribution in this area [6, 7, 63, 170].

Theorem 5.4. *For any $f \in C^2[0,1]$ and $x \in [0,1]$ the following inequalities hold*

i) $$\left|n\left[P_n^{<\frac{1}{n},\tau>}(f;x)-f(x)\right]-\frac{n}{n+1}\cdot\frac{1}{[\tau'(x)]^2}\left[f''(x)-f'(x)\frac{\tau''(x)}{\tau'(x)}\right]\varphi_\tau^2(x)\right| \leq 2\omega_{\varphi_\tau}\left((f\circ\tau^{-1})'';\frac{\sqrt{6}}{an^{1/2}}\varphi_\tau(x)\right);$$

ii) $$\left|n\left[P_n^{<\frac{1}{n},\tau>}(f;x)-f(x)\right]-\frac{n}{n+1}\cdot\frac{1}{[\tau'(x)]^2}\left[f''(x)-f'(x)\frac{\tau''(x)}{\tau'(x)}\right]\varphi_\tau^2(x)\right| \leq 2\varphi_\tau(x)\omega_{\varphi_\tau}\left((f\circ\tau^{-1})'';\frac{\sqrt{6}}{an^{1/2}}\right).$$

Proof. Let $f \in C^2[0,1]$ and $x,t \in [0,1]$. Then by Taylor's expansion, we have

$$\begin{aligned}f(t) &= \left(f\circ\tau^{-1}\right)(\tau(t)) \\ &= \left(f\circ\tau^{-1}\right)(\tau(x))+\left(f\circ\tau^{-1}\right)'(\tau(x))(\tau(t)-\tau(x))+\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\left(f\circ\tau^{-1}\right)''(u)du.\end{aligned}$$

Hence

$$\begin{aligned}&f(t)-f(x)-\left(f\circ\tau^{-1}\right)'(\tau(x))(\tau(t)-\tau(x))-\frac{1}{2}\left(f\circ\tau^{-1}\right)''(\tau(x))(\tau(t)-\tau(x))^2 \\ &= \int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\left(f\circ\tau^{-1}\right)''(u)du-\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\left(f\circ\tau^{-1}\right)''(\tau(x))du \\ &= \int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\left[\left(f\circ\tau^{-1}\right)''(u)-\left(f\circ\tau^{-1}\right)''(\tau(x))\right]du.\end{aligned}$$

Applying $P_n^{<\frac{1}{n},\tau>}$ to both sides of the above relation, we get

$$\begin{aligned}
&\left|P_n^{<\frac{1}{n},\tau>}(f;x)-f(x)-\frac{1}{2}\left[\frac{f''(x)}{(\tau'(x))^2}-f'(x)\frac{\tau''(x)}{(\tau'(x))^3}\right]\mu_{n,2}^{\tau}(x)\right|\\
&=\left|P_n^{<\frac{1}{n},\tau>}\left(\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\left[\left(f\circ\tau^{-1}\right)''(u)-\left(f\circ\tau^{-1}\right)''(\tau(x))\right]du;x\right)\right|\\
&\le P_n^{<\frac{1}{n},\tau>}\left(\left|\int_{\tau(x)}^{\tau(t)}|\tau(t)-u|\left|\left(f\circ\tau^{-1}\right)''(u)-\left(f\circ\tau^{-1}\right)''(\tau(x))\right|du\right|;x\right) \qquad (5.12)
\end{aligned}$$

For $g\in W_{\varphi_\tau}[0,1]$, we have

$$\begin{aligned}
&\left|\int_{\tau(x)}^{\tau(t)}|\tau(t)-u|\left|\left(f\circ\tau^{-1}\right)''(u)-\left(f\circ\tau^{-1}\right)''(\tau(x))\right|du\right|\\
&\le\left|\int_{\tau(x)}^{\tau(t)}|\tau(t)-u|\left|\left(f\circ\tau^{-1}\right)''(u)-\left(g\circ\tau^{-1}\right)(u)\right|du\right|\\
&+\left|\int_{\tau(x)}^{\tau(t)}|\tau(t)-u|\left|\left(g\circ\tau^{-1}\right)(u)-\left(g\circ\tau^{-1}\right)(\tau(x))\right|du\right|\\
&+\left|\int_{\tau(x)}^{\tau(t)}|\tau(t)-u|\left|\left(g\circ\tau^{-1}\right)(\tau(x))-\left(f\circ\tau^{-1}\right)''(\tau(x))\right|du\right|\\
&=\left|\int_x^t\left|\left(f\circ\tau^{-1}\right)''(\tau(y))-g(y)\right||\tau(t)-\tau(y)|\,\tau'(y)dy\right|\\
&+\left|\int_x^t|g(y)-g(x)|\,|\tau(t)-\tau(y)|\,\tau'(y)dy\right|+\left|\int_x^t\left|g(x)-\left(f\circ\tau^{-1}\right)''(\tau(x))\right||\tau(t)-\tau(y)|\,\tau'(y)dy\right|\\
&\le 2\|\left(f\circ\tau^{-1}\right)''-g\|\left|\int_x^t|\tau(t)-\tau(y)|\tau'(y)dy\right|+\left|\int_x^t\left|\int_x^y|g'(v)|dv\right||\tau(t)-\tau(y)|\tau'(y)dy\right|\\
&\le\|\left(f\circ\tau^{-1}\right)''-g\|(\tau(t)-\tau(x))^2+\|\varphi_\tau g'\|\left|\int_x^t\left|\int_x^y\frac{dv}{\varphi_\tau(v)}\right||\tau(t)-\tau(y)|\tau'(y)dy\right|.
\end{aligned}$$

Using the inequality [60, p. 141]

$$\frac{|y-v|}{v(1-v)}\le\frac{|y-x|}{x(1-x)},\ v \text{ is between } y \text{ and x },$$

we can write

$$\frac{|\tau(y)-\tau(v)|}{\tau(v)(1-\tau(v))}\le\frac{|\tau(y)-\tau(x)|}{\tau(x)(1-\tau(x))}.$$

Therefore,

$$\begin{aligned}
&\left|\int_{\tau(x)}^{\tau(t)}|\tau(t)-u|\left|\left(f\circ\tau^{-1}\right)''(u)-\left(f\circ\tau^{-1}\right)''(\tau(x))\right|du\right|\le\|\left(f\circ\tau^{-1}\right)''-g\|(\tau(t)-\tau(x))^2\\
&+\|\varphi_\tau g'\|\left|\int_x^t\left|\int_x^y\frac{|\tau(y)-\tau(x)|^{1/2}}{\tau'(v)\varphi_\tau(x)}\cdot\frac{\tau'(v)}{|\tau(y)-\tau(v)|^{1/2}}dv\right||\tau(t)-\tau(y)|\tau'(y)dy\right|\\
&\le\|\left(f\circ\tau^{-1}\right)''-g\|(\tau(t)-\tau(x))^2+2\frac{\|\varphi_\tau g'\|}{a}\varphi_\tau^{-1}(x)\left|\int_x^t|\tau(y)-\tau(x)|\,|\tau(t)-\tau(y)|\tau'(y)dy\right|
\end{aligned}$$

$$\leq \| \left(f\circ\tau^{-1}\right)'' - g\|(\tau(t)-\tau(x))^2 + 2\frac{\|\varphi_\tau g'\|}{a}\varphi_\tau^{-1}(x)\left|\int_x^t(\tau(t)-\tau(x))^2\tau'(y)dy\right|$$
$$\leq \| \left(f\circ\tau^{-1}\right)'' - g\|(\tau(t)-\tau(x))^2 + 2\frac{\|\varphi_\tau g'\|}{a}\varphi_\tau^{-1}(x)|\tau(t)-\tau(x)|^3. \tag{5.13}$$

Now combining the relations (5.12)-(5.13), applying Lemma 5.4 and the Cauchy-Schwarz inequality, we get

$$\left|P_n^{<\frac{1}{n},\tau>}(f;x) - f(x) - \frac{1}{n+1}\left[\frac{f''(x)}{(\tau'(x))^2} - f'(x)\frac{\tau''(x)}{(\tau'(x))^3}\right]\varphi_\tau^2(x)\right|$$
$$\leq \|(f\circ\tau^{-1})''-g\|P_n^{<\frac{1}{n},\tau>}\left((\tau(t)-\tau(x))^2;x\right) + 2\frac{\|\varphi_\tau g'\|}{a}\varphi_\tau^{-1}(x)P_n^{<\frac{1}{n},\tau>}\left(|\tau(t)-\tau(x)|^3;x\right)$$
$$\leq \frac{2}{n+1}\varphi_\tau^2(x)\|\left(f\circ\tau^{-1}\right)''-g\|$$
$$+\frac{2\|\varphi_\tau g'\|}{a}\varphi_\tau^{-1}(x)\left[P_n^{<\frac{1}{n},\tau>}\left((\tau(t)-\tau(x))^2;x\right)\right]^{1/2}\left[P_n^{<\frac{1}{n},\tau>}\left((\tau(t)-\tau(x))^4;x\right)\right]^{1/2}$$
$$= \frac{2}{n+1}\varphi_\tau^2(x)\|\left(f\circ\tau^{-1}\right)''-g\| + \frac{4}{n+1}\frac{\|\varphi_\tau g'\|}{a}\varphi_\tau(x)\sqrt{\frac{6(n^2-7n)\varphi_\tau^2(x)+(13n-1)}{n(n+2)(n+3)}}$$
$$= \frac{2}{n+1}\left\{\varphi_\tau^2(x)\|\left(f\circ\tau^{-1}\right)''-g\| + \frac{2\|\varphi_\tau g'\|}{a}\varphi_\tau(x)\sqrt{\frac{6(n^2-7n)\varphi_\tau^2(x)+(13n-1)}{n(n+2)(n+3)}}\right\}$$
$$\leq \frac{2}{n+1}\left\{\varphi_\tau^2(x)\|\left(f\circ\tau^{-1}\right)''-g\| + \frac{\|\varphi_\tau g'\|}{a}\varphi_\tau(x)\frac{\sqrt{6}}{n^{1/2}}\right\}.$$

Because $\varphi_\tau^2(x)\leq\varphi_\tau(x)\leq 1$, $x\in[0,1]$ we obtain

$$\left|P_n^{<\frac{1}{n},\tau>}(f;x) - f(x) - \frac{1}{(n+1)[\tau'(x)]^2}\left[f''(x) - f'(x)\frac{\tau''(x)}{\tau'(x)}\right]\varphi_\tau^2(x)\right|$$
$$\leq \frac{2}{n+1}\left\{\|(f\circ\tau^{-1})''-g\| + \frac{\sqrt{6}}{an^{1/2}}\varphi_\tau(x)\|\varphi_\tau g'\|\right\}; \tag{5.14}$$

$$\left|P_n^{<\frac{1}{n},\tau>}(f;x) - f(x) - \frac{1}{(n+1)[\tau'(x)]^2}\left[f''(x) - f'(x)\frac{\tau''(x)}{\tau'(x)}\right]\varphi_\tau^2(x)\right|$$
$$\leq \frac{2}{n+1}\varphi_\tau(x)\left\{\|(f\circ\tau^{-1})''-g\| + \frac{\sqrt{6}}{an^{1/2}}\|\varphi_\tau g'\|\right\}. \tag{5.15}$$

Taking the infimum on the right hand side of (5.14) and (5.15) over all $g\in W_{\varphi_\tau}[0,1]$, we get

$$\left|n\left[P_n^{<\frac{1}{n},\tau>}(f;x)-f(x)\right] - \frac{n}{n+1}\frac{\varphi_\tau^2(x)}{[\tau'(x)]^2}\left[f''(x)-f'(x)\frac{\tau''(x)}{\tau'(x)}\right]\right|$$
$$\leq 2K_{\varphi_\tau}\left((f\circ\tau^{-1})''; \frac{\sqrt{6}}{an^{1/2}}\varphi_\tau(x)\right)$$

and

$$\left|n\left[P_n^{<\frac{1}{n},\tau>}(f;x)-f(x)\right] - \frac{n}{n+1}\frac{\varphi_\tau^2(x)}{[\tau'(x)]^2}\left[f''(x)-f'(x)\frac{\tau''(x)}{\tau'(x)}\right]\right|$$
$$\leq 2\varphi_\tau(x)K_{\varphi_\tau}\left((f\circ\tau^{-1})''; \frac{\sqrt{6}}{an^{1/2}}\right).$$

Using (1.4) the theorem is proved. □

5.2 Bézier variant of genuine-Durrmeyer operators based on Pólya distribution

It is a well known fact that Bézier curves play an important role in computer aided designs and computer graphics systems. These curves were invented by Pierre Etienne Bézier, a French engineer at Renault. Zeng and Piriou [175] pioneered the study of Bézier variant of Bernstein operators. Subsequently, the research work on different positive linear Bézier operators motivated us to study further in this direction (cf. [31], [56], [93], [172] and [174] etc.). In this section we propose a Bézier variant of genuine-Durrmeyer operators based on Pólya distribution introduced by Agrawal et al. in [135].

5.2.1 A genuine family of Durrmeyer type operators based on Pólya distribution

Very recently, V. Gupta [96] proposed a genuine Durrmeyer-type modification of the operators (5.2) and obtained a Voronovskaja-type asymptotic theorem and a local approximation theorem. These operators are defined as follows:

$$D_n^{<\frac{1}{n}>}(f;x) := (n-1)\sum_{k=1}^{n-1} p_{n,k}^{<\frac{1}{n}>}(x)\int_0^1 p_{n-2,k-1}(t)f(t)dt + p_{n,0}^{<\frac{1}{n}>}(x)f(0) + p_{n,n}^{<\frac{1}{n}>}(x)f(1),$$

where $f \in C[0,1]$. Motivated by these results, Agrawal et al. [135] introduced a genuine Durrmeyer type integral modification of the operators given by (5.2) as follows:

$$U_n^\rho(f;x) = \sum_{k=0}^{n} F_{n,k}^\rho p_{n,k}^{(\frac{1}{n})}(x),\ \ \rho > 0, \tag{5.16}$$

where

$$F_{n,k}^\rho = \begin{cases} \int_0^1 f(t)\mu_{n,k}^\rho(t)dt, & 1 \le k \le n-1 \\ f(0), & k=0 \\ f(1), & k=n, \end{cases}$$

and

$$\mu_{n,k}^\rho(t) = \frac{t^{k\rho-1}(1-t)^{(n-k)\rho-1}}{B(k\rho,(n-k)\rho)},$$

$B(m,n)$ being the Euler's Beta-function.

For $\rho = 1$, the operators U_n^ρ reduce to the operators defined by Gupta [96] and when $\rho \to \infty$, these operators reduce to the operators considered by Lupaş and Lupaş [124], in view of the fact that $F_{n,k}^\rho \to f\left(\frac{k}{n}\right)$, as shown by Gonska and Păltănea [79].

Lemma 5.6. *[135] For $U_n^\rho(t^m;x)$, $m = 0,1,2$, one has*

i) $U_n^\rho(1;x) = 1$,

ii) $U_n^\rho(t;x) = x$,

iii) $U_n^\rho(t^2;x) = \dfrac{n\rho}{n\rho+1}\left(x^2 + \dfrac{2x(1-x)}{n+1}\right) + \dfrac{x}{n\rho+1}$.

Consequently, for every $x \in [0,1]$ it follows

$$U_n^\rho((t-x)^2;x) \le \frac{2\rho+1}{n\rho+1}\phi^2(x) = \delta_{n,\rho}^2(x),$$

where $\phi^2(x) = x(1-x)$.

Lemma 5.7. *For every $f \in C[0,1]$, we have*

$$\|U_n^\rho f\| \le \|f\|.$$

Applying Lemma 5.6, the proof of this lemma easily follows. Hence the details are omitted.

5.2.2 Bézier variant of genuine-Durrmeyer operators

We propose a Bézier variant of the operators given by (5.16) as

$$U_{n,\alpha}^\rho(f;x) = \sum_{k=0}^{n} F_{n,k}^\rho Q_{n,k}^{(\alpha)}(x), \tag{5.17}$$

where $Q_{n,k}^{(\alpha)}(x) = [J_{n,k}(x)]^\alpha - [J_{n,k+1}(x)]^\alpha$, $\alpha \ge 1$ with $J_{n,k}(x) = \sum_{j=k}^n p_{n,j}^{(1/n)}(x)$, when $k \le n$ and 0 otherwise. Clearly, $U_{n,\alpha}^\rho$ is a sequence of linear positive operators. If $\alpha = 1$, then the operators $U_{n,\alpha}^\rho$ reduce to the operators U_n^ρ.

Lemma 5.8. *a) Let $f \in C[0,1]$. Then we have $\|U_{n,\alpha}^\rho f\| \le \|f\|$.*
b) Let $x \in [0,1]$ and $f \in C[0,1]$ such that $f \ge 0$ on $[0,1]$. Then we have $U_{n,\alpha}^\rho(f;x) \le \alpha U_n^\rho(f;x)$.

Proof. a) By (5.17) and (5.16), we obtain

$$|U_{n,\alpha}^\rho(f;x)| \le \sum_{k=0}^n |F_{n,k}^\rho| Q_{n,k}^{(\alpha)}(x) \le \|f\| \sum_{k=0}^n Q_{n,k}^{(\alpha)}(x) = \|f\|,$$

because

$$U_{n,\alpha}^\rho(1;x) = \sum_{k=0}^n Q_{n,k}^{(\alpha)}(x) = \sum_{k=0}^n \{[J_{n,k}(x)]^\alpha - [J_{n,k+1}(x)]^\alpha\} = [J_{n,0}(x)]^\alpha = \left[\sum_{j=0}^n p_{n,j}^{(1/n)}(x)\right]^\alpha = 1.$$

b) Using the inequality $|a^\alpha - b^\alpha| \le \alpha|a-b|$, where $0 \le a, b \le 1$ and $\alpha \ge 1$, from the definition of $Q_{n,k}^{(\alpha)}(x)$ we obtain

$$0 < [J_{n,k}(x)]^\alpha - [J_{n,k+1}(x)]^\alpha \le \alpha(J_{n,k}(x) - J_{n,k+1}(x)) = \alpha p_{n,k}^{(1/n)}(x).$$

Hence, in view of the definition of $U_{n,\alpha}^\rho$ and the positivity of f, we get $U_{n,\alpha}^\rho(f;x) \le \alpha U_n^\rho(f;x)$, which was to be proved. □

The operators $U_{n,\alpha}^\rho$ can be expressed in an integral form as follows:

$$U_{n,\alpha}^\rho(f;x) = \int_0^1 K_{n,\alpha}^\rho(x,t) f(t)\,dt, \tag{5.18}$$

where the kernel $K_{n,\alpha}^\rho$ is given by

$$K_{n,\alpha}^\rho(x,t) = \sum_{k=1}^{n-1} Q_{n,k}^{(\alpha)}(x)\mu_{n,k}^\rho(t) + Q_{n,0}^{(\alpha)}(x)\delta(t) + Q_{n,n}^{(\alpha)}(x)\delta(1-t),$$

$\delta(u)$ being the Dirac-delta function.

Lemma 5.9. *For a fixed $x \in (0,1)$ and sufficiently large n, we have*

i) $\xi^{\rho}_{n,\alpha}(x,y) = \int_0^y K^{\rho}_{n,\alpha}(x,t)dt \leq \alpha \dfrac{\delta^2_{n,\rho}(x)}{(x-y)^2}$, $0 \leq y < x$,

ii) $1 - \xi^{\rho}_{n,\alpha}(x,z) = \int_z^1 K^{\rho}_{n,\alpha}(x,t)dt \leq \alpha \dfrac{\delta^2_{n,\rho}(x)}{(z-x)^2}$, $x < z < 1$,

where $\delta_{n,\rho}(x)$ is defined in Lemma 5.6.

Proof. (i) Using Lemmas 5.6 and 5.8, we get

$$\begin{aligned} \xi^{\rho}_{n,\alpha}(x,y) &= \int_0^y K^{\rho}_{n,\alpha}(x,t)dt \leq \int_0^y \left(\frac{x-t}{x-y}\right)^2 K^{\rho}_{n,\alpha}(x,t)dt \\ &\leq \frac{U^{\rho}_{n,\alpha}((t-x)^2;x)}{(x-y)^2} \leq \alpha \frac{U^{\rho}_{n}((t-x)^2;x)}{(x-y)^2} \leq \alpha \frac{\delta^2_{n,\rho}(x)}{(x-y)^2}. \end{aligned}$$

The proof of (ii) is similar, therefore the details are omitted. □

5.2.3 Approximation properties

The study of the rate of convergence for the functions with a derivative of bounded variation is an active area of research in approximation theory. Recently Ispir et al. [106] considered the Kantorovich modification of Lupaş operators based on Pólya distribution and estimated the rate of convergence for absolutely continuous functions having a derivative equivalent with a function of bounded variation. Very recently, the same problem has been investigated for the Bézier variant of summation integral type operators having Pólya and Bernstein basis functions and for the modified Srivastava-Gupta operators by Agrawal et al. [31] and Maheshwari [127] respectively. In this direction, significant contribution are due to (cf. [4], [30], [50], [49], [100], [105], and [118] etc.)

The aim of this section is to study some approximation properties of the operators (5.17), to investigate a direct approximation result, a global approximation theorem, a quantitative Voronovskaja type theorem and the rate of convergence for functions f whose derivative f' are of bounded variation on $[0,1]$. Lastly, we show the rate of convergence of these operators by some graphics to certain functions.

Let $W^2[0,1] = \left\{ g \in C[0,1] : g'' \in C[0,1] \right\}$, $f \in C[0,1]$ and $\delta > 0$. It is well-known the following result that gives the connection between the Peetre's K-functional K_2 and the second modulus of smoothness ω_2 (see [59]):

$$K_2(f;\delta) \leq C\omega_2(f;\sqrt{\delta}), \tag{5.19}$$

In the following we establish a direct approximation theorem for the operators $U^{\rho}_{n,\alpha}$, using the second order modulus of smoothness and the classical modulus of continuity.

Theorem 5.5. *Let $f \in C[0,1]$ and $x \in [0,1]$. Then there exists an absolute constant $C > 0$ such that*

$$|U^{\rho}_{n,\alpha}(f;x) - f(x)| \leq C\alpha\omega_2(f;\delta_{n,\rho}(x)) + \omega(f;\sqrt{\alpha}\delta_{n,\rho}(x)),$$

where $\delta_{n,\rho}(x)$ is defined in Lemma 5.6.

Proof. In view of (5.17) and (5.16), we have

$$\begin{aligned} 0 \le U^{\rho}_{n,\alpha}(t;x) &= \sum_{k=1}^{n-1} Q^{(\alpha)}_{n,k}(x) \int_0^1 t\mu^{\rho}_{n,k}(t)dt + Q^{(\alpha)}_{n,n}(x) \\ &= \sum_{k=1}^{n-1} Q^{(\alpha)}_{n,k}(x) \frac{B(k\rho+1,(n-k)\rho)}{B(k\rho,(n-k)\rho)} + Q^{(\alpha)}_{n,n}(x) \\ &= \sum_{k=1}^{n-1} \frac{k}{n} Q^{(\alpha)}_{n,k}(x) + Q^{(\alpha)}_{n,n}(x) \le \sum_{k=0}^{n} Q^{(\alpha)}_{n,k}(x) = 1 \end{aligned}$$

for all $x \in [0,1]$. This means that we can introduce the auxiliary operators $\overline{U}^{\rho}_{n,\alpha}$ defined by

$$\overline{U}^{\rho}_{n,\alpha}(f;x) = U^{\rho}_{n,\alpha}(f;x) - f(U^{\rho}_{n,\alpha}(t;x)) + f(x),\ x \in [0,1]. \tag{5.20}$$

Then the operators $\overline{U}^{\rho}_{n,\alpha}$ are linear and preserve the linear functions:

$$\overline{U}^{\rho}_{n,\alpha}(at+b;x) = ax+b.$$

Let $g \in W^2[0,1]$ and $t \in [0,1]$. Then, by Taylor's expansion, we have

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t (t-u)g''(u)du.$$

Now applying $\overline{U}^{\rho}_{n,\alpha}$ to both sides of the previous equation, we get, by (5.20), that

$$\begin{aligned} \overline{U}^{\rho}_{n,\alpha}(g;x) - g(x) &= \overline{U}^{\rho}_{n,\alpha}\left(\int_x^t (t-u)g''(u)du;x\right) \\ &= U^{\rho}_{n,\alpha}\left(\int_x^t (t-u)g''(u)du;x\right) - \int_x^{U^{\rho}_{n,\alpha}(t;x)} \left(U^{\rho}_{n,\alpha}(t;x)-u\right) g''(u)du. \end{aligned}$$

Hence

$$\begin{aligned} |\overline{U}^{\rho}_{n,\alpha}(g;x) - g(x)| &\le \|g''\| U^{\rho}_{n,\alpha}\left((t-x)^2;x\right) + \|g''\| \left(U^{\rho}_{n,\alpha}(t;x)-x\right)^2 \\ &= \|g''\| \left\{ U^{\rho}_{n,\alpha}\left((t-x)^2;x\right) + \left(U^{\rho}_{n,\alpha}(t-x;x)\right)^2 \right\}. \end{aligned}$$

Using Cauchy-Schwarz inequality and Lemmas 5.6 and 5.8, we obtain

$$|\overline{U}^{\rho}_{n,\alpha}(g;x) - g(x)| \le \|g''\| \left\{ \alpha\delta^2_{n,\rho}(x) + U^{\rho}_{n,\alpha}\left((t-x)^2;x\right) \right\} \le 2\alpha\delta^2_{n,\rho}(x)\|g''\|. \tag{5.21}$$

On the other hand, by (5.20) and Lemma 5.8, we have for each $f \in C[0,1]$ and $x \in [0,1]$ that

$$|\overline{U}^{\rho}_{n,\alpha}(f;x)| \le |U^{\rho}_{n,\alpha}(f;x)| + |f(U^{\rho}_{n,\alpha}(t;x))| + |f(x)| \le 3\|f\|.$$

Hence

$$\|\overline{U}^{\rho}_{n,\alpha}f\| \le 3\|f\|. \tag{5.22}$$

Now, combining (5.20)-(5.22), we obtain

$$\begin{aligned} |U^{\rho}_{n,\alpha}(f;x) - f(x)| &\le |\overline{U}^{\rho}_{n,\alpha}(f;x) - f(x)| + |f(U^{\rho}_{n,\alpha}(t;x)) - f(x)| \\ &\le |\overline{U}^{\rho}_{n,\alpha}(f-g;x) - (f-g)(x)| + |\overline{U}^{\rho}_{n,\alpha}(g;x) - g(x)| + \omega(f;|U^{\rho}_{n,\alpha}(t;x)-x|) \\ &\le 4\|f-g\| + 2\alpha\delta^2_{n,\rho}(x)\|g''\| + \omega(f;U^{\rho}_{n,\alpha}(|t-x|;x)) \\ &\le 4\|f-g\| + 2\alpha\delta^2_{n,\rho}(x)\|g''\| + \omega(f;(U^{\rho}_{n,\alpha}((t-x)^2;x))^{1/2}) \\ &\le 4\alpha\left\{\|f-g\| + \delta^2_{n,\rho}(x)\|g''\|\right\} + \omega(f;\sqrt{\alpha}\delta_{n,\rho}(x)). \end{aligned}$$

Taking the infimum on the right-hand side over all $g \in W^2[0,1]$ and using (5.19), we arrive at the assertion of the theorem. $\square$

Now, we establish a global approximation theorem by means of Ditzian-Totik modulus of smoothness.

Theorem 5.6. *Let f be in $C[0,1]$ and $\phi(x) = \sqrt{x(1-x)}$, then for every $x \in [0,1]$, we have*

$$|U_{n,\alpha}^{\rho}(f;x) - f(x)| \leq C\omega_{\phi}\left(f; \sqrt{\frac{2\rho+1}{n\rho+1}}\right),$$

where C is a constant independent of n and x.

Proof. Using the representation

$$g(t) = g(x) + \int_x^t g'(u)du,$$

we get

$$\left|U_{n,\alpha}^{\rho}(g;x) - g(x)\right| = \left|U_{n,\alpha}^{\rho}\left(\int_x^t g'(u)du; x\right)\right|. \tag{5.23}$$

For any $x \in (0,1)$ and $t \in [0,1]$ we find that

$$\left|\int_x^t g'(u)du\right| \leq ||\phi g'||\left|\int_x^t \frac{1}{\phi(u)}du\right|. \tag{5.24}$$

But,

$$\begin{aligned} \left|\int_x^t \frac{1}{\phi(u)}du\right| = \left|\int_x^t \frac{1}{\sqrt{u(1-u)}}du\right| &\leq \left|\int_x^t \left(\frac{1}{\sqrt{u}} + \frac{1}{\sqrt{1-u}}\right)du\right| \\ &\leq 2\left(\ |\ \sqrt{t} - \sqrt{x}\ | + |\ \sqrt{1-t} - \sqrt{1-x}\ |\ \right) \\ &= 2|t-x|\left(\frac{1}{\sqrt{t}+\sqrt{x}} + \frac{1}{\sqrt{1-t}+\sqrt{1-x}}\right) \\ &< 2|t-x|\left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{1-x}}\right) \leq \frac{2\sqrt{2}\ |t-x|}{\phi(x)}. \end{aligned} \tag{5.25}$$

Combining (5.23)-(5.25) and using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |U_{n,\alpha}^{\rho}(g;x) - g(x)| &< 2\sqrt{2}||\phi g'||\phi^{-1}(x)U_{n,\alpha}^{\rho}(|t-x|;x) \\ &\leq 2\sqrt{2}||\phi g'||\phi^{-1}(x)\left(U_{n,\alpha}^{\rho}((t-x)^2;x)\right)^{1/2} \\ &\leq 2\sqrt{2}||\phi g'||\phi^{-1}(x)\left(\alpha\ U_{n}^{\rho}((t-x)^2;x)\right)^{1/2}. \end{aligned}$$

Now using Lemma 5.6, we get

$$|U_{n,\alpha}^{\rho}(g;x) - g(x)| \leq 2\sqrt{2\alpha}\sqrt{\frac{2\rho+1}{n\rho+1}}\|\phi g'\|. \tag{5.26}$$

Using Lemma 5.8 and (5.26) we can write

$$
\begin{aligned}
\mid U_{n,\alpha}^{\rho}(f;x)-f(x)\mid &\leq \mid U_{n,\alpha}^{\rho}(f-g;x)\mid + |f(x)-g(x)| + \mid U_{n,\alpha}^{\rho}(g;x)-g(x)\mid \\
&\leq 2\sqrt{2\alpha}\bigg(||f-g|| + \sqrt{\frac{2\rho+1}{n\rho+1}}||\phi g'||\bigg).
\end{aligned}
$$

Taking infimum on the right hand side of the above inequality over all $g \in W_{\phi}[0,1]$, we get

$$
|U_{n,\alpha}^{\rho}(f;x)-f(x)| \leq 2\sqrt{2\alpha}K_{\phi}\bigg(f;\sqrt{\frac{2\rho+1}{n\rho+1}}\bigg).
$$

Using the relation (1.4) this theorem is proven. □

In the following we prove a quantitative Voronovskaja type theorem for the operator $U_{n,\alpha}^{\rho}$. This result is given using the first order Ditzian-Totik modulus of smoothness.

Theorem 5.7. *For any $f \in C^2[0,1]$ the following inequalities hold*

i) $|n\mathcal{E}\left(U_{n,\alpha}^{\rho};f;x\right)| \leq C\omega_{\phi}\left(f'',\phi(x)n^{-1/2}\right)$,

ii) $|n\mathcal{E}\left(U_{n,\alpha}^{\rho};f;x\right)| \leq C\phi(x)\omega_{\phi}\left(f'',n^{-1/2}\right)$,

where

$$
\mathcal{E}\left(U_{n,\alpha}^{\rho};f;x\right) := U_{n,\alpha}^{\rho}(f;x)-f(x)-f'(x)U_{n,\alpha}^{\rho}(t-x;x)-\frac{1}{2}f''(x)U_{n,\alpha}^{\rho}\left((t-x)^2;x\right).
$$

Proof. Let $f \in C^2[0,1]$ be given and $t,x \in [0,1]$. Then by Taylor's expansion, we have

$$
f(t)-f(x) = (t-x)f'(x) + \int_x^t (t-u)f''(u)du.
$$

Hence

$$
\begin{aligned}
f(t)-f(x)-(t-x)f'(x)-\frac{1}{2}(t-x)^2f''(x) &= \int_x^t (t-u)f''(u)du - \int_x^t (t-u)f''(x)du \\
&= \int_x^t (t-u)[f''(u)-f''(x)]du.
\end{aligned}
$$

Applying $U_{n,\alpha}^{\rho}(\cdot;x)$ to both sides of the above relation, we get

$$
\left|\mathcal{E}\left(U_{n,\alpha}^{\rho};f;x\right)\right| \leq U_{n,\alpha}^{\rho}\left(\left|\int_x^t |t-u||f''(u)-f''(x)|du\right|;x\right). \tag{5.27}
$$

The quantity $\left|\int_x^t \left|f''(u)-f''(x)\right||t-u|du\right|$ was estimated in [63, p. 337] as follows:

$$
\left|\int_x^t |f''(u)-f''(x)||t-u|du\right| \leq 2\|f''-g\|(t-x)^2 + 2\|\phi g'\|\phi^{-1}(x)|t-x|^3, \tag{5.28}
$$

where $g \in W_{\phi}[0,1]$.

We have

$$
U_n^{\rho}\left((t-x)^4;x\right) = \phi^2(x)\frac{A(\rho,n)\phi^2(x)+B(\rho,n)}{(n+1)(n+2)(n+3)(1+\rho n)(2+\rho n)(3+\rho n)},
$$

where

$$\begin{aligned}A(\rho,n) &= -3\left[-\rho(2\rho+1)^2n^4+2(14\rho^3+14\rho^2+9\rho+3)n^3+(120\rho^2+109\rho+36)n^2\right.\\&\quad\left.+6(23\rho+11)n+36\right],\\B(\rho,n) &= 2\left[(13\rho^3+18\rho^2+11\rho+3)n^3+(-\rho^3+54\rho^2+55\rho+18)n^2+33(2\rho+1)n+18\right].\end{aligned}$$

Therefore, there exists a constant $C>0$ such that

$$U_n^\rho\left((t-x)^4;x\right)\leq \frac{C}{n^2}\phi^2(x). \tag{5.29}$$

Now combining (5.27)-(5.29) and applying Lemmas 5.8 and 5.6, the Cauchy-Schwarz inequality, we get

$$\begin{aligned}&\left|\mathcal{E}\left(U_{n,\alpha}^\rho;f;x\right)\right|\leq 2\|f''-g\|U_{n,\alpha}^\rho\left((t-x)^2;x\right)+2\|\phi g'\|\phi^{-1}(x)U_{n,\alpha}^\rho\left(|t-x|^3;x\right)\\&\leq 2\|f''-g\|\alpha\delta_{n,\rho}^2(x)+2\alpha\|\phi g'\|\phi^{-1}(x)\left\{U_n^\rho(t-x)^2;x\right\}^{1/2}\left\{U_n^\rho\left((t-x)^4;x\right)\right\}^{1/2}\\&\leq 2\|f''-g\|\alpha\delta_{n,\rho}^2(x)+2\alpha\frac{C}{n}\|\phi g'\|\delta_{n,\rho}(x)\leq C\left\{\delta_{n,\rho}^2(x)\|f''-g\|+\frac{1}{n}\delta_{n,\rho}(x)\|\phi g'\|\right\}\\&=C\left\{\frac{2\rho+1}{n\rho+1}\phi^2(x)\|f''-g\|+\frac{1}{n}\sqrt{\frac{2\rho+1}{n\rho+1}}\phi(x)\|\phi g'\|\right\}\\&\leq\frac{C}{n}\left\{\phi^2(x)\|f''-g\|+n^{-1/2}\phi(x)\|\phi g'\|\right\}.\end{aligned}$$

Since $\phi^2(x)\leq\phi(x)\leq 1, x\in[0,1]$, we obtain

$$\left|\mathcal{E}\left(U_{n,\alpha}^\rho;f;x\right)\right|\leq\frac{C}{n}\left\{\|f''-g\|+n^{-1/2}\phi(x)\|\phi g'\|\right\}.$$

Also, the following inequality can be obtained

$$\left|\mathcal{E}\left(U_{n,\alpha}^\rho;f;x\right)\right|\leq\frac{C}{n}\phi(x)\left\{\|f''-g\|+n^{-1/2}\|\phi g'\|\right\}.$$

Taking the infimum on the right hand side of the above relations over $g\in W_\phi[0,1]$, we get

$$\left|n\mathcal{E}\left(U_{n,\alpha}^\rho;f;x\right)\right|\leq\begin{cases}CK_\phi\left(f'';\phi(x)n^{-1/2}\right),\\C\phi(x)K_\phi\left(f'';n^{-1/2}\right).\end{cases}$$

Using (1.4) the theorem is proved. □

Lastly we discuss the approximation of functions with a derivative of bounded variation on $[0,1]$. Let $DBV[0,1]$ denote the class of differentiable functions f defined on $[0,1]$, whose derivatives are of bounded variation on $[0,1]$. The functions $f\in DBV[0,1]$ has the following representation

$$f(x)=\int_0^x g(t)dt+f(0),$$

where $g\in BV[0,1]$, i.e., g is a function of bounded variation on $[0,1]$.

Theorem 5.8. *Let $f \in DBV[0,1]$. Then, for every $x \in (0,1)$ and sufficiently large n, we have*

$$|U^{\rho}_{n,\alpha}(f;x) - f(x)| \leq \left\{|f'(x+) + \alpha f'(x-)| + \alpha|f'(x+) - f'(x-)|\right\} \frac{\sqrt{\alpha}}{\alpha+1}\delta_{n,\rho}(x)$$

$$+\frac{\alpha\delta^2_{n,\rho}(x)}{x}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x-x/k}^{x} f'_x\right) + \frac{x}{\sqrt{n}}\left(\bigvee_{x-x/\sqrt{n}}^{x} f'_x\right)$$
$$+\frac{\alpha\delta^2_{n,\rho}(x)}{1-x}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x}^{x+(1-x)/k} f'_x\right) + \frac{1-x}{\sqrt{n}}\left(\bigvee_{x}^{x+(1-x)/\sqrt{n}} f'_x\right),$$

where $\bigvee_a^b f$ denotes the total variation of f on $[a,b]$ and f'_x is defined by

$$f'_x(t) = \begin{cases} f'(t) - f'(x-), & 0 \leq t < x \\ 0, & t = x \\ f'(t) - f'(x+) & x < t \leq 1. \end{cases} \tag{5.30}$$

Proof. Since $U^{\rho}_{n,\alpha}(1;x) = 1$, using (5.18), for every $x \in (0,1)$ we get

$$U^{\rho}_{n,\alpha}(f;x) - f(x) = \int_0^1 K^{\rho}_{n,\alpha}(x,t)(f(t) - f(x))dt = \int_0^1 K^{\rho}_{n,\alpha}(x,t)\int_x^t f'(u)dudt. \tag{5.31}$$

For any $f \in DBV[0,1]$, from (5.30) we may write

$$f'(u) = f'_x(u) + \frac{1}{\alpha+1}(f'(x+) + \alpha f'(x-)) \tag{5.32}$$
$$+\frac{1}{2}(f'(x+)-f'(x-))\left(sgn(u-x)+\frac{\alpha-1}{\alpha+1}\right)+\delta_x(u)[f'(u)-\frac{1}{2}(f'(x+)+f'(x-))],$$

where

$$\delta_x(u) = \begin{cases} 1\ ,\ u = x \\ 0\ ,\ u \neq x \end{cases}.$$

Obviously,

$$\int_0^1\left(\int_x^t\left(f'(u) - \frac{1}{2}(f'(x+) + f'(x-))\right)\delta_x(u)du\right)K^{\rho}_{n,\alpha}(x,t)dt = 0. \tag{5.33}$$

Using (5.18), we get

$$\begin{aligned} A_1 &= \int_0^1\left(\int_x^t \frac{1}{\alpha+1}(f'(x+) + \alpha f'(x-))du\right)K^{\rho}_{n,\alpha}(x,t)dt \\ &= \frac{1}{\alpha+1}(f'(x+) + \alpha f'(x-))\int_0^1 (t-x)K^{\rho}_{n,\alpha}(x,t)dt \\ &= \frac{1}{\alpha+1}(f'(x+) + \alpha f'(x-))U^{\rho}_{n,\alpha}((t-x);x) \end{aligned} \tag{5.34}$$

and

$$A_2 = \int_0^1 K^{\rho}_{n,\alpha}(x,t)\left(\int_x^t \frac{1}{2}(f'(x+)-f'(x-))\left(sgn(u-x)+\frac{\alpha-1}{\alpha+1}\right)du\right)dt \tag{5.35}$$
$$= \frac{1}{2}\left(f'(x+)-f'(x-)\right)\left[-\int_0^x\left(\int_t^x\left(sgn(u-x)+\frac{\alpha-1}{\alpha+1}\right)du\right)K^{\rho}_{n,\alpha}(x,t)dt\right.$$
$$\left.+\int_x^1\left(\int_x^t\left(sgn(u-x)+\frac{\alpha-1}{\alpha+1}\right)du\right)K^{\rho}_{n,\alpha}(x,t)dt\right]$$
$$= \frac{\alpha}{\alpha+1}\left(f'(x+)-f'(x-)\right)\int_0^1 |t-x|\,K^{\rho}_{n,\alpha}(x,t)dt$$
$$= \frac{\alpha}{\alpha+1}\left(f'(x+)-f'(x-)\right)U^{\rho}_{n,\alpha}\left(|t-x|\,;x\right).$$

Using Lemma 5.8, the equalities (5.31)-(5.35) and applying Cauchy-Schwarz inequality, we obtain

$$|U^{\rho}_{n,\alpha}(f;x)-f(x)|\leq\frac{1}{\alpha+1}|f'(x+)+\alpha f'(x-)|\sqrt{\alpha}\delta_{n,\rho}(x)+\frac{\alpha}{\alpha+1}|f'(x+)-f'(x-)|\sqrt{\alpha}\delta_{n,\rho}(x) \tag{5.36}$$
$$+\left|\int_0^x\left(\int_x^t f'_x(u)du\right)K^{\rho}_{n,\alpha}(x,t)dt\right|+\left|\int_x^1\left(\int_x^t f'_x(u)du\right)K^{\rho}_{n,\alpha}(x,t)dt\right|.$$

Now, let

$$A^{\rho}_{n,\alpha}(f'_x,x)=\int_0^x\left(\int_x^t f'_x(u)du\right)K^{\rho}_{n,\alpha}(x,t)dt,$$

and

$$B^{\rho}_{n,\alpha}(f'_x,x)=\int_x^1\left(\int_x^t f'_x(u)du\right)K^{\rho}_{n,\alpha}(x,t)dt.$$

Thus our problem is reduced to calculate the estimates of the terms $A^{\rho}_{n,\alpha}(f'_x,x)$ and $B^{\rho}_{n,\alpha}(f'_x,x)$. From the definition of $\xi^{\rho}_{n,\alpha}$ given in Lemma 5.9, we can write

$$A^{\rho}_{n,\alpha}(f'_x,x)=\int_0^x\left(\int_x^t f'_x(u)\right)\frac{\partial}{\partial t}\xi^{\rho}_{n,\alpha}(x,t)dt.$$

Applying the integration by parts, we get

$$\left|A^{\rho}_{n,\alpha}(f'_x,x)\right|\leq\int_0^x|f'_x(t)|\xi^{\rho}_{n,\alpha}(x,t)dt$$
$$\leq\int_0^{x-\frac{x}{\sqrt{n}}}|f'_x(t)|\xi^{\rho}_{n,\alpha}(x,t)dt+\int_{x-\frac{x}{\sqrt{n}}}^x|f'_x(t)|\xi^{\rho}_{n,\alpha}(x,t)dt:=I_1+I_2.$$

Since $f'_x(x)=0$ and $\xi^{\rho}_{n,\alpha}(x,t)\leq 1$, we have

$$I_2:=\int_{x-\frac{x}{\sqrt{n}}}^x\left|f'_x(t)-f'_x(x)\right|\xi^{\rho}_{n,\alpha}(x,t)dt\leq\int_{x-\frac{x}{\sqrt{n}}}^x\left(\bigvee_t^x f'_x\right)dt$$
$$\leq\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^x f'_x\right)\int_{x-\frac{x}{\sqrt{n}}}^x dt=\frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^x f'_x\right).$$

By applying Lemma 5.9 and considering $t = x - \dfrac{x}{u}$, we get

$$I_1 \leq \alpha\delta_{n,\rho}^2(x)\int_0^{x-\frac{x}{\sqrt{n}}} \left|f_x'(t)-f_x'(x)\right| \frac{dt}{(x-t)^2} \leq \alpha\delta_{n,\rho}^2(x)\int_0^{x-\frac{x}{\sqrt{n}}} \left(\bigvee_t^x f_x'\right)\frac{dt}{(x-t)^2}$$

$$= \frac{\alpha\delta_{n,\rho}^2(x)}{x}\int_1^{\sqrt{n}} \left(\bigvee_{x-\frac{x}{u}}^{x} f_x'\right) du \leq \frac{\alpha\delta_{n,\rho}^2(x)}{x}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{k}}^{x} f_x'\right).$$

Therefore,

$$|A_{n,\alpha}^{\rho}(f_x',x)| \leq \frac{\alpha\delta_{n,\rho}^2(x)}{x}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{k}}^{x} f_x'\right) + \frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x} f_x'\right). \tag{5.37}$$

Also, using integration by parts in $B_n^{\rho}(f_x',x)$ and applying Lemma 5.9 with $z = x+(1-x)/\sqrt{n}$, we have

$$\begin{aligned}
|B_{n,\alpha}^{\rho}(f_x',x)| &= \left|\int_x^1\left(\int_x^t f_x'(u)du\right)K_{n,\alpha}^{\rho}(x,t)dt\right|\\
&= \left|\int_x^z\left(\int_x^t f_x'(u)du\right)\frac{\partial}{\partial t}(1-\xi_{n,\alpha}^{\rho}(x,t))dt + \int_z^1\left(\int_x^t f_x'(u)du\right)\frac{\partial}{\partial t}(1-\xi_{n,\alpha}^{\rho}(x,t))dt\right|\\
&= \left|\left[\int_x^t(f_x'(u)du)(1-\xi_{n,\alpha}^{\rho}(x,t))\right]_x^z - \int_x^z f_x'(t)(1-\xi_{n,\alpha}^{\rho}(x,t))dt\right.\\
&\left.+\int_z^1\int_x^t(f_x'(u)du)\frac{\partial}{\partial t}(1-\xi_{n,\alpha}^{\rho}(x,t))dt\right|\\
&= \left|\int_x^z(f_x'(u)du)(1-\xi_{n,\alpha}^{\rho}(x,z)) - \int_x^z f_x'(t)(1-\xi_{n,\alpha}^{\rho}(x,t))dt\right.\\
&\left.+\left[\int_x^t(f_x'(u)du)(1-\xi_{n,\alpha}^{\rho}(x,t))\right]_z^1 - \int_z^1 f_x'(t)(1-\xi_{n,\alpha}^{\rho}(x,t))dt\right|\\
&= \left|\int_x^z f_x'(t)(1-\xi_{n,\alpha}^{\rho}(x,t))dt + \int_z^1 f_x'(t)(1-\xi_{n,\alpha}^{\rho}(x,t))dt\right|\\
&\leq \alpha\delta_{n,\rho}^2(x)\int_z^1\left(\bigvee_x^t f_x'\right)(t-x)^{-2}dt + \int_x^z\bigvee_x^t f_x' dt\\
&\leq \alpha\delta_{n,\rho}^2(x)\int_{x+(1-x)/\sqrt{n}}^1\left(\bigvee_x^t f_x'\right)(t-x)^{-2}dt + \frac{1-x}{\sqrt{n}}\left(\bigvee_x^{x+(1-x)/\sqrt{n}} f_x'\right).
\end{aligned}$$

By substituting $u = (1-x)/(t-x)$, we get

$$\begin{aligned}
|B_{n,\alpha}^{\rho}(f_x',x)| &\leq \alpha\delta_{n,\rho}^2(x)\int_1^{\sqrt{n}}\left(\bigvee_x^{x+(1-x)/u} f_x'\right)(1-x)^{-1}du + \frac{1-x}{\sqrt{n}}\left(\bigvee_x^{x+(1-x)/\sqrt{n}} f_x'\right)\\
&\leq \frac{\alpha\delta_{n,\rho}^2(x)}{1-x}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_x^{x+(1-x)/k} f_x'\right) + \frac{1-x}{\sqrt{n}}\left(\bigvee_x^{x+(1-x)/\sqrt{n}} f_x'\right).
\end{aligned} \tag{5.38}$$

Collecting the estimates (5.36) - (5.38), we get the required result. This completes the proof. □

Example 5.3. *Let us consider the following two functions*

$$f:[0,1]\to\mathbb{R}, f(x)=\begin{cases} x^2\sin\frac{1}{x}, x\neq 0 \\ 0,\ x=0 \end{cases} \quad \text{and } g:[0,1]\to\mathbb{R},\ g(x)=\begin{cases} (1-x)\cos\frac{\pi}{1-x},\ x\neq 1 \\ 0,\ x=1 \end{cases}$$

The function f is differentiable and of bounded variation on $[0,1]$, while g is continuous but is not of bounded variation on $[0,1]$.

For $n=20$, $\rho=1$ and $\alpha\in\left\{1,\frac{3}{2},2\right\}$, the convergence of $U_{n,\alpha}^{\rho}f$ to f and $U_{n,\alpha}^{\rho}g$ to g is illustrated in Figure 1 and Figure 2, respectively.

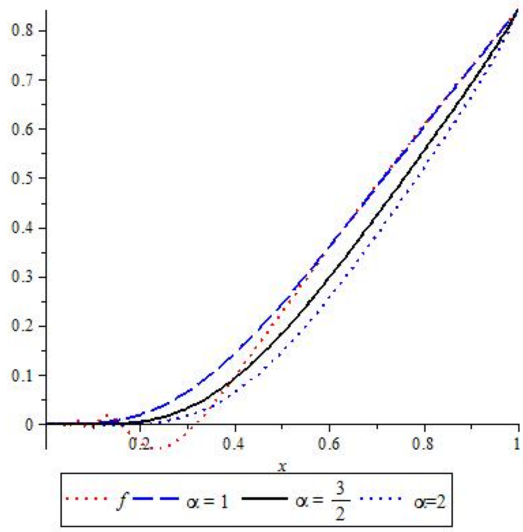

Figure 1 :The convergence of $U_{n,\alpha}^{\rho}(f;x)$ to $f(x)$

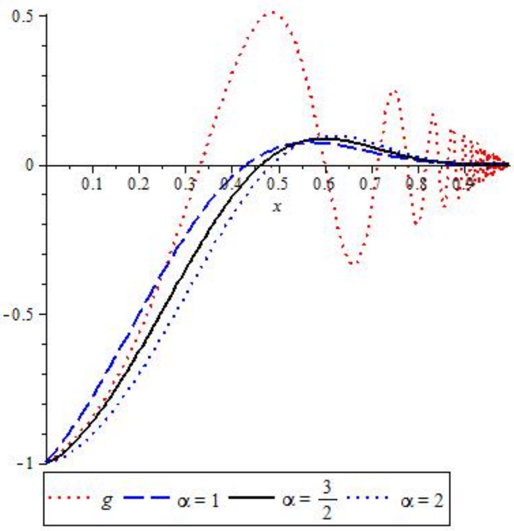

Figure 2 :The convergence of $U_{n,\alpha}^{\rho}(g;x)$ to $g(x)$

6 The Kantorovich modification of order k of certain operators

In this chapter we investigate rate of pointwise convergence of differentiated Baskakov operators, error of this pointwise approximation as well. To do this, we construct k-th order Kantorovich type modifications of Baskakov operators and obtain quantitative Voronovskaya theorem for differentiated Baskakov operators. These results are published in [3].

6.1 The k-th order Kantorovich modifications of the Baskakov operators

If B_n are the Bernstein polynomials defined in (1.6), then the Kantorovich polynomials (1.9) can be obtained replacing the values $f\left(\frac{k}{n}\right)$ by the mean values of f in the intervals $\left[\frac{k}{n+1}, \frac{k+1}{n+1}\right]$. Note that the Kantorovich operators can be written in the form

$$K_n = D \circ B_{n+1} \circ I, \tag{6.1}$$

where D is the differentiation operators and I is the antiderivative operators $I(f;x) = \int_0^x f(t)dt$.

Let $L_n : C[0,1] \to C^k[0,1]$, $k \in \mathbb{N}$ be an arbitrary sequence of linear operators, D^k and I^k be the iterates of the operators D and I, respectively. The general construction of (6.1) was considered in [156] (see also [157], [158]) as follows:

$$Q_n^k := D^k \circ L_n \circ I^k.$$

In the approximation theory by linear positive operators, the Baskakov operators are one of the most useful tool to approximate the functions on unbounded intervals. In the following we aim to investigate the operators Q_n^k which are generated using the Baskakov operators.

Given $k \in \mathbb{N}_0$, a function is said to be convex of order k if $[x_0, \ldots, x_k; f] \geq 0$ for any $x_0 < \cdots < x_k \in [0, \infty)$. A function $f \in C^k[0,\infty)$ is convex of order k if and only if $f^{(k)} \geq 0$. A linear operator L is said to be convex of order k if Lf is convex of order k whenever f is convex of order k.

We consider the operator $I_k : C[0,\infty) \to C[0,\infty)$ defined by

$$I_k(f;x) = \begin{cases} f(x) & , \quad k = 0 \\ \int\limits_0^x \frac{(x-t)^{k-1}}{(k-1)!} f(t)\,dt & , \quad k \geq 1. \end{cases}$$

The k-th order Kantorovich modifications of the Baskakov operators B_n can be defined by

$$Q_n^k := D^k \circ \mathcal{B}_n \circ I_k,$$

where $D^k = \dfrac{d^k}{dx^k}$. The k-th order Kantorovich modifications of linear positive operators is a useful constructions to describe the simultaneous approximation behaviors of the operators. For more details we refer the readers to [156, 157, 158]. Also, see the recent papers [39, 73, 102, 101].

Remark 6.1. *In the paper [36], Aral et. al. gave a new representation of q-Baskakov operators via q-forward differences. Using the relation between q-differences and divided differences in equality (2.2) of [36] and taking $q = 1$ we immediately have $D^k\mathcal{B}_n(f;x) \geq 0$ which means the convexity of order k of Baskakov operators. Hence, since $\mathcal{B}_n$ are convex of order k, the operators Q_n^k are linear and positive.*

6.2 The moments of the k-th order Kantorovich modifications of the Baskakov operators

The moments of the operators Q_n^k have a crucial role for our main results. In this section, we obtain the application of the operators to the monomials $e_i(t) = t^i$ and $(t-x)^i$ of degree up to 6. To do this we first need to following lemma.

Lemma 6.1. *For $r \in \mathbb{N}$ there holds*

$$\mathcal{B}_n(e_r;x) = \sum_{j=1}^{r} a_{r,j} x^j \frac{(n)_j}{n^r}, \tag{6.2}$$

where $(n)_j$ is Pochammer symbol given by $(n)_0 = 1$, $(n)_k = n(n+1)\ldots(n+k-1)$ and $a_{r,j}$ are positive coefficients which satisfy the equalities

$$a_{r,r} = a_{r-1,r-1} = \ldots = a_{1,1} = 1, \tag{6.3}$$

$$a_{r,1} = a_{r-1,1} = \ldots = a_{1,1} = 1, \tag{6.4}$$

$$a_{r,r-1} = r(r-1)/2, \tag{6.5}$$

$$a_{r,j} = j a_{r-1,j} + a_{r-1,j-1}, \ r,j \geq 1. \tag{6.6}$$

In particular $\mathcal{B}_n(e_r;x)$ is a polynomial of degree r without a constant term.

Proof. Using the equality

$$\frac{x(1+x)}{n} b'_{n,k}(x) = \left(\frac{k}{n} - x\right) b_{n,k}(x), \tag{6.7}$$

one can easily obtain

$$\mathcal{B}_n\left(t^{r+1};x\right) = \frac{x(1+x)}{n}\mathcal{B}'_n(t^r;x) + x\mathcal{B}_n(t^r;x).$$

On the other hand, since $\mathcal{B}_n(t;x)=x$ and $\mathcal{B}_n(t^2;x)=x^2+\dfrac{x(1+x)}{n}$, the representation (6.2) holds true for $r=1,2$ with $a_{1,1}=a_{2,1}=1$. Let us assume (6.2) to be true for r, then by (6.7) we can write

$$\begin{aligned}
\mathcal{B}_n\left(t^{r+1};x\right) &= \frac{x(1+x)}{n}\sum_{j=1}^{r} j a_{r,j}x^{j-1}\frac{(n)_j}{n^r}+x\sum_{j=1}^{r}a_{r,j}x^j\frac{(n)_j}{n^r}\\
&= a_{r,1}xn^{-r}+\sum_{j=2}^{r} j a_{r,j}x^j\frac{(n)_j}{n^{r+1}}+\sum_{j=2}^{r}(j-1)\,a_{r,j-1}x^j\frac{(n)_{j-1}}{n^{r+1}}\\
&\quad +ra_{r,r}x^{r+1}\frac{(n)_r}{n^{r+1}}+\sum_{j=2}^{r}a_{r,j-1}x^j\frac{(n)_{j-1}}{n^r}+a_{r,r}x^{r+1}\frac{(n)_r}{n^r}\\
&= a_{r,1}xn^{-r}+\sum_{j=2}^{r}x^j\frac{(n)_j}{n^{r+1}}\left(ja_{r,j}+\frac{(j-1)\,a_{r,j-1}}{(n+j-1)}+\frac{na_{r,j-1}}{(n+j-1)}\right)\\
&\quad +a_{r,r}x^{r+1}\frac{(n)_r}{n^r}\left(\frac{r}{n}+1\right)=:\sum_{j=1}^{r+1}a_{r+1,j}x^j\frac{(n)_j}{n^{r+1}}.
\end{aligned}$$

Hence (6.2) is valid for all $r\in\mathbb{N}$ since

$$\begin{aligned}
a_{r+1,j} &= ja_{r,j}+\frac{(j-1)\,a_{r,j-1}}{(n+j-1)}+\frac{na_{r,j-1}}{(n+j-1)}\\
&= ja_{r,j}+\frac{(j+n-1)\,a_{r,j-1}}{(n+j-1)}\\
&= ja_{r,j}+a_{r,j-1}.
\end{aligned}$$

□

Remark 6.2. *The coefficients $a_{r,j}$ for $r-6\le j\le r-1$ were calculated in [2] as*

$$\begin{aligned}
a_{r,r-1} &= \frac{1}{2}r(r-1)=:\beta_1(r),\\
a_{r,r-2} &= \frac{1}{24}r(3r-5)(r-1)(r-2)=:\beta_2(r),\\
a_{r,r-3} &= \frac{1}{48}r(r-1)(r-2)^2(r-3)^2=:\beta_3(r),\\
a_{r,r-4} &= \frac{1}{5760}r(r-1)(r-2)(r-3)(r-4)(15r^3-150r^2+485r-502)=:\beta_4(r),\\
a_{r,r-5} &= \frac{1}{11520}r(r-1)(r-2)(r-3)(r-4)^2(r-5)^2(3r^2-23r+38)=:\beta_5(r),\\
a_{r,r-6} &= \frac{1}{2903040}r(r-1)(r-2)(r-3)(r-4)(r-5)(r-6)\\
&\quad\times(63r^5-1575r^4+15435r^3-73801r^2+171150r-152696)=:\beta_6(r).
\end{aligned}$$

Lemma 6.2. *For $n,k\in\mathbb{N}_0$, $x\in[0,\infty)$ the following equalities hold.*

$$Q_n^k(e_0;x)=\frac{(n)_k}{n^k}, \tag{6.8}$$

$$Q_n^k(e_1;x)=\frac{2x(n)_{k+1}+k(n)_k}{2n^{k+1}}, \tag{6.9}$$

$$Q_n^k(e_2;x)=\frac{12x^2(n)_{k+2}+12(k+1)x(n)_{k+1}+k\,(3k+1)\,(n)_k}{12n^{k+2}}, \tag{6.10}$$

$$Q_n^k(e_3;x) = \frac{x^3(n)_{k+3}}{n^{k+3}} + \frac{3x^2(k+2)(n)_{k+2}}{2n^{k+3}} + \frac{x(3k+4)(k+1)(n)_{k+1}}{4n^{k+3}} + \frac{k^2(k+1)(n)_k}{8n^{k+3}}, \tag{6.11}$$

$$Q_n^k(e_4;x) = \frac{x^4\,(n)_{k+4}}{n^{k+4}} + \frac{2x^3(k+3)\,(n)_{k+3}}{n^{k+4}} + \frac{(3k+7)\,x^2(k+2)\,(n)_{k+2}}{2n^{k+4}} + \frac{(k+1)^2(k+2)x\,(n)_{k+1}}{2n^{k+4}} + \frac{k\left(5k+30k^2+15k^3-2\right)(n)_k}{240n^{k+4}}, \tag{6.12}$$

$$\begin{aligned} Q_n^k(e_5;x) &= \frac{x^5\,(n)_{k+5}}{n^{k+5}} + \frac{5x^4(k+4)\,(n)_{k+4}}{2n^{k+5}} + \frac{5x^3\,(k+3)\,(3k+10)\,(n)_{k+3}}{6n^{k+5}} \\ &+ \frac{5\,(k+3)\,(k+2)^2\,x^2\,(n)_{k+2}}{4n^{k+5}} \\ &+ \frac{(k+1)\left(110k+75k^2+15k^3+48\right)x\,(n)_{k+1}}{48n^{k+5}} \\ &+ \frac{k^2\left(7k+3k^2-2\right)(k+1)\,(n)_k}{96n^{k+5}}, \end{aligned} \tag{6.13}$$

$$\begin{aligned} Q_n^k(e_6;x) &= x^6\frac{(n)_{k+6}}{n^{k+6}} + \frac{3(k+5)x^5\,(n)_{k+5}}{n^{k+6}} + \frac{5\,(k+4)\,(3k+13)\,x^4\,(n)_{k+4}}{4n^{k+6}} \\ &+ \frac{5\,(k+4)\,(k+3)^2\,x^3\,(n)_{k+3}}{2n^{k+6}} \\ &+ \frac{(k+2)\left(305k+120k^2+15k^3+248\right)x^2\,(n)_{k+2}}{16n^{k+6}} \\ &+ \frac{\left(13k+3k^2+8\right)(k+2)\,(k+1)^2\,x\,(n)_{k+1}}{16n^{k+6}} \\ &+ \frac{k\left(-42k-91k^2+315k^3+315k^4+63k^5+16\right)(n)_k}{4032n^{k+6}}. \end{aligned} \tag{6.14}$$

Proof. Using the definition of operators Q_n^k, it follows for $r+k\geq 1$ that

$$Q_n^k e_r = \frac{r!}{(r+k)!}D^k\mathcal{B}_n e_{r+k}. \tag{6.15}$$

If we consider this equality for $r=1$, we have

$$Q_n^k e_1 = \frac{1}{(k+1)!}D^k\mathcal{B}_n e_{k+1}.$$

On the other hand, using the equality (6.2) for $r=k+1$ we have

$$\begin{aligned} Q_n^k(e_1;x) &= \frac{1}{(k+1)!}D^k\left[x^{k+1}\frac{(n)_{k+1}}{n^{k+1}} + \beta_1(k+1)x^k\frac{(n)_k}{n^{k+1}} + \text{ lower order terms in } x\right] \\ &= \frac{1}{(k+1)!}\left[(k+1)!x\frac{(n)_{k+1}}{n^{k+1}} + \frac{(k+1)!k\,(n)_k}{2n^{k+1}}\right] = \frac{2x(n)_{k+1}+k(n)_k}{2n^{k+1}}. \end{aligned}$$

By similar considerations we have

$$\begin{aligned} Q_n^k(e_2;x) &= \frac{2}{(k+2)!}D^kB_n(e_{k+2};x) \\ &= \frac{2}{(k+2)!}D^k\left[x^{k+2}\frac{(n)_{k+2}}{n^{k+2}} + \beta_1(k+2)x^{k+1}\frac{(n)_{k+1}}{n^{k+2}}\right. \\ &\left. + \beta_2(k+2)x^k\frac{(n)_k}{n^{k+2}} + \text{lower order terms in } x\right] \end{aligned}$$

$$= \frac{2}{(k+2)!}\left[\frac{(k+2)!}{2}x^2\frac{(n)_{k+2}}{n^{k+2}}+\beta_1(k+2)x(k+1)!\frac{(n)_{k+1}}{n^{k+2}}+k!\beta_2(k+2)\frac{(n)_k}{n^{k+2}}\right]$$
$$= \frac{x^2(n)_{k+2}}{n^{k+2}}+\frac{2\beta_1(k+2)x(n)_{k+1}}{(k+2)\,n^{k+2}}+\frac{2\beta_2(k+2)(n)_k}{(k+1)\,(k+2)\,n^{k+2}}.$$

Since

$$\beta_1\,(k+2)=\frac{(k+2)\,(k+1)}{2},$$
$$\beta_2\,(k+2)=\frac{1}{24}k\,(3k+1)\,(k+2)\,(k+1)\,,$$

we have

$$Q_n^k(e_2;x)=\frac{x^2(n)_{k+2}}{n^{k+2}}+\frac{x(k+1)(n)_{k+1}}{n^{k+2}}+\frac{k(3k+1)(n)_k}{12n^{k+2}}.$$

Moreover, we have

$$Q_n^k(e_3;x)=\frac{6}{(k+3)!}D^kB_n(e_{k+3};x)$$
$$=\frac{6}{(k+3)!}D^k\left[x^{k+3}\frac{(n)_{k+3}}{n^{k+3}}+\beta_1(k+3)x^{k+2}\frac{(n)_{k+2}}{n^{k+3}}\right.$$
$$\left.+\beta_2(k+3)x^{k+1}\frac{(n)_{k+1}}{n^{k+3}}+\beta_3(k+3)x^k\frac{(n)_k}{n^{k+3}}+\text{lower order terms in }x\right]$$
$$=\frac{x^3(n)_{k+3}}{n^{k+3}}+\frac{3x^2\beta_1(k+3)(n)_{k+2}}{(k+3)n^{k+3}}+\frac{6\beta_2(k+3)x(n)_{k+1}}{(k+2)(k+3)n^{k+3}}+\frac{6\beta_3(k+3)(n)_k}{(k+1)(k+2)(k+3)n^{k+3}}.$$

Since

$$\beta_1\,(k+3)=\frac{(k+2)\,(k+3)}{2},$$
$$\beta_2\,(k+3)=\frac{1}{24}\,(3k+4)\,(k+3)\,(k+2)\,(k+1)\,,$$
$$\beta_3\,(k+3)=\frac{1}{48}k^2\,(k+3)\,(k+2)\,(k+1)^2\,,$$

we get

$$Q_n^k(e_3;x)=\frac{x^3(n)_{k+3}}{n^{k+3}}+\frac{3x^2(k+2)(n)_{k+2}}{2n^{k+3}}+\frac{x(3k+4)(k+1)(n)_{k+1}}{4n^{k+3}}+\frac{k^2(k+1)(n)_k}{8n^{k+3}}.$$

Finally, we can write

$$Q_n^k(e_4;x)=\frac{24}{(k+4)!}D^kB_n(e_{k+4};x)$$
$$=\frac{24}{(k+4)!}D^k\left[x^{k+4}\frac{(n)_{k+4}}{n^{k+4}}+\beta_1(k+4)x^{k+3}\frac{(n)_{k+3}}{n^{k+4}}+\beta_2(k+4)x^{k+2}\frac{(n)_{k+2}}{n^{k+4}}\right.$$
$$\left.+\beta_3(k+4)x^{k+1}\frac{(n)_{k+1}}{n^{k+4}}+\beta_4(k+4)x^k\frac{(n)_k}{n^{k+4}}+\text{lower order terms in }x\right]$$
$$=\frac{x^4\,(n)_{k+4}}{n^{k+4}}+\frac{4x^3\beta_1(k+4)\,(n)_{k+3}}{(k+4)n^{k+4}}+\frac{12x^2\beta_2(k+4)\,(n)_{k+2}}{(k+3)(k+4)n^{k+4}}$$
$$+\frac{24\beta_3(k+4)x\,(n)_{k+1}}{(k+2)(k+3)(k+4)n^{k+4}}+\frac{24\beta_4(k+4)\,(n)_k}{(k+1)(k+2)(k+3)(k+4)n^{k+4}}.$$

Since

$$\begin{aligned}
\beta_1(k+4) &= \frac{(k+3)(k+4)}{2},\\
\beta_2(k+4) &= \frac{1}{24}(3k+7)(k+4)(k+3)(k+2),\\
\beta_3(k+4) &= \frac{1}{48}(k+4)(k+3)(k+2)^2(k+1)^2,\\
\beta_4(k+4) &= \frac{1}{5760}k(k+4)(k+3)(k+2)(k+1)\left(5k+30k^2+15k^3-2\right),
\end{aligned}$$

we get

$$\begin{aligned}
Q_n^k(e_4;x) = {} & \frac{x^4(n)_{k+4}}{n^{k+4}} + \frac{2x^3(k+3)(n)_{k+3}}{n^{k+4}} + \frac{(3k+7)x^2(k+2)(n)_{k+2}}{2n^{k+4}}\\
&+ \frac{(k+1)^2(k+2)x(n)_{k+1}}{2n^{k+4}} + \frac{k\left(5k+30k^2+15k^3-2\right)(n)_k}{240n^{k+4}}.
\end{aligned}$$

We leave the details of $Q_n^k e_5$ and $Q_n^k e_6$ for the readers. Note that the above equalities hold also true in the case $k=0$. □

With the following lemma we can now present the central moments of the operators Q_n^k up to order 6.

Lemma 6.3. *If we define the central moment operator by*

$$M_{n,r}^k(x) = Q_n^k\left((t-x)^r;x\right),\ r\in\mathbb{N}_0,$$

then we have

$$\begin{aligned}
M_{n,0}^k(x) &= \frac{(n)_k}{n^k},\\
M_{n,1}^k(x) &= \frac{k(n)_k(2x+1)}{2n^{k+1}},\\
M_{n,2}^k(x) &= \frac{(n)_k}{n^{k+2}}\left[x(x+1)n + x(x+1)k(k+1) + \frac{k(3k+1)}{12}\right],\\
M_{n,3}^k(x) &= x^3\frac{(n)_k}{n^{k+3}}\left[(3k+2)n+k(k+1)(k+2)\right]+x^2\frac{(n)_k}{2n^{k+3}}\left[(9k+6)n+3k(k+1)(k+2)\right]\\
&\quad + x\frac{(n)_k}{4n^{k+3}}\left[(6k+4)n+(3k+4)(k+1)k\right] + \frac{(n)_k}{8n^{k+3}}k^2(k+1),\\
M_{n,4}^k(x) &= x^4\frac{(n)_k}{n^{k+4}}\left[3n^2+(6k^2+14k+6)n+k(k+1)(k+2)(k+3)\right]\\
&\quad + x^3\frac{(n)_k}{n^{k+4}}\left[6n^2+(12k^2+28k+12)n+2k(k+1)(k+2)(k+3)\right]\\
&\quad + x^2\frac{(n)_k}{2n^{k+4}}\left[6n^2+(15k^2+33k+14)n+(3k+7)(k+2)k(k+1)\right]\\
&\quad + \frac{(n)_k}{240n^{k+4}}k\left(5k+30k^2+15k^3-2\right),
\end{aligned}$$

$$
\begin{aligned}
M_{n,6}^{k}(x) &= x^{6}\frac{(n)_{k}}{n^{k+6}}\left[15n^{3}+(45k^{2}+165k+130)n^{2}+(15k^{2}+130k^{3}+375k^{2}+404k+120)n\right.\\
&\quad\left.+\,k(k+1)(k+2)(k+3)(k+4)(k+5)\right]+x^{5}\frac{(n)_{k}}{n^{k+6}}\left[45n^{3}+(390+135k^{2}+495k)n^{2}\right.\\
&\quad\left.+\,(45k^{4}+390k^{3}+1125k^{2}+1212k+360)n+3k(k+1)(k+2)(k+3)(k+4)(k+5)\right]\\
&\quad+x^{4}\frac{(n)_{k}}{4n^{k+6}}\left[180n^{3}+(1660+585k^{2}+2115k)n^{2}\right.\\
&\quad+(210k^{4}+1760k^{3}+4960k^{2}+5270k+1560)n\\
&\quad\left.+\,5(k+4)(3k+13)k(k+1)(k+2)(k+3)\right]+x^{3}\frac{(n)_{k}}{2n^{k+6}}\left[30n^{3}+(360+135k^{2}+465k)n^{2}\right.\\
&\quad\left.+\,(60k^{4}+460k^{3}+1210k^{2}+1230k+360)n+5(k+4)(k+3)^{2}k(k+1)(k+2)\right]\\
&\quad+x^{2}\frac{(n)_{k}}{16n^{k+6}}\left[(180k^{2}+540k+400)n^{2}+(870k^{3}+135k^{4}+1945k^{2}+1754k+496)n\right.\\
&\quad\left.+\,(k+2)k(k+1)(305k+120k^{2}+15k^{3}+248)\right]\\
&\quad+x\frac{(n)_{k}}{16n^{k+6}}\left[(15k^{4}+70k^{3}+105k^{2}+66k+16)n+(13k+3k^{2}+8)(k+2)(k+1)^{2}k\right]\\
&\quad+\frac{(n)_{k}}{4032n^{k+6}}(-42k-91k^{2}+315k^{3}+315k^{4}+63k^{5}+16)k.
\end{aligned}
$$

Proof. Using the binomial expansion

$$M_{n,r}^{k}(x)=Q_{n}^{k}\left((t-x)^{r};x\right)=\sum_{i=0}^{r}\binom{r}{i}(-x)^{r-i}\,Q_{n}^{k}(e_{i};x)$$

and the equalities (6.8)-(6.14) we have desired results. □

Lemma 6.4. *For each $n\in\mathbb{N}$, $k\in\mathbb{N}_{0}$ and fixed $x\in[0,\infty)$, there exists a positive constant C such that*

$$\frac{M_{n,6}^{k}(x)}{M_{n,2}^{k}(x)}\leq\frac{C}{n^{2}}\left(x(x+1)\right)^{2}.$$

Proof. Using Lemma 6.3 we obtain

$$\lim_{n\to\infty}n^{2}\frac{M_{n,6}^{k}(x)}{M_{n,2}^{k}(x)}=\lim_{n\to\infty}\frac{n^{3}M_{n,6}^{k}(x)}{nM_{n,2}^{k}(x)}=15x^{2}(x+1)^{2}.$$

Therefore, we have the desired result. □

Proposition 6.1. *We have*

$$
\begin{aligned}
&\lim_{n\to\infty}n\left(Q_{n}^{k}(e_{0};x)-1\right)=\frac{k(k-1)}{2},\\
&\lim_{n\to\infty}nQ_{n}^{k}(e_{1}-x;x)=\frac{k(1+2x)}{2},\\
&\lim_{n\to\infty}nQ_{n}^{k}\left((e_{1}-x)^{2};x\right)=x(1+x).
\end{aligned}
\tag{6.16}
$$

Proof. Indeed, since

$$\begin{aligned} n\left(Q_n^k\left(e_0;x\right)-1\right) &= \frac{n\left(n\right)_k-n^{k+1}}{n^k} \\ &= \frac{n^2\left(n+1\right)\ldots\left(n+k-1\right)-n^{k+1}}{n^k} \\ &= \frac{n^k\sum\limits_{i=1}^{k-1}i+\text{lower order terms in } n}{n^k} \end{aligned}$$

taking the limit with $n\to\infty$ we have $n\left(Q_n^k\left(e_0;x\right)-1\right)\to\sum\limits_{i=1}^{k-1}i=k\left(k-1\right)/2$. The others can be obtained by similar way. □

6.3 Voronovskaya theorem for differentiated Baskakov operators

The goal of this section is to obtain quantitative Voronovskaya theorem for differentiated Baskakov operators.

Let $B_2\left[0,\infty\right)$ be the set of all functions f defined on $\left[0,\infty\right)$ satisfying the condition $\left|f\left(x\right)\right|\leq M\left(1+x^2\right)$ with positive constant M which may depend only on f and $C_2\left[0,\infty\right)$ denotes the subspace of all continuous functions in $B_2\left[0,\infty\right)$. By $C_2^*\left[0,\infty\right)$, we denote the subspace of all functions $f\in C_2\left[0,\infty\right)$ for which $\lim\limits_{x\to\infty}\dfrac{f\left(x\right)}{1+x^2}$ is finite. $B_2\left[0,\infty\right)$ is a linear normed space with the norm $\left\|f\right\|_*=\sup\limits_{x\geq0}\dfrac{\left|f\left(x\right)\right|}{1+x^2}$.

For every $f\in C_2^*\left[0,\infty\right)$, the weighted modulus of continuity $\Omega\left(.;\delta\right)$ has the properties:

$$\lim_{\delta\to0}\Omega\left(f;\delta\right)=0$$

and

$$\Omega\left(f;\lambda\delta\right)\leq2\left(1+\lambda\right)\left(1+\delta^2\right)\Omega\left(f;\delta\right),\ \lambda>0. \tag{6.17}$$

For $f\in C_2\left[0,\infty\right)$, from (1.5) and (6.17) we can write

$$\begin{aligned} \left|f\left(t\right)-f\left(x\right)\right| &\leq \left(1+\left(t-x\right)^2\right)\left(1+x^2\right)\Omega\left(f;\left|t-x\right|\right) \\ &\leq 2\left(1+\frac{\left|t-x\right|}{\delta}\right)\left(1+\delta^2\right)\Omega\left(f;\delta\right)\left(1+\left(t-x\right)^2\right)\left(1+x^2\right). \end{aligned} \tag{6.18}$$

All concepts mentioned above can be found in Section 1.4. Very recently in [1], the authors presented the quantitative Voronovskaya theorem for positive linear operators acting on unbounded intervals as:

Theorem 6.1. *Let E be a subspace of $C\left[0,\infty\right)$ which contains the polynomials, and $L_n:E\to C\left[0,\infty\right)$ be a sequence of l.p.o such that $L_ne_i=e_i, i=0,1$. If $f\in E$ and $f''\in C_2^*\left[0,\infty\right)$, then we have for $x\in\left[0,\infty\right)$ that*

$$\left|L_n\left(f;x\right)-f\left(x\right)-\frac{1}{2}f''\left(x\right)\mu_{n,2}^L\left(x\right)\right|\leq16\left(1+x^2\right)\Omega\left(f'';\left(\frac{\mu_{n,6}^L\left(x\right)}{\mu_{n,2}^L\left(x\right)}\right)^{1/4}\right)\mu_{n,2}^L\left(x\right),$$

where $\mu_{n,m}^L\left(x\right)$, $m\in\mathbb{N}$ is the moment of order m of L_n.

Corollary 6.1. *The above result may be stated without the assumptions $L_n e_i = e_i$, $i = 0, 1$, as*

$$\begin{aligned}
&\left| L_n(f;x) - f(x) L_n(e_0;x) - f'(x) L_n(e_1 - x;x) - \frac{1}{2} f''(x) L_n\left((e_1-x)^2;x\right) \right| \\
\leq\ & 16\left(1+x^2\right)\Omega\left(f''; \left(\frac{\mu_{n,6}^L(x)}{\mu_{n,2}^L(x)}\right)^{1/4}\right)\mu_{n,2}^L(x).
\end{aligned}$$

Theorem 6.2. *Let $k \geq 0$, $n \in \mathbb{N}$. If $f^{(k)}$, $f^{(k+1)}$ exist at any point $x \in [0,\infty)$ and $f^{(k+2)} \in C_2^*[0,\infty)$, then it follows for $x \in [0,\infty)$ that*

$$\begin{aligned}
&\left| n\left[D^k \mathcal{B}_n(f;x) - f^{(k)}(x)\right] - \frac{1}{2}\frac{d^k}{dx^k}\left\{x(1+x)f''(x)\right\}\right| \\
\leq\ & \alpha_n\left|f^{(k)}(x)\right| + \gamma_n(x)\left|f^{(k+1)}(x)\right| + \tau_n(x)\left|f^{(k+2)}(x)\right| \\
& + Cx(x+1)16\left(1+x^2\right)\Omega\left(f^{(k+2)}; \sqrt{\frac{x(x+1)}{n}}\right),
\end{aligned}$$

where

$$\begin{aligned}
\alpha_n &= \left| n\left(\frac{(n)_k}{n^k} - 1\right) - \frac{k(k-1)}{2}\right|, \\
\gamma_n(x) &= \frac{k(2x+1)}{2}\left(\frac{n(n)_k}{n^{k+1}} - 1\right), \\
\tau_n(x) &= \frac{x(1+x)}{2}\left(\frac{(n)_k(n+k^2+k)}{n^{k+1}} - 1\right) + \frac{k(3k+1)(n)_k}{24n^{k+1}}.
\end{aligned}$$

Corollary 6.2. *Since for a fixed $x \in [0,\infty)$, $\alpha_n \to 0$, $\gamma_n(x) \to 0$, $\tau_n(x) \to 0$ as $n \to \infty$ and for the function $f \in C_2^*[0,\infty)$, $\Omega(f;\delta) \to 0$ as $\delta \to 0$, we immediately have*

$$\lim_{n\to\infty} n\left[D^k \mathcal{B}_n(f;x) - f^{(k)}(x)\right] = \frac{1}{2}\frac{d^k}{dx^k}\left\{x(1+x)f''(x)\right\}.$$

Proof. In the Corollary 6.1 if we put Q_n^k instead of L_n and $f^{(k)}$ instead of f we can write

$$\begin{aligned}
&\left| n\left[D^k \mathcal{B}_n(f;x) - f^{(k)}(x)\right] - \frac{1}{2}\frac{d^k}{dx^k}\left\{x(1+x)f''(x)\right\}\right| \\
&\leq \left| n\left(Q_n^k(e_0;x) - 1\right) f^{(k)}(x) + nQ_n^k(e_1-x;x) f^{(k+1)}(x) \right. \\
&\left. + \frac{1}{2} nQ_n^k\left((e_1-x)^2;x\right) f^{(k+2)}(x) - \frac{1}{2}\frac{d^k}{dx^k}\left\{x(1+x)f''(x)\right\}\right| \\
&+ nQ_n^k\left((e_1-x)^2;x\right) 16\left(1+x^2\right)\Omega\left(f^{(k+2)}; \left(\frac{Q_n^k\left((e_1-x)^6;x\right)}{Q_n^k\left((e_1-x)^2;x\right)}\right)^{1/4}\right).
\end{aligned}$$

On the other hand, by the Leibniz' rule we observe

$$\frac{1}{2}\frac{d^k}{dx^k}\left\{x(1+x)f''(x)\right\} = \frac{x(1+x)}{2} f^{(k+2)}(x) + \frac{k(1+2x)}{2} f^{(k+1)}(x) + \frac{k(k-1)}{2} f^{(k)}(x).$$

Using this equality we have

$$\left| n\left[D^k \mathcal{B}_n(f;x) - f^{(k)}(x) \right] - \frac{1}{2}\frac{d^k}{dx^k}\left\{ x(1+x) f''(x) \right\} \right| \tag{6.19}$$

$$\le \left| n\left(Q_n^k(e_0;x) - 1 \right) - \frac{k(k-1)}{2} \right| \left| f^{(k)}(x) \right|$$

$$+ \left| nQ_n^k(e_1 - x;x) - \frac{k(1+2x)}{2} \right| \left| f^{(k+1)}(x) \right|$$

$$+ \left| \frac{n}{2} Q_n^k\left((e_1 - x)^2;x \right) - \frac{x(1+x)}{2} \right| \left| f^{(k+2)}(x) \right|$$

$$+ nQ_n^k\left((e_1-x)^2;x \right) 16\left(1+x^2\right) \Omega\left(f^{(k+2)}; \left(\frac{Q_n^k\left((e_1-x)^6;x \right)}{Q_n^k\left((e_1-x)^2;x \right)} \right)^{1/4} \right)$$

$$:= I_1 \left| f^{(k)}(x) \right| + I_2 \left| f^{(k+1)}(x) \right| + I_3 \left| f^{(k+2)(x)} \right|$$

$$+ nQ_n^k\left((e_1-x)^2;x \right) 16\left(1+x^2\right) \Omega\left(f^{(k+2)}; \left(\frac{Q_n^k\left((e_1-x)^6;x \right)}{Q_n^k\left((e_1-x)^2;x \right)} \right)^{1/4} \right).$$

Let us estimate the terms I_1, I_2, I_3. Using the equalities from Lemma 6.3, we have

$$I_1 = \left| n\left(\frac{(n)_k}{n^k} - 1 \right) - \frac{k(k-1)}{2} \right| := \alpha_n, \tag{6.20}$$

$$I_2 = \frac{k(2x+1)}{2}\left(\frac{(n)_k}{n^k} - 1 \right) := \gamma_n(x), \tag{6.21}$$

$$I_3 = \frac{x(1+x)}{2}\left(\frac{(n)_k(n+k^2+k)}{n^{k+1}} - 1 \right) + \frac{k(3k+1)(n)_k}{24n^{k+1}} := \tau_n(x). \tag{6.22}$$

On the other hand by (6.16), there exists a positive constant C such that

$$nQ_n^k\left((e_1-x)^2;x \right) \le Cx(x+1). \tag{6.23}$$

By Lemma 6.4 we get

$$\left(\frac{Q_n^k\left((e_1-x)^6;x \right)}{Q_n^k\left((e_1-x)^2;x \right)} \right)^{1/4} \le \frac{C}{\sqrt{n}}\sqrt{x(x+1)}. \tag{6.24}$$

Putting (6.23) and (6.20)-(6.24) in (6.19), we get the desired result. □

7 Academic future plans

The purpose of this chapter is to present some of the lines that describe the present and future projects in scientific research and the teaching career.

I shall continue my research in the field of approximation by linear positive operators. At the same time, I shall focus on the certain type of inequalities and their applications in theory of linear positive operators. I shall continue to elaborate new scientific papers in all fields quated above, or other areas of mathematics, especially related to complex analysis and probability theory.

I intend to write two scientific monographs related to my contributions in two main directions. One of these summarizing my results in approximation formulas of definite integrals and I shall focus on the methods to obtain quadrature formulas using spline functions. The second monograph will be a survey on Grüss and Ostrowski type inequalities and applications of these inequalities in context of approximation by linear positive operators will be considered.

I would like to publish two books for students in engineering, computer science, mathematics, economics and finance. One of these refers to financial and actuarial mathematics and will contain interest calculation, anuities, life tables, force of mortality and standard life insurance.The second book will be related to the probability and statistics topic.

Some of my future research projects are described in the following.

7.1 Baskakov-Szász type operators based on inverse Pólya-Eggenberger distribution

The inverse Pólya-Eggenberger distribution [113] is defined by

$$Pr(N=n+k)=\binom{n+k-1}{k}\frac{u(u+s)\cdots(u+(n-1)s)v(v+s)\cdots(v+(k-1)s)}{(u+v)(u+v+s)\cdots(u+v+(n+k-1)s)}, k=0,1,\cdots$$

Stancu [166] proposed Baskakov operators based on inverse Pólya-Eggenberger distribution depending on a non-negative parameter $\alpha=\alpha(n)\to 0$ as $n\to\infty$,

$$V_n^{[\alpha]}(f;x)=\sum_{k=0}^{\infty} v_{n,k}^{[\alpha]}(x) f\left(\frac{k}{n}\right),\quad x\in[0,\infty), \tag{7.1}$$

where $v_{n,k}^{[\alpha]}(x)=\binom{n+k-1}{k}\dfrac{1^{[n,-\alpha]}x^{[k,-\alpha]}}{(1+x)^{[n+k,-\alpha]}}$ and $t^{[n,h]}=t(t-h)\cdots(t-(n-1)h)$.

In 1989, Razi [151] introduced a Bernstein-Kantorovich operator based on Pólya-Eggenberger distribution and studied the rate of convergence and degree of approximation for these operators. Very Recently, Deo et al. [57] considered a Stancu-Kantorovich operators based on inverse Pólya-Eggenberger distribution of the operators (7.1) and established some direct results. Păltănea

[142] introduced a generalization of the well known Phillips operators by considering the generalized basis functions under integration depending on certain parameter $\rho > 0$. Gupta and Rassias [97] proposed a Durrmeyer modification of certain Szász type operators and obtained some approximation properties, as asymptotic formula, weighted approximation and error estimation in terms of modulus of continuity. Very recently, Goyal et al. [91] defined a one parameter family of hybrid operators and studied local, weighted and simultaneous approximation properties for these operators.

Inspired by the above work, I would like to start a new joint project with Purshottam Agrawal and Arun Kajla related to the linear positive operators based on inverse Pólya-Eggenberger distribution. We would like to propose a new sequence of summation-integral type operators as follows:

For $\gamma > 0$ and $C_\gamma[0,\infty) := \left\{f \in C[0,\infty) : |f(t)| \leq N_f e^{\gamma t},\ \text{for some } N_f > 0\right\}$, we define

$$\mathcal{R}_{n,\rho}^{[\alpha]}(f;x) = \sum_{k=1}^{\infty} v_{n,k}^{[\alpha]}(x) \int_0^\infty s_{n,k}^{\rho}(t) f(t) dt + v_{n,0}^{[\alpha]}(x) f(0), \tag{7.2}$$

where $s_{n,k}^{\rho}(t) = n\rho e^{-n\rho t}\dfrac{(n\rho t)^{k\rho-1}}{\Gamma(k\rho)}$ and $v_{n,k}^{[\alpha]}(x)$ is defined as above. We note that the operators (7.2) preserve only the constant functions.

Let $e_l(u) = u^l, l \in \mathbb{N} \cup \{0\}$. Applying the definition of Gamma function, we get

$$\begin{aligned} \int_0^\infty s_{n,k}^{\rho}(t) t^l dt &= \int_0^\infty n\rho e^{-n\rho t}\frac{(n\rho t)^{k\rho-1}}{\Gamma(k\rho)} t^l dt = \frac{\Gamma(k\rho+l)}{\Gamma(k\rho)(n\rho)^l} \\ &= \frac{(k\rho+l-1)(k\rho+l-2)\cdots(k\rho)}{(n\rho)^l}. \end{aligned} \tag{7.3}$$

For $f \in C_\gamma[0,\infty)$, we consider

$$L_n(f;t) = \sum_{k=0}^{\infty} b_{n,k}(t) f\left(\frac{k}{n}\right), \tag{7.4}$$

where

$$b_{n,k}(t) = \binom{n+k-1}{k}\frac{t^k}{(1+t)^{n+k}},$$

and

$$K_{n,\rho}(f;t) = \sum_{k=1}^{\infty} b_{n,k}(t) \int_0^\infty s_{n,k}^{\rho}(u) f(u) du + b_{n,0}(t) f(0). \tag{7.5}$$

In order to calculate more easier the moments of the operators $\mathcal{R}_{n,\rho}^{[\alpha]}$, we note that the following representation holds:

Lemma 7.1. *For $\alpha > 0$ and $x \in \mathbb{R}^+$, we have*

$$\mathcal{R}_{n,\rho}^{[\alpha]}(f;x) = \frac{1}{B\left(\frac{x}{\alpha}, \frac{1}{\alpha}\right)} \int_0^\infty \frac{t^{\frac{x}{\alpha}-1}}{(1+t)^{\frac{1+x}{\alpha}}} K_{n,\rho}(f;t) dt,$$

where $B(r,s), r, s > 0$ is the Beta function.

Proof. We know that

$$B(m,n)=\int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}dt=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)},$$
$$\Gamma(w)=\int_0^\infty u^{w-1}e^{-w}du,$$
$$\Gamma(w+n)=w(w+1)\cdots(w+n-1)\Gamma(w),\ \text{for every natural number } n.$$

Hence,

$$\begin{aligned}
B\left(\frac{x}{\alpha}+k,\frac{1}{\alpha}+n\right)&=\frac{\Gamma(\frac{x}{\alpha}+k)\Gamma\left(\frac{1}{\alpha}+n\right)}{\Gamma(\frac{1+x}{\alpha}+k+n)}\\
&=\frac{\frac{x}{\alpha}\left(\frac{x}{\alpha}+1\right)\cdots\left(\frac{x}{\alpha}+k-1\right)\Gamma(\frac{x}{\alpha})\frac{1}{\alpha}\left(\frac{1}{\alpha}+1\right)\cdots\left(\frac{1}{\alpha}+n-1\right)\Gamma(\frac{1}{\alpha})}{\left(\frac{1+x}{\alpha}\right)\left(\frac{1+x}{\alpha}+1\right)\cdots\left(\frac{1+x}{\alpha}+n+k-1\right)\Gamma(\frac{1+x}{\alpha})}\\
&=v_{n,k}^{[\alpha]}(x)\binom{n+k-1}{k}^{-1}B\left(\frac{x}{\alpha},\frac{1}{\alpha}\right).
\end{aligned}\tag{7.6}$$

Using the relation (7.6) the operator $\mathcal{R}_{n,\rho}^{[\alpha]}$ can be written as follows:

$$\begin{aligned}
\mathcal{R}_{n,\rho}^{[\alpha]}(f;x)&=\sum_{k=1}^{\infty}\binom{n+k-1}{k}\frac{B\left(\frac{x}{\alpha}+k,\frac{1}{\alpha}+n\right)}{B\left(\frac{x}{\alpha},\frac{1}{\alpha}\right)}\int_0^\infty s_{n,k}^{\rho}(u)f(u)du+\frac{B\left(\frac{x}{\alpha},\frac{1}{\alpha}+n\right)}{B\left(\frac{x}{\alpha},\frac{1}{\alpha}\right)}f(0)\\
&=\frac{1}{B\left(\frac{x}{\alpha},\frac{1}{\alpha}\right)}\left\{\sum_{k=1}^{\infty}\binom{n+k-1}{k}\left(\int_0^\infty s_{n,k}^{\rho}(u)f(u)du\right)\int_0^\infty\frac{t^{\frac{x}{\alpha}+k-1}}{(1+t)^{\frac{1+x}{\alpha}+n+k}}dt\right.\\
&\left.+\,f(0)\int_0^\infty\frac{t^{\frac{x}{\alpha}-1}}{(1+t)^{\frac{1+x}{\alpha}+n}}\right\}=\frac{1}{B\left(\frac{x}{\alpha},\frac{1}{\alpha}\right)}\int_0^\infty\frac{t^{\frac{x}{\alpha}-1}}{(1+t)^{\frac{1+x}{\alpha}}}K_{n,\rho}(f;t)dt.
\end{aligned}$$

This completes the proof. □

Using (7.3)-(7.5), Lemma 7.1 and ([160], Lemma 1), the values of the first three moments of the operators given by (7.2) can be obtained immediately:

Lemma 7.2. *For $\rho>0$, we have*

i) $\mathcal{R}_{n,\rho}^{[\alpha]}(e_0;x)=1$;

ii) $\mathcal{R}_{n,\rho}^{[\alpha]}(e_1;x)=\dfrac{x}{1-\alpha}$;

iii) $\mathcal{R}_{n,\rho}^{[\alpha]}(e_2;x)=\dfrac{(n+1)x^2}{(1-\alpha)(1-2\alpha)n}+\left[\dfrac{\alpha(n+1)}{(1-\alpha)(1-2\alpha)n}+\dfrac{\rho+1}{n\rho(1-\alpha)}\right]x.$

Remark 7.1. *By simple computations, from Lemma 7.2 we get*

$$\mathcal{R}_{n,\rho}^{[\alpha]}(t-x;x)=\frac{\alpha x}{1-\alpha},$$
$$\mathcal{R}_{n,\rho}^{[\alpha]}((t-x)^2;x)=\frac{(1+n\alpha+2n\alpha^2)x^2}{n(1-\alpha)(1-2\alpha)}+\frac{(1-2\alpha+\rho+(n-1)\alpha\rho)x}{n\rho(1-\alpha)(1-2\alpha)}.$$

Applying Bohman-Korovkin Theorem, the following result holds.

Theorem 7.1. *Let $f\in C_\gamma[0,\infty)$ and $\alpha=\alpha(n)\to 0$ as $n\to\infty$. Then $\lim\limits_{n\to\infty}\mathcal{R}_{n,\rho}^{[\alpha]}(f;x)=f(x)$, uniformly in each compact subset of $[0,\infty)$.*

Using the above results a Voronovskaja type asymptotic formula can be obtained.

Theorem 7.2. *Let* $f \in C_\gamma[0,\infty)$ *and* $\alpha = \alpha(n) \to 0$, *as* $n \to \infty$. *If* f'' *exists at a point* $x \in [0,\infty)$ *and* $\lim_{n\to\infty} n\alpha(n) = l \in \mathbb{R}$, *then we have*

$$\lim_{n\to\infty} n\left[\mathcal{R}_{n,\rho}^{[\alpha]}(f;x) - f(x)\right] = lxf'(x) + \frac{1}{2}\left[(1+l)x^2 + \frac{1+\rho+l\rho}{\rho}x\right] f''(x).$$

Proof. Applying Taylor's expansion, we can write

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + \varepsilon(t,x)(t-x)^2, \tag{7.7}$$

where $\lim_{t\to x} \varepsilon(t,x) = 0$. By using the linearity of the operator $\mathcal{R}_{n,\rho}^{[\alpha]}$, we get

$$\mathcal{R}_{n,\rho}^{[\alpha]}(f;x) - f(x) = \mathcal{R}_{n,\rho}^{[\alpha]}((t-x);x)f'(x) + \frac{1}{2}\mathcal{R}_{n,\rho}^{[\alpha]}((t-x)^2;x)f''(x) + \mathcal{R}_{n,\rho}^{[\alpha]}(\varepsilon(t,x)(t-x)^2;x).$$

Applying the Cauchy-Schwarz inequality, we obtain

$$n\mathcal{R}_{n,\rho}^{[\alpha]}(\varepsilon(t,x)(t-x)^2;x) \leq \sqrt{\mathcal{R}_{n,\rho}^{[\alpha]}(\varepsilon^2(t,x);x)}\sqrt{n^2\mathcal{R}_{n,\rho}^{[\alpha]}((t-x)^4;x)}.$$

In view of Theorem 7.1, $\lim_{n\to\infty} \mathcal{R}_{n,\rho}^{[\alpha]}(\varepsilon^2(t,x);x) = \varepsilon^2(x,x) = 0$, since $\varepsilon(t,x) \to 0$ as $t \to x$. By simple computations for every $x \in [0,\infty)$, we obtain

$$\lim_{n\to\infty} n^2\mathcal{R}_{n,\rho}^{[\alpha]}\left((t-x)^4;x\right) = 3(l+1)^2x^4 + \frac{6(l+1)(\rho l+\rho+1)}{\rho}x^3 + \frac{3(\rho l+\rho+1)^2}{\rho^2}x^2. \tag{7.8}$$

Hence,

$$\lim_{n\to\infty} n\mathcal{R}_{n,\rho}^{[\alpha]}(\varepsilon(t,x)(t-x)^2;x) = 0.$$

Using Remark 7.1 we get

$$\begin{aligned}
&\lim_{n\to\infty} n\mathcal{R}_{n,\rho}^{[\alpha]}\left(t-x;x\right) = lx, \\
&\lim_{n\to\infty} n\mathcal{R}_{n,\rho}^{[\alpha]}\left((t-x)^2;x\right) = (1+l)x^2 + \frac{1+\rho+l\rho}{\rho}x.
\end{aligned} \tag{7.9}$$

Combining the results from above the theorem is proved. □

We will intend to study some approximation properties of the operators $\mathcal{R}_{n,\rho}^{[\alpha]}$, for example the rate of convergence in terms of the moduli of continuity and a Lipschitz type space. Also, the weighed approximation properties for these operators and the rate of convergence for functions with derivatives of bounded variation are another interesting problems that will be studied. Furthermore, the convergence of these operators to certain functions we intend to show by illustrative graphics using MAPLE algorithms.

7.2 Direct results for certain summation-integral type operators

To obtain the generalization of the well known Szász-Mirakjan operators, Jain in [109], introduced the following operators

$$B_n^{\beta}(f,x) = \sum_{k=0}^{\infty} L_{n,k}^{(\beta)}(x)\, f(k/n), \qquad x \in [0,\infty), \tag{7.10}$$

where $0 \le \beta < 1$, and the basis functions are defined as

$$L_{n,k}^{(\beta)}(x) = \frac{nx(nx+k\beta)^{k-1}}{k!}\, e^{-(nx+k\beta)},$$

where it is seen that $\sum_{k=0}^{\infty} L_{n,k}^{(\beta)}(x) = 1$. As a special case when $\beta = 0$, these operators in (7.10) reduce to the well-known Szász-Mirakyan operators. Gupta and Agarwal [98] compiled the results concerning convergence behaviour of different operators for which this convergence is justified.

Recently Gupta and Greubel [94], proposed the Durrmeyer type modification of the operators given in (7.10) and established some direct results.

Inspired by the above work, I would like to start a new joint project with Vijay Gupta related to certain summation-integral type operators Baskakov-Szász. We propose a new sequence of summation-integral type operators as follows:

For $0 \le \beta < 1$ and $c \ge 0$, we propose mixed Durrmeyer type operators for $x \in [0,\infty)$ as

$$\begin{aligned} P_n^{\beta,c}(f,x) &= \sum_{k=1}^{\infty} \left(\int_0^{\infty} L_{n,k-1}^{(\beta)}(t)\, dt \right)^{-1} p_{n,k}(x;c) \int_0^{\infty} L_{n,k-1}^{(\beta)}(t) f(t)\, dt + p_{n,0}(x;c) f(0) \\ &= \sum_{k=1}^{\infty} \frac{< L_{n,k-1}^{(\beta)}, f >}{< L_{n,k-1}^{(\beta)}, 1 >} p_{n,k}(x;c) + p_{n,0}(x;c) f(0), \end{aligned} \tag{7.11}$$

where $< f, g >= \int_0^{\infty} f(t)\, g(t) dt$ and

$$p_{n,k}(x;c) = \frac{(-x)^k}{k!} \phi_{n,c}^{(k)}(x).$$

One has the following special cases:
(a) If $\phi_{n,c}(x) = e^{-nx}, (c = 0), x \in [0,\infty)$ and $\beta = 0$, we get Phillips operators.
(b) In case $\phi_{n,c}(x) = (1+cx)^{-n/c}, (c > 0), x \in [0,\infty)$ and $\beta = 0$, we get the Baskakov-Szász type operators.

Our aim is to study some approximation properties of the operators defined in (7.11) for the case $\phi_{n,c}(x) = (1+cx)^{-n/c}, c > 0, x \in [0,\infty)$.

Lemma 7.3. *Let us define for $m \in \mathbb{N}, c \ge 0$*

$$V_{n,m}^c(x) = \sum_{k=0}^{\infty} p_{n,k}(x;c) \left(\frac{k}{n} \right)^m .$$

Then by simple computation we have

$$nV_{n,m+1}^c(x) = x(1+cx)[V_{n,m}^c(x)]' + nxV_{n,m}^c(x).$$

In particular:

$$V_{n,0}^c(x) = 1, \quad V_{n,1}^c(x) = x, \quad V_{n,2}^c(x) = x^2 + \frac{x(1+cx)}{n}.$$

Let $e_n : [a,b] \to \mathbb{R}$, $e_n(x) = x^n$, $n \in \mathbb{N}_0$ be the n-th monomial. If we denote

$$P_r(k;\beta) := \frac{\langle L_{n,k-1}^{(\beta)}, e_r \rangle}{\langle L_{n,k-1}^{(\beta)}, e_0 \rangle},$$

then $P_r(k;\beta)$ is a polynomial of order r in variable k.

Lemma 7.4. *[94] For $0 \leq \beta < 1$, $r \geq 0$, the polynomials $P_r(k;\beta)$ satisfy the recurrence relationship*

$$n^2 P_{r+2}(k;\beta) = n[(1-\beta)k + r + 2]P_{r+1}(k;\beta) + \beta k(r+2)P_r(k;\beta). \tag{7.12}$$

Using the relation (7.12) the polynomials $P_r(k;\beta)$ can be determined. We will recall in the following the result obtained in [95].

Lemma 7.5. *[95] For $0 \leq \beta < 1$, we have*

$$\begin{aligned}
&P_0(k-1;\beta) = 1, \\
&P_1(k-1;\beta) = \frac{1}{n}\left[(1-\beta)\,k + \frac{\beta\,(2-\beta)}{1-\beta}\right], && (7.13) \\
&P_2(k-1;\beta) = \frac{1}{n^2}\left[(1-\beta)^2\,k^2 + a_1^2\,k + \frac{\beta^2\,(3-\beta)}{1-\beta}\right], && (7.14) \\
&P_3(k-1;\beta) = \frac{1}{n^3}\left[(1-\beta)^3\,k^3 + 3\,a_1^3\,k^2 + \frac{a_2^3\,k}{1-\beta} + \frac{\beta^3\,(4-\beta)}{1-\beta}\right], \\
&P_4(k-1;\beta) = \frac{1}{n^4}\left[(1-\beta)^4\,k^4 + 2\,a_1^4\,k^3 + a_2^4\,k^2 + \frac{2\,a_3^4\,k}{1-\beta} + \frac{\beta^4\,(5-\beta)}{1-\beta}\right], \\
&P_5(k-1;\beta) = \frac{1}{n^5}\left[(1-\beta)^5\,k^5 + 5\,a_1^5\,k^4 + 5\,a_2^5\,k^3 + \frac{5\,a_3^5\,k^2}{1-\beta} + \frac{a_4^5\,k}{1-\beta} + \frac{\beta^5\,(6-\beta)}{1-\beta}\right],
\end{aligned}$$

where

$$\begin{aligned}
&a_1^2 = 1 + 4\beta - 2\beta^2, \\
&a_1^3 = 1 + \beta - 3\beta^2 + \beta^3, \quad a_2^3 = 2 + 4\beta + 6\beta^2 - 12\beta^3 + 3\beta^4, \\
&a_1^4 = 3 - 2\beta - 7\beta^2 + 8\beta^3 - 2\beta^4, \quad a_2^4 = 11 + 16\beta + 6\beta^2 - 24\beta^3 + 6\beta^4, \\
&a_3^4 = 3 + 5\beta + 5\beta^2 + 5\beta^3 - 10\beta^4 + 2\beta^5, \\
&a_1^5 = (1-\beta)^3\,(2 + 2\beta - \beta^2), \quad a_2^5 = (1-\beta)\,(7 + 8\beta - 8\beta^3 + 2\beta^4), \\
&a_3^5 = 10 + 6\beta - 3\beta^2 - 8\beta^3 - 12\beta^4 + 12\beta^5 - 2\beta^6, \\
&a_4^5 = 24 + 36\beta + 30\beta^2 + 20\beta^3 + 15\beta^4 - 30\beta^5 + 5\beta^6.
\end{aligned}$$

Let $T_{n,r}^{\beta,c}(x) := P_n^{\beta,c}(e_r, x), r = 0, 1, \ldots$ be the r-th order moment of the operators (7.11). Therefore,

$$T_{n,r}^{\beta,c}(x) = \sum_{k=1}^{\infty} \left(\int_0^{\infty} L_{n,k-1}^{(\beta)}(t)\,dt \right)^{-1} p_{n,k}(x;c) \int_0^{\infty} L_{n,k-1}^{(\beta)}(t)\, t^r\, dt$$
$$= \sum_{k=1}^{\infty} P_r(k-1;\beta)\, p_{n,k}(x;c), \text{ for } r = 1,2,\ldots.$$

Lemma 7.6. *For $r = 0,1,2$ the moments of the operators (7.11) are*

$$T_{n,0}^{\beta,c}(x) = 1, \qquad T_{n,1}^{\beta,c}(x) = (1-\beta)x + \frac{\beta(2-\beta)}{n(1-\beta)}(1-\phi_{n,c}(x)),$$
$$T_{n,2}^{\beta,c}(x) = (1-\beta)^2 \left[x^2 + \frac{x(1+cx)}{n} \right] + \frac{(1+4\beta-2\beta^2)x}{n} + \frac{\beta^2(3-\beta)}{n^2(1-\beta)}(1-\phi_{n,c}(x)).$$

Proof. Obviously by (7.11), we have $T_{n,0}^{\beta,c}(x) = 1$. Using Lemma 7.3 and Lemma 7.5, we have

$$T_{n,1}^{\beta,c}(x) = \sum_{k=1}^{\infty} P_1(k-1;\beta)\, p_{n,k}(x;c) = \sum_{k=1}^{\infty} p_{n,k}(x;c)\, \frac{1}{n} \left[(1-\beta)k + \frac{\beta(2-\beta)}{1-\beta} \right]$$
$$= (1-\beta) V_{n,1}^{c}(x) + \frac{\beta(2-\beta)}{n(1-\beta)} \left(V_{n,0}^{c}(x) - \phi_{n,c}(x) \right)$$
$$= (1-\beta)x + \frac{\beta(2-\beta)}{n(1-\beta)}(1-\phi_{n,c}(x)).$$

$$T_{n,2}^{\beta,c}(x) = \sum_{k=1}^{\infty} P_2(k-1;\beta)\, p_{n,k}(x;c)$$
$$= \sum_{k=1}^{\infty} \frac{1}{n^2} \left[(1-\beta)^2 k^2 + (1+4\beta-2\beta^2)\, k + \frac{\beta^2(3-\beta)}{1-\beta} \right] p_{n,k}(x;c)$$
$$= (1-\beta)^2\, V_{n,2}^{c}(x) + \frac{1+4\beta-2\beta^2}{n}\, V_{n,1}^{c}(x) + \frac{\beta^2(3-\beta)}{n^2(1-\beta)} \left(V_{n,0}^{c}(x) - \phi_{n,c}(x) \right)$$
$$= (1-\beta)^2 \left[x^2 + \frac{x(1+cx)}{n} \right] + \frac{(1+4\beta-2\beta^2)x}{n} + \frac{\beta^2(3-\beta)}{n^2(1-\beta)}(1-\phi_{n,c}(x)).$$

A continuation of this process will provide $T_{n,r}^{\beta}(x)$ for cases of $r \geq 3$. ☐

Remark 7.2. *If we denote the central moment as $\mu_{n,r}^{\beta,c}(x) = P_n^{\beta,c}((t-x)^r, x)$, then*

$$\mu_{n,1}^{\beta,c}(x) = -\beta x + \frac{\beta(2-\beta)}{n(1-\beta)}(1-\phi_{n,c}(x)),$$
$$\mu_{n,2}^{\beta,c}(x) = \left[\beta^2 + \frac{c(1-\beta)^2}{n} \right] x^2 + \frac{2-4\beta-\beta^2+\beta^3+2\beta(2-\beta)\phi_{n,c}(x)}{n(1-\beta)} x + \frac{\beta^2(3-\beta)(1-\phi_{n,c}(x)}{n^2(1-\beta)}.$$

Lemma 7.7. *If $\beta = \beta(n) \to 0$, as $n \to \infty$ and $\lim_{n\to\infty} n\beta(n) = l \in \mathbb{R}$, then*

i) $\lim_{n\to\infty} n\mu_{n,1}^{\beta,c}(x) = -lx,$

ii) $\lim_{n\to\infty} n\mu_{n,2}^{\beta,c}(x) = x(cx+2),$

iii) $\lim_{n\to\infty} n^2\mu_{n,4}^{\beta,c}(x) = 3x^2(cx+2)^2.$

Furthermore, we can obtain the rate of convergence in terms of the moduli of continuity.

Remark 7.3. *Let f be a continuous function on $[0,\infty)$. For $n \to \infty$, the sequence $\{P_n^{\beta,c}(f,x)\}$ converges uniformly to $f(x)$ in $[a,b] \subset [0,\infty)$, provided $\beta = \beta(n) \to 0$ for sufficiently large n, which follows from the well known Bohman-Korovkin theorem.*

Also, we can establish Voronovskaja type result for the $P_n^{\beta,c}$ operators.

Theorem 7.3. *Let f be a bounded integrable function on $[0,\infty)$ and f'' exists at a point $x \in [0,\infty)$ and $c \geq 0$. If $\beta = \beta(n) \to 0$, as $n \to \infty$ and $\lim\limits_{n\to\infty} n\beta(n) = l \in \mathbb{R}$, then*

$$\lim_{n\to\infty} n\left[P_n^{\beta,c}(f,x) - f(x)\right] = -lxf'(x) + \frac{cx^2+2x}{2}f''(x).$$

Proof. By the Taylor's expansion of f, we have

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + r(t,x)(t-x)^2, \tag{7.15}$$

where $\lim\limits_{t\to x} r(t,x) = 0$. Operating $P_n^{\beta,c}$ to the identity (7.15), we obtain

$$P_n^{\beta,c}(f,x) - f(x) = P_n^{\beta,c}(t-x,x)f'(x) + P_n^{\beta,c}\left((t-x)^2, x\right)\frac{f''(x)}{2} + P_n^{\beta,c}\left(r\left(t,x\right)\left(t-x\right)^2, x\right).$$

Using the Cauchy-Schwarz inequality, we have

$$P_n^{\beta,c}\left(r\left(t,x\right)\left(t-x\right)^2, x\right) \leq \sqrt{P_n^{\beta,c}\left(r^2\left(t,x\right), x\right)}\sqrt{P_n^{\beta,c}\left((t-x)^4, x\right)}. \tag{7.16}$$

In view of Remark 7.3, we have

$$\lim_{n\to\infty} P_n^{\beta}\left(r^2\left(t,x\right), x\right) = r^2\left(x,x\right) = 0. \tag{7.17}$$

Now from (7.16), (7.17) and from Lemma 7.7, we get

$$\lim_{n\to\infty} nP_n^{\beta,c}\left(r\left(t,x\right)\left(t-x\right)^2, x\right) = 0.$$

Thus

$$\lim_{n\to\infty} n\left(P_n^{\beta,c}(f,x) - f(x)\right) = \lim_{n\to\infty} n\Big[P_n^{\beta,c}(t-x,x)f'(x) + \frac{1}{2}f''(x)P_n^{\beta,c}((t-x)^2, x)$$
$$+ P_n^{\beta,c}(r\left(t,x\right)\left(t-x\right)^2, x)\Big].$$

The result follows immediately by applying Lemma 7.7. □

We intend to study some approximation properties of the operators $P_n^{\beta,c}$, for example the rate of convergence in terms of the moduli of continuity. Also, the weighed approximation properties for these operators and the rate of convergence for functions with derivatives of bounded variation are another interesting problems that will be studied.

7.3 Chlodowsky variant of Jakimovski-Leviatan-Durrmeyer type operators

Let $g(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(1) \neq 0$ be an analytic function in the disc $|z| < R$, $R > 0$ and p_k be the Appell polynomials defined by the relation

$$g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k.$$

Therefore, the polynomials are

$$p_k(x) = \sum_{i=0}^{k} a_i \frac{x^{k-i}}{(k-i)!}, \ k \in \mathbb{N}.$$

Let $C_\gamma[0,\infty)$ be the class of functions with exponential growth on $[0,\infty)$, namely $|f(x)| \leq N_f e^{\gamma x}$ for some finite $N_f, \gamma > 0$.

In 1969, Jakimovski and Leviatan [110] defined a sequence of operators as follows

$$P_n(f;x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right). \tag{7.18}$$

Büyükyazici et al. [52] introduced the Chlodowsky variant of the operators defined by (7.18) as

$$P_n^*(f;x) = \frac{e^{-nx/b_n}}{g(1)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) f\left(\frac{k}{n}b_n\right), \tag{7.19}$$

where b_n is a positive increasing sequence with the properties $\lim_{n\to\infty} b_n = \infty$ and $\lim_{n\to\infty} \frac{b_n}{n} = 0$ and proved a Korovkin-type theorem, Voronovskaya-type asymptotic approximation theorem and gave the rate of approximation of these operators in a weighted space.

Very recently, Karaisa [112] introduced the Durrmeyer type Jakimovski-Leviatan operators defined as follows

$$L_n(f;x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} \frac{p_k(nx)}{B(n+1,k)} \int_0^{\infty} \frac{t^{k-1}}{(1+t)^{n+k+1}} f(t)dt, \ x \geq 0, \tag{7.20}$$

where $B(x,y)$ is the Beta function, and established some approximation properties of these operators.

Inspired by the above work, I would like to start a new joint project with Purshottam Agrawal and Trapti Neer related to this subject. We intend to introduce the Chlodowsky variant of the operators defined by (7.20) as follows:

$$L_n^*(f;x) = \frac{e^{-nx/b_n}}{g(1)} \sum_{k=0}^{\infty} \frac{p_k\left(\frac{n}{b_n}x\right)}{B(n+1,k)} \int_0^{\infty} \frac{t^{k-1}}{(1+t)^{n+k+1}} f(b_n t)dt. \tag{7.21}$$

We want to give the rate of approximation in terms of first order modulus of continuity and the Ditzian-Totik modulus of smoothness. Also we will introduce a Voronovskaja-type asymptotic formula and we will give some approximation results for a weighted space.

7.4 Estimates for the differences of positive linear operators and their derivatives

In [9] we obtained some inequalities for a positive linear functional using Taylor's formula. These results led us to new estimates of the differences of certain positive linear operators. Some applications for some known positive linear operators were given.

I would like to continue this joint project with Ioan Raşa by giving estimates for such differences of certain positive linear operators involving their derivatives. For the biginning of this study we considered Bernstein operators and Durrmeyer operators.

Proposition 7.1. *For Bernstein operators the following property holds:*

$$\left\|(B_n f)^{(k)} - B_{n-k}\left(f^{(k)}\right)\right\| \leq \frac{(k-1)k}{2n}\|f^{(k)}\| + \omega\left(f^{(k)}, \frac{k}{n}\right),\ f \in C^k[0,1],\ k = 0, 1, \ldots n.$$

Proof. The differences of Bernstein operators and their derivatives can be written as

$$\begin{aligned}
&(B_n(f;x))^{(k)} - B_{n-k}\left(f^{(k)}(x)\right)\\
&= n(n-1)\ldots(n-k+1)\sum_{i=0}^{n-k} p_{n-k,i}(x)\Delta^k_{\frac{1}{n}} f\left(\frac{i}{n}\right) - \sum_{i=0}^{n-k} p_{n-k,i}(x) f^{(k)}\left(\frac{i}{n-k}\right)\\
&= \sum_{i=0}^{n-k} p_{n-k,i}(x)\left\{n(n-1)\ldots(n-k+1)\Delta^k_{\frac{1}{n}} f\left(\frac{i}{n}\right) - f^{(k)}\left(\frac{i}{n-k}\right)\right\}\\
&= \sum_{i=0}^{n-k} p_{n-k,i}(x)\left\{\frac{n(n-1)\ldots(n-k+1)}{n^k} k!\left[\frac{i}{n}, \ldots, \frac{i+k}{n}; f\right] - f^{(k)}\left(\frac{i}{n-k}\right)\right\}\\
&= \sum_{i=0}^{n-k} p_{n-k,i}(x)\left\{\frac{n(n-1)\ldots(n-k+1)}{n^k} f^{(k)}(\xi_i) - f^{(k)}\left(\frac{i}{n-k}\right)\right\}\\
&= \sum_{i=0}^{n-k} p_{n-k,i}(x)\left\{\left(\frac{n(n-1)\ldots(n-k+1)}{n^k} - 1\right) f^{(k)}(\xi_i) + f^{(k)}(\xi_i) - f^{(k)}\left(\frac{i}{n-k}\right)\right\},
\end{aligned}$$

where $\dfrac{i}{n} \leq \xi_i \leq \dfrac{i+k}{n}$.
But,

$$0 \leq 1 - \frac{n(n-1)\ldots(n-k+1)}{n^k} \leq \frac{k(k-1)}{2n} \text{ and } \frac{i}{n} \leq \frac{i}{n-k} \leq \frac{i+k}{n},\ \text{for } 0 \leq i \leq n-k.$$

Therefore,

$$\left\|(B_n f)^{(k)} - B_{n-k}\left(f^{(k)}\right)\right\| \leq \frac{(k-1)k}{2n}\|f^{(k)}\| + \omega\left(f^{(k)}, \frac{k}{n}\right).$$

□

In order to give the estimate for the difference of the Durrmeyer operators and their derivatives we need the following result:

Lemma 7.8. *Let $F : C[0,1] \to \mathbb{R}$ be a positive linear functional with $F(e_0) = 1$ and $F(e_1) = b$. Then, for all $\varphi \in C^2[0,1]$ there is $\xi \in [0,1]$ such that*

$$F(\varphi) - \varphi(b) = (F(e_2) - e_2(b))\frac{\varphi''(\xi)}{2}.$$

Proposition 7.2. *For Durrmeyer operators the following property holds:*

$$\left\| \frac{(n+r+1)!(n-r)!}{(n+1)!n!} (M_n f)^{(r)} - M_{n-r}\left(f^{(r)}\right) \right\| \le \frac{1}{4} \|f^{(r+2)}\| \frac{n+3}{(n+3)^2 - r^2} + \omega\left(f^{(r)}, \frac{r(n-r)}{(n+2)^2 - r^2}\right),$$

for $f \in C^{r+2}[0,1],\ r = 0, 1, \dots n.$

Proof. In [58], M.M. Derriennic introduced the following relation for the derivative of Durrmeyer operators:

$$(M_n(f;x))^{(r)} = \frac{(n+1)!n!}{(n-r)!(n+r)!} \sum_{k=0}^{n-r} p_{n-r,k}(x) \int_0^1 p_{n+r,k+r}(t) f^{(r)}(t)dt.$$

Therefore, we can write

$$\frac{(n+r+1)!(n-r)!}{(n+1)!n!} (M_n(f;x))^{(r)} - M_{n-r}\left(f^{(r)};x\right)$$

$$= \sum_{k=0}^{n-r} p_{n-r,k}(x) \int_0^1 \left((n+r+1)p_{n+r,k+r}(t) - (n-r+1)p_{n-r,k}(t)\right) f^{(r)}(t)dt.$$

Let $\varphi \in C^2[0,1]$. With fixed $0 \le r \le n$, consider the functional

$$A_k(\varphi) := \int_0^1 \left((n+r+1)p_{n+r,k+r}(t) - (n-r+1)p_{n-r,k}(t)\right) \varphi(t)dt = B_k(\varphi) - C_k(\varphi),$$

where

$$B_k(\varphi) := \int_0^1 \frac{(n+r+1)!}{(k+r)!(n-k)!} t^{k+r}(1-t)^{n-k}\varphi(t)dt,$$
$$C_k(\varphi) = \int_0^1 \frac{(n-r+1)!}{k!(n-k-r)!} t^k(1-t)^{n-k-r}\varphi(t)dt.$$

By simple calculations, we get

$$B_k(e_0) = 1, \quad B_k(e_1) = \frac{k+r+1}{n+r+2}, \quad B_k(e_2) = \frac{(k+r+1)(k+r+2)}{(n+r+2)(n+r+3)},$$
$$C_k(e_0) = 1, \quad C_k(e_1) = \frac{k+1}{n-r+2}, \quad C_k(e_2) = \frac{(k+1)(k+2)}{(n-r+2)(n-r+3)}.$$

Therefore,

$$|A_k(\varphi)| = |B_k(\varphi) - C_k(\varphi)| \le \left| B_k(\varphi) - \varphi\left(\frac{k+r+1}{n+r+2}\right) \right|$$
$$+ \left| C_k(\varphi) - \varphi\left(\frac{k+1}{n-r+2}\right) \right| + \left| \varphi\left(\frac{k+r+1}{n+r+2}\right) - \varphi\left(\frac{k+1}{n-r+2}\right) \right|$$
$$\le \frac{1}{2}\|\varphi''\| \left(\frac{(k+r+1)(n-k+1)}{(n+r+2)^2(n+r+3)} + \frac{(k+1)(n-r-k+1)}{(n-r+2)^2(n-r+3)} \right) + \omega\left(\varphi, r\frac{|n-r-2k|}{(n+2-r)(n+2+r)}\right)$$
$$\le \frac{1}{4}\|\varphi''\| \frac{n+3}{(n+3)^2 - r^2} + \omega\left(\varphi, \frac{r(n-r)}{(n+2)^2 - r^2}\right).$$

Thus, for $f \in C^{r+2}[0,1]$ the Proposition is proved. ∎

7.5 Estimation of P_n-simple functionals

The divided difference $[x_1, x_2, \ldots, x_n; f]$ of a function $f : [a,b] \to \mathbb{R}$ on the distinct knots $x_1, x_2, \ldots, x_n \in [a,b]$ is defined by

$$[x_1, x_2, \ldots, x_n; f] = \sum_{k=1}^{n} \frac{f(x_k)}{\omega'(x_k)},$$

where $\omega(x) = (x - x_1)(x - x_2)\ldots(x - x_n)$.

In [147] T. Popoviciu introduced the concept of a P_n-simple functional ($n \in \mathbb{Z}, n \geq -1$).

Definition 7.1. *[147] Let $F : C[a,b] \to \mathbb{R}$ be a linear functional. One says that F is a P_n-simple functional if the following requirements are satisfied:*

i) $F(e_{n+1}) \neq 0$, where $e_i(x) = x^i, i \in \mathbb{N}$;

ii) For any function $f \in C[a,b]$ there exist distinct points $t_i = t_i(f) \in [a,b]$, $i \in \{1, \ldots, n+2\}$, such that

$$F(f) = F(e_{n+1})[t_1, t_2, \ldots, t_{n+2}; f].$$

Definition 7.2. *A linear functional $F : C[a,b] \to \mathbb{R}$ has degree of exactness n if it vanishes for $e_j, j \in \{0, 1, \ldots, n\}$.*

It is easy to see that if F is a P_n-simple functional, then it has degree of exactness n.

In [146] T.Popoviciu obtained the following result.

Theorem 7.4. *[146] Let $F : C[a,b] \to \mathbb{R}$ be a linear and bounded P_n-simple functional, $n \geq 0$. If $F_1 : C^{n+1}[a,b] \to \mathbb{R}$ is defined by*

$$F_1(f) = F(f) - F(e_{n+1}) \frac{f^{(n+1)}(c)}{(n+1)!} \ , \ f \in C^{n+1}[a,b],$$

where $c \in (a,b)$ is given by the following equation

$$F(e_{n+2}) - (n+2)F(e_{n+1})c = 0,$$

then F_1 is a P_{n+2}-simple functional.

In [125] A. Lupaş gave some representations of a linear and positive functional defined on $C[a,b]$ using its values on test functions e_j. These representations involve divided differences.

Theorem 7.5. *[125] If $F : C[a,b] \to \mathbb{R}$ is a linear and positive functional that verifies*

$$F(e_0) = 1, F(e_j) = a_j, \ j = 1, 2,$$

then, for any $f \in C[a,b]$, there exist the distinct points $\theta_i = \theta_i(f) \in [a,b]$, $i = 1, 2$, such that

$$F(f) = f(a_1) + (a_2 - a_1^2)[\theta_1, \frac{\theta_1 + \theta_2}{2}, \theta_2; f].$$

The above mean value theorem was stated for $C^2[a,b]$ as follows.

Corollary 7.1. *[125] If $F : C^2[a,b] \to R$ is a linear and positive functional that verifies*

$$F(e_0) = 1, \quad F(e_i) = a_i, \quad i = 1, 2, 3, 4, \quad a_2 \neq a_1^2,$$

then for any $f \in C^2[a,b]$ there are two distinct points $\theta_i = \theta_i(f) \in [a,b]$, $i = 1, 2$, such that

$$F(f) = f(a_1) + \frac{a_2 - a_1^2}{2} f'' \left(\frac{a_3 - a_1^3}{3(a_2 - a_1^2)} \right)$$

$$+ \frac{a_1^6 - 3a_1^4 a_2 + 4a_1^3 a_3 - 3a_1^2 a_4 + 3a_2 a_4 - 2a_3^2}{3(a_2 - a_1^2)} [\theta_1, \theta_{31}, \theta_{22}, \theta_{13}, \theta_2; f],$$

where

$$\theta_{ij} = \frac{i}{4}\theta_1 + \frac{j}{4}\theta_2, \; i = 1, 2, 3, \; j = 4 - i.$$

Furthermore, some similar results were given for linear and positive operators.

Theorem 7.6. *[125] If the linear and positive operators $L_n : C[a,b] \to C[a,b]$ verify $L_n(e_0; x) = e_0(x)$, $L_n(e_k; x) = a_{k,n}(x), k = 1, 2, 3, 4,$ then*

i) for all $x \in [a,b]$ and $f \in C[a,b]$, there are the distinct points $\theta_{i,n} = \theta_{i,n}(f, x) \in [a,b]$, $i = 1, 2$, such that

$$L_n(f; x) = f(a_{1,n}(x)) + \left[a_{2,n}(x) - a_{1,n}^2(x)\right] [\theta_{1,n}, \frac{\theta_{1,n} + \theta_{2,n}}{2}, \theta_{2,n}; f]; \tag{7.22}$$

ii) for all $x \in M := \left\{x \in [a,b]; a_{2,n}(x) \neq a_{1,n}^2(x)\right\}$ and $g \in C^2[a,b]$ there exist distinct points $\xi_{i,n} = \xi_{i,n}(g, x) \in [a,b]$, $i = 1, 2$, such that

$$L_n(g; x) = g(a_{1,n}(x)) + \frac{a_{2,n}(x) - a_{1,n}^2(x)}{2} g''(z_n(x)) +$$

$$+ K_n(x) \left[\xi_{1,n}, \frac{3\xi_{1,n} + \xi_{2,n}}{4}, \frac{\xi_{1,n} + \xi_{2,n}}{2}, \frac{\xi_{1,n} + 3\xi_{2,n}}{4}, \xi_{2,n}; g \right],$$

where

$$z_n = \frac{a_{3,n} - (a_{1,n})^3}{3[a_{2,n} - (a_{1,n})^2]},$$
$$K_n = a_{4,n} - (a_{1,n})^4 - 6[a_{2,n} - (a_{1,n})^2] z_n^2.$$

In [8] we propose the following generalization of these results.

Let $F : C^{2n-2}[a,b] \to \mathbb{R}$ be a linear positive functional that verifies the conditions

$$F(e_0) = 1, F(e_i) = a_i, i \in \{1, \ldots, 2n\}. \tag{7.23}$$

Define

$$u_{1,j} = a_j - a_1^j, j \in \{1, \ldots, 2n\},$$

$$u_{k,j} = \begin{cases} u_{k-1,j}, \; j \in \{1, \ldots, 2k-3\}, \\ u_{k-1,j} - u_{k-1,2k-2} \binom{j}{2k-2} \left[\dfrac{u_{k-1,2k-1}}{(2k-1) u_{k-1,2k-2}} \right]^{j-2k+2}, \\ j \in \{2k-2, \ldots, 2n\}, \end{cases}$$

for $k \geq 2$.

Theorem 7.7. *[8] If $F : C^{2n-2}[a,b] \to \mathbb{R}$ is a linear and positive functional that verifies conditions (7.23) and $u_{k,2k} \neq 0, k = \{1,\ldots,n-1\}$, then for any $f \in C^{2n-2}[a,b]$ there are two distinct points $\theta_i = \theta_i(f)$, $i = 1,2$, such that*

$$F(f) = \sum_{k=0}^{n-1} c_k f^{(2k)}(z_k) + K[\theta_1, \theta_{2n-1,1}, \theta_{2n-2,2}, \cdots, \theta_{1,2n-1}, \theta_2; f], n \geq 1,$$

where

$$\theta_{i,j} = \frac{i\theta_1 + j\theta_2}{2n}, \; j \in \{1,\ldots,2n-1\}, \; i = 2n-j,$$
$$c_0 = 1, z_0 = a_1, c_k = \frac{u_{k,2k}}{(2k)!}, z_k = \frac{u_{k,2k+1}}{(2k+1)u_{k,2k}}, \text{ for } k \geq 1,$$
$$K = u_{n,2n}.$$

Proof. Define $R_1(f) := F(f) - f(a_1)$. Using Theorem 7.5 one has

$$R_1(f) = (a_2 - a_1^2)[\theta_1, \frac{\theta_1+\theta_2}{2}, \theta_2; f].$$

Since $R_1(e_2) = a_2 - a_1^2 \neq 0$, we obtain that R_1 is a P_1-simple functional. From Theorem 7.4 it follows that

$$R_2(f) := R_1(f) - R_1(e_2)\frac{f''(z_1)}{2!}$$

is a P_3-simple functional, where $z_1 = \dfrac{R_1(e_3)}{3R_1(e_2)}$.

It can be proved by induction that

$$R_n(f) := R_{n-1}(f) - R_{n-1}(e_{2n-2})\frac{f^{(2n-2)}(z_{n-1})}{(2n-2)!} \tag{7.24}$$

is a P_{2n-1}-simple functional, where $z_{n-1} = \dfrac{R_{n-1}(e_{2n-1})}{(2n-1)R_{n-1}(e_{2n-2})}$. Since R_n is a P_{2n-1}-simple functional, for any function $f \in C[a,b]$, there exist distinct points $t_i = t_i(f) \in [a,b]$, $i \in \{1,\ldots,2n+1\}$, such that

$$R_n(f) = R_n(e_{2n})[t_1, t_2, \ldots, t_{2n+1}; f].$$

Therefore,

$$R_{n-1}(f) - R_{n-1}(e_{2n-2})\frac{f^{(2n-2)}(z_{n-1})}{(2n-2)!} = R_n(e_{2n})[t_1, t_2, \ldots, t_{2n+1}; f]. \tag{7.25}$$

From the relations (7.24) and (7.25) it follows that

$$F(f) - \sum_{k=0}^{n-1} c_k f^{(2k)}(z_k) = R_n(e_{2n})[t_1, t_2, \ldots, t_{2n+1}; f], \tag{7.26}$$

where $c_k = \dfrac{R_k(e_{2k})}{(2k)!}$.

Using the relation (7.24) for any $k \geq 2$ we infer that

$$R_k(e_j) = R_{k-1}(e_j),\ j \in \{0, \ldots, 2k-3\},$$
$$R_k(e_j) = R_{k-1}(e_j) - R_{k-1}(e_{2k-2}) \binom{j}{2k-2} \left(\frac{R_{k-1}(e_{2k-1})}{(2k-1)R_{k-1}(e_{2k-2})} \right)^{j-2k+2}, j \in \{2k-2, \ldots, 2n\}.$$

From the above relations and the definition of $u_{k,j}$, it follows that

$$R_k(e_j) = u_{k,j}. \tag{7.27}$$

T. Popoviciu established the following result (see [145], p. 176):

If $f \in C[a, b]$ and $x_0, x_1, ..., x_n$ are distinct points which belong to $[a, b]$, then there exist two distinct points y_1, $y_2 \in [a, b]$, such that

$$[x_0, x_1, x_2, \ldots, x_n; f] = [y_1, y_1 + \frac{1}{n}(y_2 - y_1), y_1 + \frac{2}{n}(y_2 - y_1), \ldots, y_2; f].$$

Using this result and the relations (7.26)-(7.27), the theorem is proved. □

Remark 7.4. *[8] Let $w_j := a_j - a_1^j, j \in \{1, \ldots, 2n\}$. From Theorem 7.7 it follows that*

i) for n=1,

$$F(f) = f(a_1) + w_2[\theta_1, \frac{\theta_1 + \theta_2}{2}, \theta_2; f].$$

This result was obtained in [125]; see the above Theorem 7.5.

ii) for n=2,

$$F(f) = f(a_1) + c_1 f''(z_1) + K[\theta_1, \theta_{3,1}, \theta_{2,2}, \theta_{1,3}, \theta_2; f],$$

where

$$\theta_{i,j} = \frac{i\theta_1 + j\theta_2}{4}, j \in \{1, 2, 3\}, i = 4 - j,$$
$$z_1 = \frac{1}{3}\frac{w_3}{w_2}, c_1 = \frac{1}{2}w_2,$$
$$K = \frac{1}{3}\frac{3w_4w_2 - 2w_3^2}{w_2}.$$

This result was given in [125, Corollary 2]; see also the above Corollary 7.1.

iii) for n=3,

$$F(f) = f(a_1) + c_1 f''(z_1) + c_2 f^{(4)}(z_2) + K[\theta_1, \theta_{5,1}, \theta_{4,2}, \theta_{3,3}, \theta_{2,4}, \theta_{1,5}, \theta_2; f],$$

where

$$c_1 = \frac{1}{2}w_2,\ c_2 = \frac{1}{72}\frac{3w_4w_2 - 2w_3^2}{w_2},\ \ \theta_{i,j} = \frac{i\theta_1 + j\theta_2}{6}, j \in \{1, \ldots, 5\}, i = 6 - j,$$
$$z_1 = \frac{w_3}{3w_2},\ \ z_2 = \frac{27w_2^2w_5 - 10w_3^3}{45w_2(3w_2w_4 - 2w_3^2)},$$
$$K = w_6 - \frac{5w_3^4}{27w_2^3} - \frac{9w_2}{5(3w_2w_4 - 2w_3^2)} \left(w_5 - \frac{10}{27}\frac{w_3^3}{w_2^2} \right)^2.$$

This representation was considered by F. Sofonea in [161].

A similar result can be obtained for linear positive operators as follows:

Theorem 7.8. *[8] If the linear positive operators* $L_n : C^{2p-2}[a,b] \to C[a,b]$ *verify* $L_n(e_0;x) = e_0(x)$, $L_n(e_k;x) = a_{k,n}(x), k \in \{1,\ldots,2p\}$, *then for any* $x \in M := \Big\{x \in [a,b]; u^n_{k,2k}(x) \neq 0$, $k = 1,\ldots,p-1\}$ *and* $f \in C^{2p-2}[a,b]$ *there are two distinct points* $\theta_i^n = \theta_i^n(f;x) \in [a,b]$, $i = 1,2$, *such that*

$$L_n(f;x) = \sum_{k=0}^{p-1} c_{k,n}(x) f^{(2k)}(z_{k,n}(x)) + K_n(x)[\theta_1^n, \theta^n_{2p-1,1}, \theta^n_{2p-2,2}, \ldots, \theta^n_{1,2p-1}, \theta_2; f], p \geq 1, \quad (7.28)$$

where

$$\theta^n_{i,j} = \frac{i\theta^n_1 + j\theta^n_2}{2p}, j \in \{1,\ldots,2p-1\},\ i = 2p-j,$$

$$u^n_{1,j}(x) = a_{j,n}(x) - a^j_{1,n}(x), j \in \{1,\ldots,2p\},$$

$$u^n_{k,j}(x) = \begin{cases} u^n_{k-1,j}(x),\ j \in \{1,\ldots,2k-3\}, \\ u^n_{k-1,j}(x) - u^n_{k-1,2k-2}(x)\binom{j}{2k-2} \cdot \left[\dfrac{u^n_{k-1,2k-1}(x)}{(2k-1)u^n_{k-1,2k-2}(x)}\right]^{j-2k+2}, \\ j \in \{2k-2,\ldots,2p\}, \end{cases}$$

for $k \geq 2$,

$$c_{0,n}(x) = 1, z_{0,n}(x) = a_{1,n}(x), c_{k,n}(x) = \frac{u^n_{k,2k}(x)}{(2k)!}, z_{k,n}(x) = \frac{u^n_{k,2k+1}(x)}{(2k+1)u^n_{k,2k}(x)},\ \textit{for } k \geq 1,$$

$$K_n(x) = u^n_{p,2p}(x).$$

Since the coefficients and the knots in these approximation processes are given recursively, we intend to obtain an explicit form of them. Some estimates for P_n-simple functionals using the least concave majorant of the modulus of continuity were considered by Gavrea (see [68], [69]) and Raşa [150]. These results motivate us to obtain some approximation properties for linear positive operators expressed in terms of modulus of smoothness. We will apply these results in order to find the remainder term in various approximation processes as: the Bernstein operators, the Durrmeyer operators, the Beta-type operators, the genuine Bernstein-Durrmeyer operators, the Kantorovich operators.

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