Curvature Problems in Global Riemann-Finsler Geometry

Habilitation thesis

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Contents

Preface

The present work has as central idea different aspects of positive curvature. The first Part contains results related to topological implication of different types of positive curvature (flag, Ricci, kk-Ricci). In the first chapter 2 are studied connectedness problems in positively curved Finsler manifolds. These chapter are mainly based on the papers sappro si geom phys

The chapter 3 present some compactedness criteria For submanifolds of Finsler manifolds in the presence of some positive curvature conditions (k-Ricci) and an intersection theorem. These chapter is based on the papers etc, etc....

The chapter 4 deals with growth of the injectivity radius in the case of parallel Ricci tensor. In this situation is proved an upper bound type relation between Ricci curvature and injectivity radius.

The chapter 5 contains generalization of Weinstein fixed point theorem to positively curved Finsler manifolds.

In the second Part is proved an Hardy-Brezis-Marcus type result for domains in a Minkowski space $(\mathbb{R}^n, F)$. The results are based on a deep study of the distance function. It is proved that the distance function to the boundary of a domain is superharmonic if the mean curvature (here the distance is the Minkowski disrance and the mean curvature is induced by the Minkowski norm). It is proved the Hardy-Brezis-Marcus inequality in this setting and it also a method to obtain such kind of results are given. Finally the main inequality is used to obtain an existence and unicity result for a singular Poisson type equation.

In the last decades Finsler geometry produced remarkable development.

Many papers and books on this topic have been published. Specially, a lot of results from Riemannian geometry have been extended for Finsler manifolds.

Probably the first work in Finsler geometry was the PhD thesis of Paul Finsler (1918). But more one half of a century before Riemann (in 1854) pointed the difference between the case of what is known as Riemannian geometry and the general case (see [Spi75] for an English translation). He states in his address: "The study of the metric which is the fourth root of a quartic differential form is quite time-consuming (zeitraubend) and does not throw new light to the problem."

After Einstein's formulation of general relativity, Riemannian geometry became widely used and the Levi-Civita connection came to the forefront. This connection is both torsion free and metric-compatible.

Though Finsler geometry was originated in calculus of variations, geometrically a Finsler manifold means that at each tangent space a norm, varying smoothly, is given, not necessarily induced by an inner product. In the first half on the 20-th century the tools and techniques appropriate for treatment of Finsler geometry were developed.

On a Finsler manifold there does not exist, in general, a linear metrical connection. The generalizations of the Levi-Civita connection induced by a Riemannian metric live just in the vertical bundle π^*TM or TTM, however, there are several ones. The differences between these connections are in the level of the metric compatibility and the torsion. The first of these generalizations were proposed by J.L. Synge (1925), J.H. Taylor (1925), L. Berwald (1928) [Ber28] and, most important, Elie Cartan(1934) [Car34] — the last one is metric compatible, but has the largest number of non-vanishing torsion tensors —; after a short time, S.S. Chern [Che43; Che48; Che96] proposed a different generalization, which is identical with the connection proposed later by Rund (see[Ana96])— it is not fully metric compatible but it has less number of non-vanishing torsion tensors. These connections can be used to prove many results from Riemannian geometry in Finslerian context (see [AP94; BCS00]). Another useful connection in Finsler geometry is the Berwald connection ([Ber28; BCS00; Mat86])— it has no torsion but it has a great deviance from the compatibility with the metric. In [Aba96] and

[MA94] one can find nice characterizations of these connections, illustrating their similarities and differences.

The last decades have meant a great development of global Riemannian geometry. It is an important project to try to generalize these to Finsler settings. It is a remarkable fact that the Jacobi equation, the second variation formula and the index form for Finsler manifolds look exactly like their counterparts in Riemannian case. These enable one to prove in Finslerian context the Cartan-Hadamard theorem,the Bonnet-Myers theorem and the Synge theorem [AP94; BCS00]. The Morse Index Theorem was also generalized to Finsler manifolds. That was due to Lehmann [Leh64]; see Matsumoto for an exposition [Mat86]. On the other hand, in the Riemannian and semi-Riemannian case, the Morse Index Theorem where the ends are submanifolds is also proved by many authors (Ambrose [Amb57], Bolton [Bol77], Kalish [Aku89], Piccione and Tausk [PT99]).

In this part it is extensively used the Morse Index Theorem and the Morse Index Theorem for variable endpoints in the case of Finsler manifolds (published in [Pet06]). We show that, despite the fact that the second fundamental form is not symmetric, the Morse Index Form is symmetric and this fact is crucial in the proofs.

Like in Riemannian geometry the Finsler spaces of constant curvature (constant flag curvature) constitute an important class of Finsler spaces. Finsler spaces of constant negative curvature are studied by Akbar-Zadeh [Akb88]. The structure of that kind of spaces is well clarified however Finsler manifolds of positive curvature have not been completely understood yet. Recently, results on Finsler spaces of positive (constant) curvature are obtained by Shen (see [She96]) and by Bryant (see [Bry96]). The latter gave examples of non-Riemannian Finsler structures with constant positive curvature on the 2-sphere.

Chapter 2 deals with connectedness problems in Finsler geometry. Here the Morse Index Theorem and generally, Morse Theory, is extensively used.

The manifolds of positive sectional curvature are far to be understood, even in Riemannian case. For the Finslerian counterpart, in the case of

Finsler manifolds of positive flag curvature there are few results.

The connectedness principles are well known in algebraic and Riemannian geometry. Recently such kinds of principles are developed for the Riemann or Kähler manifolds (see [FM05; FMR05]). In this work we present the generalization to Finsler setting of connectedness principles in the case of positive flag curvature, the case of positive k-Ricci curvature is also considered ([Pet07], [Pet09]. Namely we develop a connectedness principle in the case of positive k Ricci curvature for embedded submanifolds with large asymptotic index.

In the Finslerian category the situation is much more complicated than in the Riemannian context. The variation of the energy applied to a geodesic with the ends on submanifolds gives rise naturally to a second fundamental form (see [Pet06]). A submanifold is totally geodesic (that is geodesics of the submanifold are also geodesics for the ambient manifold) is equivalent to the statement that the second fundamental form vanishes holds only for Berwald spaces, because the reference vector of the second fundamental form (which appears in the connection coefficients) is not tangent to the submanifold.

We define the asymptotic index using the second fundamental form and the results are proved using the asymptotic index. But the results concerning totally geodesic submanifolds are true for Berwald spaces (in these spaces the asymptotic index is equal to the dimension of submanifold iff the submanifold is totally geodesic).

In the Riemannian case results of this type are obtained in terms of asymptotic index, totally geodesic submanifolds or extrinsic curvature of a submanifold (see [CK52; FM05; Flo94] in the last notion). In this work, using Morse theory in the Finslerian, the results involving asymptotic index are proved in the Finsler spaces, the results concerning totally geodesic submanifolds are proved in the Berwald-Finsler category and the results involving the extrinsic curvature are not treated because even in Berwald spaces, where the reference vector is irrelevant for the connection coefficients, and further for the curvature tensor, the inner products which appear in the flag curvature have a dependence of the reference vector.

The classical Gauss-Bonnet Theorem opened a series of results that are extracting topological properties of a differentiable manifold from the various properties of certain differential geometric invariants of that manifold. The basic topics in this framework consist of the Hopf-Rinow Theorem, the theory of Jacobi fields and the relationship between geodesics and curvature, the Theorems of Hadamard, Myers, Synge, the Rauch Comparison Theorem, the Morse Index Theorem and others. In the Finslerian setting the most recent account of results of this type is due to D. Bao, S.S. Chern and Z. Shen in [BCS00], Ch. 6-9. For a weakened version of the Myers theorem we refer to [Ana07].

In Chapter 3 are presented various results concerning various curvature integral conditions which implies the compactness of the ambient manifold which generalize in many aspects the celebrated Myers Theorem. The last results proves an intersection theorem.

The last chapter of this part Chapter 5 contains a fixed point theorem for Finsler manifolds of positive flag curvature, generalizing a celebrated Weinstein result for Riemannian manifolds.

Chapter 4 deals with the growth of the injectivity radius in the case of parallel Ricci tensor. We prove an estimate of the sup of Ricci scalr in the h parallel hypothesis on the Ricci curvature. Also several nontrivial examples are provided.

All the contents of the first chapter is devoted to various positive curvature assumptions (flag, k-Ricci, Ricci) which are topics of great interest (there are few results concerning positive curvature).

The second part contains a Hardy-Brezis-Marcus ineqaulity for Minkowski spaces, involving the mean curvature of a domain. The proofs are based on a deep study of the distance function and a result which proves the the superharmonicity of the Laplacian of the distance function in the distributional sense. There is presented a technique to obtain classes of Hardy inequalities.

Finally the results are used to obtain an existence and unicity result for a singular Laplace type proble.

Part I

Different aspects of positive curvature in global Riemann-Finsler geometry.

Chapter 1

Preliminaries

1.1 Fundamentals of real Finsler geometry

Let M be a real manifold M of dimension n, (TM, π, M) the tangent bundle of M. The vertical bundle of the manifold M is the vector bundle $\overline{\pi}\colon \mathcal{V} \to TM$ given by $\mathcal{V} = \ker d\pi \subset T(TM)$. (x^i) will denote local coordinates on an open subset U of M, and (x^i, y^i) the induced coordinates on $\pi^{-1}(U) \subset TM$. The radial vertical vector field ι is locally given by $\iota(u^a \frac{\partial}{\partial x^a}) = u^a \frac{\partial}{\partial y^a}|_u$. It follows that local coordinates on M are given by $(x^1, \ldots, x^n, u^1, \ldots, u^n)$ and a local frame for $T(TM)$ is given by $\{\partial_1, \ldots, \partial_n, \dot{\partial}_1, \ldots, \dot{\partial}_n\}$.

We denote by $o : M \to TM$ the zero section of TM where $(o(p) = o_p \in T_pM)$ is the origin of T_pM, and we consider $\widetilde{M} = TM \setminus 0(M)$, the slit tangent bundle. M carries a natural projection $\pi : \widetilde{M} \to M$, the restriction of the canonical projection $\pi : TM \to M$. The bundle $T\widetilde{M} \subset T(TM)$ has a natural projection $\tilde{\pi} : T\widetilde{M} \to TM$, the restriction of the natural projection $\tilde{\pi} = d\pi : T(TM) \to TM$.

A coordinate patch (U_1, φ_1) in M generates a coordinate patch $(\tilde{U}_1, \tilde{\varphi}_1)$ in TM and $\widetilde{M}$ by setting $\tilde{U}_1 = \pi^{-1}(U_1)$ and $\tilde{\varphi}_1(u) = d\varphi_1(u)$, $u \in \varphi(U_1)$.

Let $\varphi_1 = (x_1^1, \ldots, x_1^n)$. Then $\{(\frac{\partial}{\partial x_1^j})|_p\}$ is a basis of T_pM for any $p \in U_1$. For $u = u_1^i(\frac{\partial}{\partial x_1^i})$,

$$\tilde{\varphi}_1(u) = (x_1^1, \ldots x_1^n, u_1^1, \ldots, u_1^n).$$

Let consider another coordinate patch (U_2, φ_2) around p. On $U_1 \cap U_2$ we have

$$dx_2^i = \frac{\partial x_2^i}{\partial x_1^j} dx_1^j, \qquad \frac{\partial}{\partial x_2^i} = \frac{\partial x_2^j}{\partial x_1^i} \frac{\partial}{\partial x_1^j}.$$

The projection $d\pi$ gives rise to the vertical bundle. Namely, the vertical bundle of a manifold M is the vector bundle $\tilde{\pi} : \mathcal{V} \to TM$, of rank $m = \dim M$ given by

$$\mathcal{V} = \ker d\pi \subset T(TM).$$

There exists a natural isomorphism between $T_{\pi(u)}$ and $\mathcal{V}_u$ given by

$$\iota_u = d(j_{\pi(u)}) \circ k_u : T_{\pi(u)} \to \mathcal{V}_u,$$

where $j_p : T_pM \to TM$ is the inclusion and $k_u ; T_pM \to T_u(T_pM)$ is the usual identification, for $u \in T_pM$.

A horizontal bundle is a subbundle $\mathcal{H}$ of $T(TM)$ such that

$$T(TM) = \mathcal{V} \oplus \mathcal{H},$$

and a horizontal map is a bundle map $\Theta : \mathcal{V} \to T(TM)$ such that $(d\pi \circ \Theta)_u = \iota_u^{-1}$. Horizontal bundles and horizontal maps are interesting only over $\widetilde{M}$ (in fact over the zero section we have the natural splitting $T_{o_p}(TM) = \mathcal{V}_{o_p} \oplus \mathcal{H}_{o_p}$).

A non-linear connection is a map $\tilde{D} : \mathfrak{X}(TM) \to \mathfrak{X}(T^*M \otimes TM)$ satisfying

$$\tilde{D}\xi'_p - \tilde{D}\xi_p = \iota_u^{-1}(d\xi'_p - d\xi_p)$$

and

$$\tilde{D}o = 0,$$

$\forall \xi', \xi \in \mathfrak{X}(TM)$. $\tilde{D}\xi$ is called the covariant differential of the vector field $\xi \in \mathfrak{X}(TM)$ and $\tilde{D}\xi(u)$ (denoted also by $\tilde{\nabla}_u \xi$) is called the covariant derivative of ξ is the direction $u \in T_pM$.

Let $\mathcal{H}$ be a horizontal bundle. The direct sum decomposition $T\widetilde{M} = \mathcal{H} \oplus \mathcal{V}$ implies the existence of a vertical projection $k : T\widetilde{M} \to \mathcal{V}$. We can further define a non-linear connection $\tilde{D}_{\mathcal{H}}$ on M by setting

$$\tilde{D}_{\mathcal{H}} = \iota^{-1}_{\xi(p)} \circ k_{\xi(p)} \circ d\xi_p$$

for any $p \in M$ and $\xi \in \mathfrak{X}(TM)$.

Conversely let $\tilde{D} : \mathfrak{X}(TM) \to \mathfrak{X}(T^*M \otimes TM)$ be a non-linear connections. Let $u \in \widetilde{M}_p$ and $\xi \in \mathfrak{X}(TM)$ such that $\xi(p) = u$. Define

$$\Theta^{\tilde{D}}_u : \mathcal{V}_u \to T_u\widetilde{M}$$

by

$$\Theta^{\tilde{D}}_u = d\xi_p \circ \iota_u^{-1} - \iota_u \circ \tilde{D}\xi_p \circ \iota_u^{-1}.$$

It can be verified that $\Theta^{\tilde{D}}$ is a horizontal map. Next, to a horizontal map Θ we can associate an horizontal bundle $\mathcal{H}^{\Theta}$ by

$$\mathcal{H}^{\Theta}_u = \Theta_u(\mathcal{V}_u).$$

In this way is defined a correspondence among horizontal bundles, horizontal maps and non-linear connections.

Proposition 1.1. *Let M be a manifold. The maps $\mathcal{H} \mapsto \tilde{d}_{\mathcal{H}}$, $\tilde{D} \mapsto \Theta^{\tilde{D}}$ and $\Theta \mapsto \mathcal{H}^{\Theta}$ define a one to one correspondence among horizontal bundles, non-linear connections and horizontal maps.*

A non-linear connection $\tilde{D}$ is positive homogeneous if

$$\tilde{D}(\lambda\xi) = \lambda\tilde{D}\xi$$

for all $\lambda \in \mathbb{R}_+$ and $\xi \in \mathfrak{X}(TM)$.

For $\lambda \in \mathbb{R}_+$ define

$$\nu_\lambda : TM \to TM \text{ by } \nu_\lambda(p; u) = (p; \lambda u).$$

A horizontal bundle is called positive homogeneous if

$$\mathcal{H}_{\nu_\lambda(u)} = d(\nu_\lambda)_u(\mathcal{H}_u)$$

and a horizontal map is positive homogeneous if

$$d(\nu_\lambda)_u \circ \Theta_u \circ \iota_u = \Theta_{\nu_\lambda(u)} \circ \iota_{\nu_\lambda(u)}$$

for all $u \in TM$ and $\lambda \in \mathbb{R}_+$.

Proposition 1.2. *Let M be a manifold. The correspondence defined in the Proposition 1.1 preserves homogeneity.*

Let $\tilde{D}$ be a non-linear connection on a manifold M, and $\tilde{\nabla}$ its associated covariant differentiation. Let $\xi \in \mathfrak{X}(TM)$, $p \in M$ and set $u = \xi(p)$. One obtains

$$\tilde{\nabla}_v \xi(p) = v^h [\frac{\partial \xi^k}{\partial x^h} + \Gamma^k_h(\xi(p))] \frac{\partial}{\partial x^k} \|_p,$$

where Γ^k_h are the Christoffel symbols of the non-linear connection. The non-linear connection is smooth if the Christoffel symbols are smooth on $\widetilde{M}$.

For the horizontal bundle induced by the non-linear connection one obtains

$$\delta_j|_u = \partial_j|_u - \Gamma^k_j(u)\dot{\partial}_k|_u$$

a local frame for $\mathcal{H}$.

Horizontal bundles, horizontal maps and non-linear connections are locally identified by the coefficients Γ^k_h.

A Finsler metric on M is a a function $F\colon TM \to \mathbb{R}_+$ satisfying the following properties:

1. F^2 is smooth on $\widetilde{M}$, where $\widetilde{M} = TM \setminus \{0\}$,
2. $F(u) > 0$ for all $u \in \widetilde{M}$,
3. $F(\lambda u) = \lambda F(u)$ for all $u \in TM$, $\lambda \in \mathbb{R}_+$,
4. For any $p \in M$ the indicatrix $I_x(p) = \{u \in T_pM \mid F(u) < 1\}$ is strongly convex.

A manifold endowed with a Finsler metric F is called a Finsler manifold.

From the condition (4) it follows that the quantities $g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 F^2(x,y)}{\partial y^i \partial y^j}$ means positive definite matrix, so a Riemannian metric $\langle\ ,\ \rangle$ can be introduced in the vertical bundle $(\mathcal{V}, \overline{\pi}, TM)$.

We shall use the following notations:

$$G = F^2, \quad G_{\ ;i} = \frac{\partial G}{\partial x^i}, \quad G_i = \frac{\partial g}{\partial y^i}, \quad G_{i;j} = \frac{\partial^2 G}{\partial y^j \partial x^i}$$

The function F is homogeneous of order two,

$$G(x, \lambda y) = \lambda^2 G(x,y), \text{ for } x \in M \text{ and } y \in T_x M$$

so by the Euler theorem we have

$$\begin{aligned} G_i(x,y)y^i &= 2G(x,y) \\ G_{ij}(x,y)y^j &= G_i(x,y) \\ G_{ij}(x,y)y^i y^j &= 2G(x,y) \\ G_{ijk}y^k &= 0 \end{aligned}$$

for all $x \in M,\ y \in T_x M$.

The following lemma justifies the first condition in the definition of the Finsler metric (see [AP94]) :

Lemma 1.3. *Let $F : TM \to \mathbb{R}_+$ be a Finsler metric on a manifold M. Then $G = F^2$ is smooth on TM if and only if F comes from a Riemannian metric on M.*

In this work we use the Cartan connection, following [AP94]. As mentioned before, the last condition in the definition of the Finsler metric induces a Riemannian structure on the vertical bundle $\mathcal{V}$ by setting

$$\langle V, W \rangle_v = \frac{1}{2} G_{ij} V^i W^j, \ \forall V, W \in \mathcal{V}_v.$$

The homogeneity property of F implies that

$$G \equiv \langle \iota, \iota \rangle,$$

that is, the Finsler metric is recovered by embedding $\widetilde{M}$ by ι. The Riemannian structure on ι induced in this way is called induced by the Finsler metric.

Theorem 1.4. *Let M be a real manifold, $F : TM \to \mathbb{R}_+$ be a Finsler metric on M, and $\langle \ , \ \rangle$ be the Riemannian metric on the vertical bundle induced by the Finsler metric. There exists a unique vertical connection $D : \mathfrak{X}(\mathcal{V}) \to \mathfrak{X}(T^*\widetilde{M} \otimes \mathcal{V})$ with the properties:*

1. *D is good;*
2. *$X\langle V, W\rangle = \langle \nabla_X V, W\rangle + \langle V, \nabla_X W\rangle$ for all $X \in T\widetilde{M}$ and $V, W \in \mathfrak{X}(\mathcal{V})$;*
3. *$\theta(V, W) = 0$ for all $V, W \in \mathcal{V}$, is the torsion of the linear connection on $\widetilde{M}$ induced by D;*
4. *$\theta(H, K) \in \mathcal{V}$ for all $H, K \in \mathcal{H}$.*

The connection stated in the theorem is just the Cartan connection. This is a good vertical connection in $\mathcal{V}$, i.e. a $\mathbb{R}$-linear map

$$\nabla \colon \mathfrak{X}(\widetilde{M}) \times \mathfrak{X}(\mathcal{V}) \to \mathfrak{X}(\mathcal{V})$$

having the usual properties of a covariant derivations, metrical with respect to g, and 'good' in the sense that the bundle map $\Lambda \colon T\widetilde{M} \to \mathcal{V}$ defined by $\Lambda(X) = \nabla_X \iota$ is a bundle isomorphism when restricted to $\mathcal{V}$. The latter property induces the horizontal subspaces $H_u = \ker \Lambda$ for all $u \in \widetilde{M}$, which is direct summand of the vertical subspaces $V_u = \mathrm{Ker}\,(d\pi)_u$:

$$T\widetilde{M} = \mathcal{H} \oplus \mathcal{V}$$

$\Theta \colon \mathcal{V} \to \mathcal{H}$ denotes the horizontal map associated to the horizontal bundle $\mathcal{H}$. For a tangent vector field X on M we have its vertical lift X^V and its horizontal lift X^H to $\widetilde{M}$.

Proof. As usual we assume that the connection exists, and we recover the connection form ω_i^j, showing its uniqueness.

The second property asserted in the theorem yields

$$\begin{aligned} G_{ijk} &= \dot\partial_i(G_{jk}) = 2\dot\partial_i\langle\dot\partial_j,\dot\partial_k\rangle = 2\langle\nabla_{\dot\partial_i}\dot\partial_j,\dot\partial_k\rangle + 2\langle\dot\partial_j,\nabla_{\dot\partial_i}\dot\partial_k\rangle \\ &= 2\langle\omega_k^r(\dot\partial_i)\dot\partial_r,\dot\partial_j\rangle + 2\langle\dot\partial_j,\omega_k^s(\dot\partial_i)\dot\partial_s\rangle = G_{rj}\Gamma^r_{ji} + G_{kr}\Gamma^r_{ki} \end{aligned}$$

and the analogous formulas for G_{kij} and G_{jki}. The third property stated in the theorem (torsion freeness) implies that

$$\Gamma^r_{ij} = \Gamma^r_{ji}\ ;$$

and this implies

$$\Gamma^r_{ij} = \frac{1}{2}G^{rs}G_{ijs},$$

where, as usual, (G^{rs}) denoted the inverse matrix of (G_{rs}). From the last formula and the Euler identities it follows that $\Gamma^r_{ij}y^j = 0$.

By the second property stated in the theorem we obtain:

$$\begin{aligned} \delta_i(G_{jk}) &= 2\delta_i\langle\dot\partial_j,\dot\partial_k\rangle = 2\langle\nabla_{\delta_i}\dot\partial_j,\dot\partial_k\rangle + 2\langle\dot\partial_j,\nabla\delta_i\dot\partial_k\rangle \\ &= GG_{sk}\Gamma^s_{j;i} + G_{js}\Gamma^s_{k;i}, \end{aligned}$$

and the analogous formulas for $\delta_k(G_{ij})$ and $\delta_j(G_{ki})$, and one obtains that $\Gamma^h_{i;j} = \Gamma^h_{j;i}$ by the last property stated in the theorem, and we get

$$\begin{aligned} \Gamma^h_{i;j} &= \frac{1}{2}G^{hk}[\delta_j(G_{ki}) + \delta_i(G_{kj}) - \delta_k(G_{ij})] \\ &= \gamma^h_{ij} - \frac{1}{2}G^{hk}[G_{ilk}\Gamma^k_j + G_{jlk}\Gamma^k_i - G_{ijk}\Gamma^k_l] \end{aligned}$$

where we denote

$$\gamma^h_{ij} = \frac{1}{2}G^{hk}[G_{ki;j} + G_{kj;i} - Gij;k] = \gamma^h_{ji}$$

The coefficients Γ_i^k are still unknown, the next step is to determine them. For this we compute

$$\gamma_{ij}^h u^j = \frac{1}{2} G^{hk} [G_{ki;j} u^j + G_{k;i} - G_{i;k}]$$

and

$$\gamma_{ij}^h u^i u^j = G^{hk} [G_{k;j} - G_{\ ;k}]$$

It follows the relation

$$\Gamma_j^h = \Gamma_{i;j}^h u^i = \gamma_{ij}^h u^i - \Gamma_{jk}^h \Gamma_i^k u^i$$

and

$$\Gamma_j^h u^j = \gamma_{ij}^h u^i u^j = \Gamma^{hk} [\Gamma_{k;j} u^j - G_{\ ;k}]$$

and finally

$$\Gamma_j^h = \frac{1}{2} G^{hk} [G_{kj;i} + G_{k;j} - G_{j;k}] - \Gamma_{jk}^h G^{kl} [G_{l;i} u^i - G_{\ ;l}]$$

Hence, the coefficients Γ_j^h are determined and that means that we know the connections forms

$$\omega_b^a = \Gamma_{b;i}^a dx^i + \Gamma_{bc}^a \psi^c = \widetilde{\Gamma}_{b;i}^a dx^i + \Gamma_{bc}^a du^c$$

are the connection forms of a good vertical connection satisfying the conditions in the theorem.

□

Using Θ first we get the radial horizontal vector field $\chi = \Theta \circ \iota$. Secondly we can extend the covariant derivation ∇ of the vertical bundle to the whole tangent bunlde of $\widetilde{M}$. Denoting it with the same letter, for horizontal vector fields H we have

$$\nabla_X H = \Theta(\nabla_X(\Theta^{-1}(H))) \quad \forall\, X \in \mathfrak{X}(\widetilde{M}),$$

and then, an arbitrary vector field $Y \in \mathfrak{X}\widetilde{M}$ is decomposed into vertical and horizontal parts, so

$$\nabla_X Y = \nabla_X Y^V + \nabla_X Y^H.$$

Thus $\nabla : \mathfrak{X}(T\widetilde{M}) \times \mathfrak{X}(T\widetilde{M}) \to \mathfrak{X}(T\widetilde{M})$ is a linear connection on $\widetilde{M}$ induced by a good vertical connection. Its torsion θ and curvature R are defined as usual:

$$\begin{aligned} \nabla_X Y - \nabla_Y X &= [X,Y] + \theta(X,Y) \;\; \forall X, Y \in \mathfrak{X}T\widetilde{M} \\ R_Z(X,Y) &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \;\; \forall X, Y, Z \in \mathfrak{X}T\widetilde{M} \end{aligned}$$

and the torsion has the property that for horizontal vectors $\theta(X,Y)$ is a vertical vector [AP94]. The curvature operator Ω is a global $T^*\widetilde{M} \otimes T\widetilde{M}$-valued 2-form. That means that $\Omega(X,Y)$ is a global $T\widetilde{M}$-valued 1-form for any $X, Y \in T\widetilde{M}$ by the relation $\Omega(X,Y)Z = R_Z(X,Y)$ for any $X, Y, Z \in \mathfrak{X}(T\widetilde{M})$, and Ω is well defined. Specially the sectional curvature of ∇ along a curve σ is given as follows:

$$R_{\dot\sigma}(U^H, U^H) = \langle \Omega(\dot\sigma^H, U^H)U^H, \dot\sigma^H \rangle$$

for any $U \in \mathfrak{X}(M)$. This is called the horizontal flag curvature in [AP94]. The horizontal flag curvature is the most important contraction of the curvature operator because it appears in the second variation formula.

We often use that the torsion of two horizontal vectors is a vertical one, that is $\theta(X,Y) \in \mathcal{V}$ for all $X, Y \in \mathcal{H}$ [AP94].

The metrical property of the Cartan connection is also important [AP94]:

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.$$

In the following we shall present the first and second variation of the length, as in [AP94].

Definition 1.5. *A regular curve $\sigma : [a,b] \to M$ is a C^1 curve such that*

$$\forall t \in [a,b] \qquad \dot{\sigma}(t) = d\sigma_t(\frac{d}{dt}) \neq 0.$$

The *length* with respect to the Finsler metric $F : TM \to \mathbb{R}^+$, of the regular curve σ is given by

$$L(\sigma) = \int_a^b F(\dot{\sigma}(t))dt$$

A geodesic for the Finsler metric F is a curve which is a critical point of the energy functional. We present now the one parameter variation of a curve:

Definition 1.6. *Let $\sigma_0 : [a,b] \to M$ be a curve with $F(\dot{\sigma}_0) = c_0$. A regular variation of σ_0 is a C^1-map*

$$\Sigma : (-\varepsilon, \varepsilon) \times [a,b] \to M$$

such that

1. $\sigma_0(t) = \Sigma(0,t), \forall t \in [a,b]$
2. $\forall s \in (-\varepsilon, \varepsilon)$ *the curve* $\sigma_s(t) = \Sigma(s,t)$ *is a regular curve in* M*;*
3. $F(\dot{\sigma}_s) = c_s > 0, \forall s \in (-\varepsilon, \varepsilon)$.

   A regular variation Σ is fixed if it moreover satisfies

4. $\sigma_s(a) = \sigma_0(a)$ *and* $\sigma_s(b) = \sigma_0(b)$ *for all* $s \in (-\varepsilon, \varepsilon)$.

For a regular variation Σ of σ_0 we define the function $l_\Sigma : (-\epsilon, \epsilon) \to \mathbb{R}^+$ by

$$l_\Sigma(s) = L(\sigma_s).$$

Definition 1.7. *A regular curve σ_0 is a geodesic for F iff*

$$\frac{dl_\Sigma}{ds}(0) = 0$$

for all fixed regular variations Σ of σ_0.

In [AP94] there is derived the first and the second variation of the length functional. It is also derived the differential equation of geodesics and it is shown that every geodesic for F is also a geodesic for the Cartan connection, and conversely, the geodesics of the Cartan connection are geodesics of the Finsler metric.

It is used there the pulled-back of the Cartan connection along a curve. The pulled-back bundle does not live on TM, but on $T\widetilde{M}$. Anyway the construction is not very complicated and it is clear. We briefly present it here.

Let $\Sigma : (-\epsilon, \epsilon) \times [a, b] \to M$ be a regular variation of a curve $\sigma_0 : [a, b] \to M$. Let

$$p : \Sigma^*(TM) \to (-\epsilon, \epsilon) \times [a, b]$$

be the pull back bundle, and $\gamma : \Sigma^*(TM) \to TM$ be the fiber map which identifies each $\Sigma^*(TM)_{(s,t)}$ with $T_{\Sigma(s,t)}M$ for all $(s, t) \in (-\epsilon, \epsilon) \times [a, b]$. A local frame for $\Sigma^*(TM)$ is given by the local fields

$$\frac{\partial}{\partial x^i}\Big|_{(s,t)} = \gamma^{-1}\Big(\frac{\partial}{\partial x^i}\Big|_{\Sigma(s,t)}\Big)$$

for $i = 1, \dots n$. An element $\xi \in \mathfrak{X}(\Sigma^*(TM))$ can be written locally by

$$\xi(s, t) = u^i(s, t)\frac{\partial}{\partial x^i}\Big|_{(s,t)},$$

and a local frame on $T(\Sigma^*(TM))$ is given by $\partial_s, \partial_t, \dot{\partial}_i$, where $\partial_s = \frac{\partial}{\partial s}, \partial_t = \frac{\partial}{\partial t}$ and $\dot{\partial}_i = \frac{\partial}{\partial u^i}$.

There are two particularly important sections of $\Sigma^*(TM)$:

$$\begin{aligned} T &= \gamma^{-1}(d\Sigma\ (\frac{\partial}{\partial t})) = \frac{\partial \Sigma^i}{\partial t}\frac{\partial}{\partial x^i} \\ \text{and} & \\ U &= \gamma^{-1}(d\Sigma\ (\frac{\partial}{\partial s})) = \frac{\partial \Sigma^i}{\partial s}\frac{\partial}{\partial x^i} \end{aligned}$$

Definition 1.8. *The section U is the transversal vector of Σ.*

By setting $\Sigma^*\widetilde{M} = \gamma^{-1}(\widetilde{M})$, we have that $T \in \mathfrak{X}(\Sigma^*\widetilde{M})$ and $T(s,t) = \gamma^{-1}(\dot{\sigma}_s(t))$.

We may pull-back $T\widetilde{M}$ over $\Sigma^*\widetilde{M}$ by using γ, obtaining the map $\tilde{\gamma} : \gamma^*(T\widetilde{M}) \to T\widetilde{M}$ which identifies, for any $u \in \Sigma^*\widetilde{M}_{(s,t)} = \gamma^{-1}(\widetilde{M}_{\Sigma(s,t)})$, $\gamma^*(t\widetilde{M})_u$ with $T_{\gamma(u)}\widetilde{M}$.

We shall enounce now the first and the second variation of the length for Finsler metric.

Theorem 1.9 ([AP94]). *Let $F : TM \longrightarrow \mathbb{R}^+$be a Finsler metric on a manifold M. Take a regular curve $\sigma_0 : [a,b] \longrightarrow M$, with $F(\dot{\sigma}_0) \equiv c_0 \geq 0$, and let $\Sigma : (-\varepsilon, \varepsilon) \times [a,b] \to M$ be a regular variation of σ_0. Then*

$$\frac{dl_\Sigma}{ds}(0) = \frac{1}{c_0}\{\langle U^H, T^H\rangle_{\dot{\sigma}_0}|_a^b - \int_a^b \langle U^H, \nabla_{T^H}T^H\rangle_{\dot{\sigma}_0} dt\}.$$

In paricular if the variation is fixed we have

$$\frac{dl_\Sigma}{ds}(0) = -\frac{1}{c_0}\int_a^b \langle U^H, \nabla_{T^H}T^H\rangle_{\dot{\sigma}_0} dt.$$

The equation of geodesics is obtained as a corollary:

Corollary 1.10. *Let $F : TM \longrightarrow \mathbb{R}^+$be a Finsler metric on a manifold M and $\sigma_0 : [a,b] \longrightarrow M$ a regular curve. Then σ is a geodesic for F iff $\nabla_{T^H}T^H \equiv 0$ where $T^H(u) = \chi_u(\dot{\sigma}(t)) \in \mathcal{H}_u$ for all $u \in \tilde{M}_{\sigma(t)}$.*

Now it follows the second variation of arc-length.

Theorem 1.11 ([AP94]). *Let $F : TM \longrightarrow \mathbb{R}_+$ be a Finsler metric on a manifold M. Take a geodesic $\sigma_0 : [a,b] \longrightarrow M$, with $F(\dot{\sigma}_0) \equiv 1$, and let $\Sigma : (-\varepsilon, \varepsilon) \times [a,b] \to M$ be a regular variation of σ_0. Then*

$$\begin{aligned} \frac{d^2 l_\Sigma}{ds^2}(0) &= \langle \nabla_{U^H}U^H, T^H\rangle_{\dot{\sigma}_0}|_a^b \\ &+ \int_a^b [\|\nabla_{T^H}U^H\|^2_{\dot{\sigma}_0} - \langle \Omega(T^H, U^h), T^H\rangle_{\dot{\sigma}_0} \\ &- |\frac{\partial}{\partial t}\langle U^H, T^H\rangle_{\dot{\sigma}_0}|^2] dt \end{aligned}$$

where $\|H\|_u^2 = \langle H, H\rangle_u$ *for all* $u \in \tilde{M}$ *and* $H \in \mathcal{H}_u$. *In particular, if the variation* Σ *is fixed we have*

$$\begin{aligned}\frac{d^2 l_\Sigma}{ds^2}(0) &= \int_a^b [\|\nabla_{T^H} U^H\|^2_{\dot\sigma_0} - \langle \Omega(T^H, U^h), T^H\rangle_{\dot\sigma_0} \\ &\quad -|\frac{\partial}{\partial t}\langle U^H, T^H\rangle_{\dot\sigma_0}|^2]dt\end{aligned}$$

Chapter 2

Connectedness principles in positively curved Finsler manifolds

2.1 Introduction

The connectedness principle are well known in algebraic and Riemannian geometry. Recently such kinds of principles are developed in the Riemann or Kähler manifolds (see [FM05; FMR05]). [Pet02]. In this chapter we develop connectedness principles in the case of positive flag k-Ricci curvature [Pet07],[Pet09]. Namely we develop a connectedness principle in the case of positive k Ricci curvature for embedded submanifolds with large asymptotic index.

In the Finslerian category the situation is much more complicated than in the Riemannian context. The variation of the energy applied to a geodesic with the ends on submanifolds gives rise naturally to a second fundamental form (see [Pet06]). A submanifold is totally geodesic (that is geodesics of the submanifold are also geodesics for the ambient manifold) is equivalent to the statement that the second fundamental form vanishes holds only for Berwald spaces, because the reference vector of the second fundamental form (which appears in the connection coefficients) is not tangent to the submanifold.

We define the asymptotic index using the second fundamenatal form and the results are proved using the asymptotic index. But the results concerning totally geodesic submanifolds are true for Berwald spaces (in these spaces the asymptotic index is equal to the dimension of submanifold if and only if the submanifold is totally geodesic).

In the Riemannian case results of this type are obtained in terms of asymptotic index, totally geodesic submanifolds or extrinsic curvature of a submanifold (see [CK52; FM05; Flo94] in the last notion). In this chapter, using Morse theory in the Finslerian, the results involving asymptotic index are proved in the Finsler spaces, the results concerning totally geodesic submanifolds are proved in the Berwald-Finsler category and the results involving the extrinsic curvature are not treated because even in Berwald spaces, where the reference vector is irrelevant for the connection coefficients, and further for the curvature tensor, the inner products which appear in the flag curvature have a dependence of the reference vector.

In the recent years the global behaviour of Ricci and flag curvature was extensively studied by Bao and Robles [BR04], Rademacher [Rad04b; Rad04a] , and Shen [She01].

2.2 The main results

Now we state the main connectedness theorems of the paper. Their proofs will follow after Theorems 2.5 and 2.7, Section 2.5.

Theorem 2.1 ([Pet09]). *Let M be an m-dimensional Finsler manifold of positive k-th Ricci curvature, and let $f = (f_1, f_2) : N_1 \times N_2 \to M \times M$, with $f_j : N_j \to M$ isometric immersion of a compact manifold with asymptotic index ν_{f_j}, $j = 1, 2$. Then the following properties are true (we denote $\nu_f = \nu_{f_1} + \nu_{f_2}$):*

1. *If $\nu_f > m + k - 1$, then $f^{-1}(\Delta) \neq \emptyset$.*

2. *If $\nu_f > m + k$ and M is simply connected, then $f^{-1}(\Delta)$ is connected.*

3. *For $\nu_f > m+k+i-1$ there is an exact sequence*

$$\pi_i(f^{-1}(\Delta)) \longrightarrow \pi_1(N) \xrightarrow{(p_1 f)_* - (p_2 f)_*} \pi_i(M) \longrightarrow \pi_{i-1}(f^{-1}(\Delta))$$

In the case when f is a pair of immersions we have:

Theorem 2.2 ([Pet09]). *Let M be an m-dimensional Finsler manifold of positive k-th Ricci curvature, and let $f = (f_1, f_2) : N_1 \times N_2 \to M \times M$, with $f_j : N_j \to M$ isometric immersion of a compact manifold with asymptotic index ν_{f_j}, $j = 1, 2$. Then the following properties are true (we denote $\nu_f = \nu_{f_1} + \nu_{f_2}$):*

1. *If $\nu_f \geq m+k-1$, then $f^{-1}(\Delta) \neq \emptyset$.*
2. *If $\nu_f \geq m+k$ and M is simply connected, then $f^{-1}(\Delta)$ is connected.*
   If $f = (f_1, f_1)$ where f_1 is an embedding, then
3. *For $\nu_f \geq m+k+i-1$ there is an exact sequence*

   $$\pi_i(f^{-1}(\Delta)) \longrightarrow \pi_1(N) \xrightarrow{(p_1 f)_* - (p_2 f)_*} \pi_i(M) \longrightarrow \pi_{i-1}(f^{-1}(\Delta))$$

4. *for $i \leq \nu_f - m - k + 1$, there are natural isomorphisms*

   $$\pi_i(N_1, f^{-1}(\Delta)) \to \pi_i(M, N_1)$$

   for $i \leq \nu_f - m$ and a surjection for $i = \nu_f - m - k + 2$. Here $\pi_i(N_j, f^{-1}(\Delta))$ is understood as the i-th homotopy group of the composition map

   $$f^{-1}(\Delta) \hookrightarrow N \xrightarrow{p_j} N_j.$$

For positive flag curvature from Theorems 2.5 and 2.8 we have

Theorem 2.3 ([Pet07]). *Let M be an m-dimensional compact Finsler manifold of positive flag curvature and Δ the diagonal of $M \times M$. Consider an isometric immersion $f : N \to M \times M$ of a closed manifold with asymptotic index ν_f. The following statements hold:*

1. *If $\nu_f > m$, then $f^{-1}(\Delta) \neq \emptyset$*

2. *If* $\nu_f > m+1$ *and* M *is simply connected, then* $f^{-1}(\Delta)$ *is connected.*

3. *For* $\nu_f > m+i$ *the following sequence of homotopy groups*

   $\pi_i(f^{-1}(\Delta)) \longrightarrow \pi_i(N) \xrightarrow{(p_1 f)_* - (p_2 f)_*} \pi_i(M) \longrightarrow \pi_{i-1}(f^{-1}(\Delta)) \longrightarrow \ldots$ *is exact.*

In the case where f is not a correspondence but a pair of immersions we have the following stronger result:

Theorem 2.4 ([Pet07]). *Under the assumptions of Theorem 2.3 if in addition* $N = N_1 \times N_2$ *and* $f = (f_1, f_2)$ *with asymptotic index* ν_f*, then*

1. *If* $\nu_f \geq m$ *then* $f^{-1}(\Delta) \neq \emptyset$

2. *If* $\nu_f \geq m+1$ *and* M *is simply connected, then it follows that* $f^{-1}(\Delta)$ *is connected. If* $f = (f_1, f_1)$ *with* f_1 *embedding then*

3. *For* $\nu_f \geq m+i$ *the following sequence of homotopy groups*

   $\pi_i(f^{-1}(\Delta)) \longrightarrow \pi_1(N) \xrightarrow{(p_1 f)_* - (p_2 f)_*} \pi_i(M) \longrightarrow \pi_{i-1}(f^{-1}(\Delta))$
   is exact.

4. *We have the natural isomorphism*

$$\pi_i(N_1, f^{-1}(\Delta)) \to \pi_i(M, N_1)$$

   for $i \leq \nu_f - m$ *and a surjection for* $i = \nu_f - m - k + 2$*. Here* $\pi_i(N_j, f^{-1}(\Delta))$ *is understood as the* i*-th homotopy group of the composition map*

   $f^{-1}(\Delta) \hookrightarrow N \xrightarrow{p_j} N_j.$

2.3 Preliminaries

We recall from the previous chapter some notions that are needed here. Let N be a submanifold of M of dimension $p < m$. We consider the set

$$A = \{(x,v) | x \in N, v \in T_x M \setminus \{0\}\} = \{\widetilde{x} \in \widetilde{M} | \pi(\widetilde{x}) \in N\}.$$

We consider $H_{\widetilde{x}}T_xM$ and $H_{\widetilde{x}}T_xN$ be the horizontal liftings of T_xM and T_xN respectively along $\widetilde{x}$ and

$$H_NTM = \bigcup_{\widetilde{x}\in A} H_{\widetilde{x}}T_xM$$

and

$$H_NTN = \bigcup_{\widetilde{x}\in A} H_{\widetilde{x}}T_xN.$$

Let $N_{\widetilde{x}}^{\perp}$ be the $\langle\cdot,\cdot\rangle_{\widetilde{x}}$ orthogonal complement of $H_{\widetilde{x}}TN$ in $H_{\widetilde{x}}TM$. Let $X, Y \in H_NTN$ and let X^*, Y^* be their prolongations to H_NTM (that is if $X, Y \in H_{\widetilde{x}}T_xN$ for some $\tilde{x} \in TM$ it follows that $X^*, Y^* \in H_{\widetilde{x}}T_xM$). The restriction of $\nabla_{X^*}Y^*$ to $\widetilde{N}$ does not depend on the choice of the prolongation. By the orthogonal decomposition induced by the inner product induced by the Finsler metric

$$H_{\widetilde{x}}T_xM = H_{\widetilde{x}}T_xN \oplus N_{\widetilde{x}}^{\perp}$$

we obtain that

$$\nabla_{X^*}Y^* = \nabla^*_X Y + \mathbb{I}(X,Y).$$

We will call $\mathbb{I}(X,Y)$ the second fundamental form at X and Y. Note that for $\widetilde{x} = (x,v) \in A$ with $v \in T_xM \setminus T_xN$ we have

$$\langle \nabla_{X^*}Y^*, v^H\rangle_v = \mathbb{I}_v(X,Y)$$

and we call it the second fundamental form of X and Y in the direction of v (note that v is also the reference vector in the covariant derivative). We will be interested mostly in the sign of the second fundamental form.

Let $f : N \to M$ be an immersion. The asymptotic index of the immersion f in the direction is defined by

$$\nu_f = \min_{x\in N} \nu_f(x)$$

where $\nu_f(x)$ is the maximal dimension of a subspace of T_xN on which the second fundamental form vanishes in every direction $v \in T_xM \setminus T_xN$. The submanifold N will be called a totally geodesic submanifold (in the analytic sense) if and only if $\nu(f) = \dim N$.

In the last part of this section we introduce the k-Ricci curvature, following [She01]. For a $(k+1)$-dimensional subspace $\mathcal{V} \in T_xM$ the Ricci curvature $\mathbf{Ric}_y\mathcal{V}$ on $\mathcal{V}$ is the trace of the Riemann curvature restricted to $\mathcal{V}$, with flagpole y, and is given by:

$$\mathbf{Ric}_y(\mathcal{V}) = \sum_{i=1}^{k} \langle R_y(b_i), b_i \rangle_y = \sum_{i=1}^{k} \langle \Omega(y, b_i) b_i, b_i \rangle_y,$$

where $(b_i)_{i=\overline{1,\ldots,k+1}}$, $b_{k+1} = y$ is an arbitrary orthonormal basis for $(\mathcal{V}, \langle , \rangle_y)$.We will call $\mathbf{Ric}_y(\mathcal{V})$ the Ricci curvature on $\mathcal{V}$. $\mathbf{Ric}_y(\mathcal{V})$ is positively homogeneous of degree two on $\mathcal{V}$,

$$\mathbf{Ric}_{\lambda y}(\mathcal{V}) = \lambda^2 \mathbf{Ric}_y(\mathcal{V}), \quad \text{for} \quad \lambda > 0, y \in \mathcal{V}.$$

It is clear from the definition that $\mathbf{Ric}(y) = \mathrm{trace}(\mathbf{Ric}_i^i(y))$, for $y \in T_xM$.

If $\mathcal{V} = P \subset T_xM$ is a tangent plane, the flag curvature is given by

$$K(P, y) = \frac{g_y(R_y(u), u)}{g_y(y, y) g_y(u, u) - g_y^2(u, y)},$$

where $u \in P \setminus \{0\}$, $\mathrm{span}(y, u) = P$. This is independent of the choice of $u \in P \setminus \{0\}$, and for u being g_y orthogonal to y and of g_y norm 1 it becames

$$\mathbf{K}(P, y) = \frac{\mathbf{Ric}_y P}{F^2(y)}, \quad y \in P.$$

Consider

$$\mathbf{Ric}_k := \inf_{\dim(\mathcal{V})=k+1} \inf_{y \in \mathcal{V}} \frac{\mathbf{Ric}_y(\mathcal{V})}{F^2(y)},$$

the infimum being considered over all $(k+1)$-dimensional subspaces $\mathcal{V} \subset T_xM$ and $y \in \mathcal{V} \setminus \{0\}$. From the above definitions it can be seen that

$$\mathbf{Ric}_1 \leq \cdots \leq \frac{\mathbf{Ric}_k}{k} \leq \cdots \leq \frac{\mathbf{Ric}_{n-1}}{n-1},$$

and

$$\mathbf{Ric}_1 = \inf_{(P,y)} \mathbf{K}(P,y) \quad \text{and} \quad \mathbf{Ric}_{(n-1)} = \inf_{F(y)=1} \mathbf{Ric}(y)$$

.

We will say tha the Finsler manifold (M, F) has positive k-Ricci curvature if $\mathbf{Ric}_k > 0$.

In [She01], Shen proves various results under assumptions on k-Ricci curvature, related to vanishing of homotopy groups, injectivity radius and conjugate radius under the assumptions that the k-Ricci curvature satisfies $\mathbf{Ric}_k \geq k$.

Generally, for the notions and facts from algebraic topology we used the books of A. Hatcher [Hat] and G. W. Whitehead [Whi78].

2.4 Morse Theory on path space

In this section we use Morse theory on the path space, the main tool in the proofs of the connectedness principles developed in the paper.

Let M be a connected Finsler manifold and $P(M)$ denote the path space of the manifold with the topology induced by the metric

$$d(\alpha_0, \alpha_1) = (\int_0^1 (F(\dot{\alpha_0}) - F(\dot{\alpha_1}))^2 dt)^{\frac{1}{2}} + \max_{t\in[0,1]} \mathrm{d}_M(\alpha_0(t), \alpha_1(t)),$$

which is well defined even if α_0 and α_1 have finitely many cusps (here d_M denote the metric induced on M by the Finsler metric).

Consider the projection map $p : P(M) \to M \times M$ given by $p(\alpha) = (\alpha(0), \alpha(1))$ which defines a Serre fibration given by $\Omega(M) \to P(M) \to M \times M$ where the fiber $\Omega(M)$ is the loop space of M with a fixed basepoint.

For a manifold N and a smooth map $f : N \to M \times M$ we consider the pullback fibration by f, $\Omega(M) \to P(M, f) \to N$. $P(M, f) \subset N \times P(M)$ consists of (x, α) such that $f(x) = (\alpha(0), \alpha(1))$ and has the induced topology.

We study the space $P(M, f)$ from the Morse theory of the energy functional $E(x, \alpha) = \frac{1}{2} \int_0^1 F^2(\dot{\alpha}(t))dt$. By the results from [Kob96] any critical point (x, α) of the energy functional E is a geodesic for which $(\dot{\alpha}(0), -\dot{\alpha}(1))^H$ is $\langle\ ,\ \rangle_{\dot{\alpha}}$ orthogonal to $(f_*(T_x(N)))^H$. We will restrict in this section to compact manifolds.

Theorem 2.5 ([Pet07]). *Let M and N be compact Finsler manifolds, and $f : N \to M \times M$ an isometric immersion, and let $\Delta \subset M \times M$ be the diagonal. Assume that every nontrivial critical point (x, α) of E has index $I_\alpha \geq \lambda_0$. Then the following assertions are true.*

1. *If $\lambda_0 \geq 1$, then $f^{-1}(\Delta) \neq \emptyset$.*
2. *If $\lambda_0 \geq 2$ and M is simply connected, then $f^{-1}(\Delta)$ is connected.*

   If in addition $f = f_1 \times f_1 : N = N_1 \times N_1 \to M \times M$, where f is an embedding, then
3. *$\pi_i(P(M, f), f^{-1}(\Delta)) = 0$ for all $i < \lambda_0$.*
4. *For $\lambda_0 \geq i$, then there is an exact sequence of homotopy groups,*

   $$\pi_i(f^{-1}(\Delta)) \longrightarrow \pi_i(N) \xrightarrow{(p_1 f)_* - (p_2 f)_*} \pi_i(M) \longrightarrow \pi_{i-1}(f^{-1}(\Delta)) \longrightarrow \dots,$$

   where p_1 and p_2 are the projections to the factors.

Proof. (1) $f^{-1}(\Delta) = \emptyset$ implies that E has an absolute minimum at some non-trivial critical point (x, α), hence its index should be zero, this is a contradiction.

(2) At this step we use the finite approximation of the path space proved for the Finsler setting by Dazord ([Daz69] p. 129-134).

Let $P_c(M, f) = E^{-1}([0, c))$ be an open subset of $P(M, f)$ and $B_c^k(M, F) \subset P_c(M, f)$ the space of piecewise smooth geodesics with k-cusps (with each piece of length less than the injectivity radius). For k sufficiently large $B_c^k(M, f)$ and $P_c(M, f)$ are homotopy equivalent. Being the space $B_c^k(M, f)$ formed by k-broken geodesic it can be identified with an open submanifold of the product $N \times M \times \ldots \times M$ (k copies of M). E is a proper function when restricted to $B_c^k(M, f)$ and furthermore $E|_{B_c^k(M,f)}$ and $E|_{P_c(M,f)}$ have the same critical points with identical indices ([Daz69] p. 129-134).

Suppose, by contrary, that $f^{-1}(\Delta)$ is not connected. In this case there exist disjoint non-empty compact subsets A and B such that $A \cup B = f^{-1}(\Delta)$. We can think at $f^{-1}(\Delta)$ as the set of constant paths in $P(M, f)$. Let $p \in A$ and $q \in B$.

The manifold M being simply connected it follows that the loop space $\Omega(M)$ is path connected, and it implies that $P(M, f)$ is also path connected, and, furthermore, there exists a path α_0 in $P(M, f)$ joining p and q. By the previous observations related to $B_c^k(M, f)$ we can choose a path $\alpha_0 \in B_{c_0}^k(M, f)$ joining p and q for some constant $c_0 > 0$ and some $k \in \mathbb{N}$, enough large such that $B_{c_0}^k(M, f)$ has the same homotopy type as $P_c(M, f)$.

Let $X = B_{c_0}(M, f)$ with the induced product metric [KPV01] from $N \times M \times \ldots M$ (k times of M) and consider $g = E|_X$. By an above consideration we identify $f^{-1}(\Delta)$ with $g^{-1}(0)$. We will prove (3) under the assumption that there exists a sequence of connected paths $\alpha_k : [0, 1] \to g^{-1}[0, \frac{1}{k}]$, $(k \geq 1)$ in X with $\alpha_k(0) = p$ and $\alpha_k(1) = q$, in the homotopy class of $[\alpha_0]$, keeping the endpoints fixed. By the compactness of A and B the distance $d_X(A, B) > 0$. Consider the function $a(x) = d(x, A) - d(x, B)$. We can see that $a|_{\alpha_k}$ satisfies $a(p) < 0$ and $a(q) > 0$, so there exists a point $x_k \in \alpha_k$ with $a(x_k) = 0$. Now $(x_k)_{k \in \mathbb{N}}$ contains a convergent subsequence, so we can assume that (x_k) itself is convergent to a point x, and $\lim_{k \to \infty} g(x_k) = 0$. Thus we obtain a contradiction $x \in g^{-1}(0)$ and $a(x) = 0$, since $A \cap B \neq \emptyset$.

By Corollary 6.8 in [Mil63] and from the fact that $\lambda_0 \geq 2$, there exists a Morse function h on X such that $|g - h| \leq \frac{1}{10k}$ on the sublevel set $X^{\leq c_0 + \frac{1}{2}} = \{x \in X, h(x) \leq c_0 + \frac{1}{2}\}$, such that the critical points of h in the set $h^{-1}([\frac{1}{2k}, c_0 + \frac{1}{2}])$ have Morse index greater than 1, so from Morse Theory it follows that $h^{-1}(-\infty, c_0 + \frac{1}{2}]$ is homotopy equivalent to $h^{-1}(-\infty, \frac{1}{2k}]$ by gluing cells of dimensions at least 2. But this implies that the relative homotopy group $\pi_1((-\infty, c_0 + \frac{1}{2}], (-\infty, \frac{1}{2k}]) = 0$, so α_0 is homotopic to a path α_k in $h^{-1}(-\infty, \frac{1}{2k}]$ with the same endpoints p and q fixed.

(3) It is enough to show that $E^{-1}(0) = f^{-1}(\Delta) \subset B_c(M, f)$ has an open neighborhood $U \in B_c(M, f)$ which is a deformation retract of $f^{-1}(\Delta)$. The existence of such a neighborhood will be given in Proposition 2.6.

(4) We have that $\pi_i(\Omega M) \cong \pi_{i+1}(M)$. Further we have the diagram

$$\begin{tikzcd}
\dots\pi_i(P(M,f)) \arrow[r] \arrow[d, "\tilde{f}_*"] & \pi_i(N) \arrow[r, "\phi"] \arrow[d, "f_*"] & \pi_i(M) \arrow[r] \arrow[d, "="] & \pi_{i-1}(P(M,f))\dots \arrow[d, "\tilde{f}_*"] \\
\dots\pi_i(P(M)) \arrow[r, "p_*"] & \pi_i(M\times M) \arrow[r, "\delta_*"] & \pi_i(M) \arrow[r] & \pi_{i-1}(P(M))\dots
\end{tikzcd}$$

because $\Omega(M) \to P(M,f) \to N$ is the pullback of the Serre fibration $\Omega(M) \to P(M) \to M\times M$ via the immersion $f: N \to M\times M$. Being $P(M)$ homotopic to M it follows that $p: M \to M\times M$ is homotopic to the diagonal map. Denoting p_i the projection of $M\times M$ to the i-th factor, then $\delta_* = (p_1)_* - (p_2)_*$. From the diagram we have the homomorphism $\phi = (p_1 f)_* - (p_2 f)_*$. □

The next proposition states the existence of the neighborhood used in the proof Theorem 2.5 point (3)(for the proof see [Pet02]). Let X be a complete Finsler manifold and let $f: N \to X\times X$ be a isometric immersion, where N is a compact manifold. Let S be the subset

$$S = \{(x, f(x), \dots, f(x)) \in N\times(X\times X)\times\cdots\times(X\times X) | x\in f^{-1}(\Delta)\},$$

k times $X\times X$.

Proposition 2.6 ([Pet07])**.** *Let X be a complete Finsler manifold and $f: N \to X\times X$ be an isometric immersion as above. The subset S in $N\times(X\times X)\times\dots(X\times X)$, ($k$-copies of $(X\times X)$) is a deformation retract of an open neighborhood U if one of the following conditions holds:*

- *f is a totally geodesic map*
- *$N = N_1\times N_1$ and $f = f_1\times f_1$, where f_1 is an embedding.*

2.5 Index estimates via the asymptotic index

Now we are going to prove some index estimates for the energy functional E of the Finsler metric on $P(M,f)$. At a point $(x,\alpha)\in P(M,f)$ the tangent space consists of vectors (v,W), with $v\in T_xM$ and W piecewise smooth vector field along α such that $f_*(v) = (W(0), W(1))$. Being f an immersion the tangent space can be identified with the space $(W(0), W(1))$. For a parallel vector field along α, by the second variation formula of Finsler energy, the Hessian of the energy function satisfies

$$\begin{aligned} E_{**} &= \int_0^1 -\langle \Omega(\dot\alpha^H, W^H)W^H, \dot\alpha^H\rangle_{\dot\alpha} dt \\ &\quad + \langle \mathbb{I}_{\dot\alpha}(f_*(v), f_*(v))^H, (-\dot\alpha(0), \dot\alpha(1))^H\rangle_{\dot\alpha} \end{aligned}$$

Theorem 2.7 ([Pet09])**.** *Let M be a compact Finsler manifold of positive k-Ricci curvature and let $f : N \to M \times M$ be an isometric immersion with asymptotic index ν_f. Let (x, α) be a non-trivial critical point of E, with Morse Index I_α. Then,*

1. $I_\alpha \geq \nu_f - m - k + 1$
2. *If $f = (f_1, f_2) : N = N_1 \times N_2 \to M \times M$ such that $f_i : N_i \to M$ is an immersion, $i = 1, 2$, then $I_\alpha \geq \nu_f - m - k + 2$.*

Proof. (1).Consider the set V of vector fields along α such that v^H is parallel along and $\langle\ ,\ \rangle_{\dot\alpha}$ orthogonal to $\dot\alpha^H$. It is clear that $\dim V = m - 1$ and we can identify V with $\{v(0), v(1), v \in V\}$.

Consider the symmetric quadratic form on W given by

$$\langle A(v), v\rangle = -\int_0^1 -\langle \Omega(\dot\alpha^H, W^H)W^H, \dot\alpha^H\rangle_{\dot\alpha} dt,$$

where $v = (W(0), W(1))$.

Now let V_1 be a subspace of maximal dimension such that the quadratic form is negative definite on V_1. $V1$ can be view as a subspace of $T_{(f(x))}(M \times M)$ with $f(x) = (\alpha(0), \alpha(1))$.

Consider further $\mathcal{N}_x$ the maximal subspace of T_xN such that the second fundamental form in the direction $\dot\alpha$ is zero, so $\dim \mathcal{N}_x \geq \nu_f$. Being both V^h and $f_*(\mathcal{N}_x)^H$ $\langle,\rangle_{\dot\alpha}$ orthogonal to $(\dot\alpha(0), -\dot\alpha(1))$ we have by the dimension theorem that

$$\dim\left((f_*(\mathcal{N}_x))^H \cap V\right) \geq \nu_f + \dim V - 2m + 1.$$

$$E_{**}(W, W) = \langle A(v), v \rangle < 0$$

for non zero vectors $v \in (f_*(\mathcal{N}_x))^H \cap V$. We will show that $\dim V \geq m-k$.

We consider an orthonormal basis for V which are eigenvectors of A, $v_1, v_2, \ldots, v_{m-1}$.

From the curvature assumption

$$\begin{aligned} \sum_{i=1}^{k} \langle A(v_i), v_i \rangle &= \sum_{i=1}^{k} \int_0^1 -\langle \Omega(\dot{\alpha}^H, v_i^H) v_i^H, \dot{\alpha}^H \rangle_{\dot{\alpha}} dt \\ &= \int_0^1 \sum_{i=1}^{k} -\langle \Omega(\dot{\alpha}^H, v_i^H) v_i^H, \dot{\alpha}^H \rangle_{\dot{\alpha}} dt < 0 \end{aligned}$$

It follows that there exists $i \in 1, \ldots k$, $\langle A(v_i), v_i \rangle < 0$, let's say that $i = 1$. Analogous we have

$$\begin{aligned} \sum_{i=2}^{k+1} \langle A(v_i), v_i \rangle &= \sum_{i=2}^{k+1} \int_0^1 -\langle \Omega(\dot{\alpha}^H, v_i^H) v_i^H, \dot{\alpha}^H \rangle_{\dot{\alpha}} dt \\ &= \int_0^1 \sum_{i=2}^{k+1} -\langle \Omega(\dot{\alpha}^H, v_i^H) v_i^H, \dot{\alpha}^H \rangle_{\dot{\alpha}} dt < 0, \end{aligned}$$

so we can choose v_2 with $\langle A(v_2), v_2 \rangle < 0$. By repeating this operation we can choose $v_1, \ldots, v_{m-k}$ with $\langle A(v_i), v_i \rangle < 0$ for $1 \leq i \leq m-k$, that is $\langle A(v), v \rangle < 0$ for v in the subspace spanned by $v_1, \ldots, v_{m-k}$, and the first assertion of the theorem follows.

(2). α is a geodesic in M such that $\dot{\alpha}^h(0)$ is $\langle\ ,\ \rangle_{\dot{\alpha}(0)}$ orthogonal to $((f_1)_*(T_{x_1}N_1))^H$ and $\dot{\alpha}^H(1)$ is $\langle\ ,\ \rangle_{\dot{\alpha}(1)}$ orthogonal to $((f_2)_*(T_{x_2}N_2))^H$, where $x = (x_1, x_2)$. Now both $(\dot{\alpha}(0), 0)^H$ and $(0, \dot{\alpha}(1))^H$ are normal to $f_*(T_xN)^H$ and V, so we have for the dimension of their intersection

$$\dim \left((f_*(\mathcal{N}_x))^H \cap V \right) \geq \nu_f + \dim V - 2m + 2 = \nu_f - m - k + 2.$$

$\square$

In the positive flag curvature case it follows the next theorem:

Theorem 2.8 ([Pet07]). *Let M be a compact Finsler manifold of positive flag curvature and let $f : N \to M \times M$ be an isometric immersion with asymptotic index ν_f. Let (x, α) be a non-trivial critical point of E, with Morse Index I_α. Then,*

1. $I_\alpha \geq \nu_f - m$

2. *If $f = (f_1, f_2) : N = N_1 \times N_2 \to M \times M$ such that $f_i : N_i \to M$ is an immersion, $i = 1, 2$, then $I_\alpha \geq \nu_f - m + 1$.*

Proof. (1) Consider the set V of vector fields along α such that v^H is parallel along and $\langle\ ,\ \rangle_{\dot\alpha}$ orthogonal to $\dot\alpha^H$. It is clear that $\dim V = m - 1$ and we can identify $V = \{v(0), v(1)\}$.

Consider further $\mathcal{N}_x$ the maximal subspace of T_xN such that the second fundamental form in the direction $\dot\alpha$ is zero, so $\dim \mathcal{N}_x \geq \nu_f$. Being both V^h and $f_*(\mathcal{N}_x)^H$ $\langle,\rangle_{\dot\alpha}$ orthogonal to $(\dot\alpha(0), -\dot\alpha(1))$ we have by the dimension theorem that

$$\dim\left((f_*(\mathcal{N}_x))^H \cap V\right) \geq \nu_f + \dim V - 2m + 1 = \nu_f - m.$$

(2) α is a geodesic in M such that $\dot\alpha^h(0)$ is $\langle\ ,\ \rangle_{\dot\alpha(0)}$ orthogonal to $((f_1)_*(T_{x_1}N_1))^H$ and $\dot\alpha^H(1)$ is $\langle\ ,\ \rangle_{\dot\alpha(1)}$ orthogonal to $((f_2)_*(T_{x_2}N_2))^H$, where $x = (x_1, x_2)$. Now both $(\dot\alpha(0), 0)^H$ and $(0, \dot\alpha(1))^H$ are normal to $f_*(T_xN)^H$ and V, so we have for the dimension of their intersection

$$\dim\left((f_*(\mathcal{N}_x))^H \cap V\right) \geq \nu_f + \dim V - 2m + 2 = \nu_f - m + 1.$$

□

Now, from theorems 2.5 and 2.7 follow the theorems announced in the Section 2.2

2.6 Some consequences of the main results

Theorem 2.9 ([Pet07]). *Let M be a compact simply connected Finsler manifold of positive flag curvature. Let $f_i : N_i \to M$ be a compact isometric immersion with asymptotic index $\nu_{f\,i}, i = 1, 2$. For $\nu_{f_1} + \nu_{f_2} \geq m + 1$ then both $f_1^{-1}(f_2(N_2))$ and $f_2^{-1}(f_1(N_1))$ are connected.*

Proof. Let us consider the immersion $(f_1, f_2) : N_1 \times N_2 \to M \times M$. By the Theorem 2.4, $(f_1, f_2)^{-1}(\varDelta)$ is connected and it follows that $f_1^{-1}(f_2(N_2)) = p_1((f_1, f_2)^{-1}(\varDelta))$ and $f_2^{-1}(f_1(N_2)) = p_2((f_1, f_2)^{-1}(\varDelta))$ are connected. □

The next theorem is a Frankel type result about intersection of submanifolds, see [Fra61; KP00; Pet02] for different versions in the Riemann and Finsler setting.

The original Frankel theorem states that for a complete connected Riemannian manifold M of positive sectional curvature, two totally geodesic submanifolds V and W have nonempty intersection, $V \cap W \neq \emptyset$, provided that $\dim V + \dim W \geq \dim M$.

Theorem 2.10 ([Pet07]). *Let M be an m dimensional connected Finsler manifold of positive flag curvature and let $f_i : N_i \to M$ be isometric immersions of a compact submanifold with asymptotic indexes ν_{f_i}. If $\nu_{f_1} + \nu_{f_2} \geq m$, then $f_1(N_1) \cap f_2(N_2) \neq \emptyset$.*

Proof. Consider $f = (f_1, f_2) : N_1 \times N_2 \to M \times M$. Now $\nu_f = \nu_{f_1} + \nu_{f_2} \geq m$, so from Theorem 2.3 it follows that $f^{-1}(\varDelta) \neq \emptyset$, that is $f_1(N_1) \cap f_2(N_2) \neq \emptyset$. □

A map $f : N \to M$ is said to be $(i + 1)$- connected if it induces an isomorphism up to the i-th homotopy group and a surjective homomorphism on the $(i + 1)$-th homotopy group.

In what follows we prove some results related to the above defined connectedness notion.

Theorem 2.11 ([Pet07]). *Let M be a compact simply connected Finsler manifold of positive flag curvature and let $f : N \to M$ be an immersion* *of a compact manifold. If the asymptotic index* $\nu_f > \frac{dim M}{2}$*, then* f *is an embedding.*

Proof. It is enough to show that f is one-to-one map since f is an immersion. We have that

$$f^{-1}(\Delta) = \{(x,x), x \in N\} \cup \{(x,y)|f(x) = f(y), x \neq y\}.$$

If f is not injective it follows that $f^{-1}(\Delta)$ is not connected, this is a contradiction. □

Theorem 2.12 ([Pet07])**.** *Let* M *be a compact Finsler manifold of positive flag curvature and* N *be a compact embedded submanifold with asymptotic index* ν*. We have* $\pi_i(M,N) = 0$ *for* $i \leq 2\nu - dim M$*.*

Proof. In Theorem 2.4 consider $N_1 = N_2$ and $f_i = ß : N \hookrightarrow M$ the inclusion. Now $f^{(-1)}(\Delta) = N = N_1 \cap N_2$ and the result follows. □

The next theorem is a result related to embeddings, similar to the results of B. Wilking [Wil03], which states that the for a positively curved n-dimensional manifold M of positive sectional curvature and a $(n-k)$-dimensional totally geodesic compact submanifold N the inclusion $\iota : N \to M$ is $n - 2k + 1$ connected.

Theorem 2.13 ([Pet07])**.** *Let* M *be a compact simply connected Finsler manifold of positive flag curvature and let* $f : N \to M$ *be an embedding of a compact manifold with asymptotic index* ν_f*. Then* f *is* $(2\nu_f - dim M + 1)$*-connected.*

Proof. The result is a consequence of Theorems 2.4 and 2.12. □

Theorem 2.14 ([Pet07])**.** *Let* M *be a compact simply connected manifold of positive flag curvature. If* $f : N \to M$ *is an isometric immersion of a compact manifold with asymptotic index* ν_f *then* f *is* $(2\nu_f - dim M + 1)$ *connected.*

Proof. We have that $(2\nu_f - \dim M + 1) > 1$, so $2\nu_f > \dim M$. Unsing Theorems 2.11 and 2.13 the proof is concluded. □

2.7 Applications for totally geodesic submanifolds of Berwald manifolds

In Riemannian geometry the fact that the second fundamental form of a submanifold is zero is equivalent to the property that the submanifold is totally geodesic.

In our case the situation is more subtle. The second fundamental form, defined from the variational approach (second variation of the energy of a geodesic where the ends of the geodesics are in two submanifolds) is rather an analytical definition. The reference vector of the second fundamental form is not tangent to the submanifold, so it is an extrinsic characteristic of the submanifold (it is related to the way the submanifold lies in the ambient manifold). The condition that the second fundamental form vanishes is not equivalent to the geometric property of a submanifold being totally geodesic. For Berwald spaces the above equivalence is true, due to the fact that the reference vector of the covariant derivative is irrelevant. So, we have two notions of a totally geodesic submanifold. One of them is analytical, that is, the second fundamental form in a non-tangent direction vanishes, and the other is geometrical, that is, the geodesics of the submanifold are geodesics in the ambient manifold.

In the case when the manifold M is of Berwald type, the connection of M lives on the tangent level (the reference vector is irrelevant) and is linear. In this case, for a submanifold, the condition that the second fundamental form defined in Section 2.3 vanishes is equivalent to the property that the submanifold is totally geodesic, that is the geodesic of the submanifold are geodesics for the ambient manifold. Furthermore in this case a submanifold N of the manifold M is totally geodesic in both senses, analytic and geometric if and only if $\nu_f = \dim N$ (in the Berwald category the reference vector is irrelevant).

In the case of Berwald spaces, the previous characterization of totally geodesic submanifolds is also implied by Szabó's structure theorems on Berwald spaces (see [Sza81b]). One of Szabó's results says that if (M, L, ∇) is a manifold endowed with a Berwald metric L and ∇ is the Berwald connection, then the connection is Riemann metrizable, i.e., there exists a non-unique Riemannian metric g on M such that ∇ is the Riemannian connection of g. This implies that the geodesics of the Berwald metric L and the non-unique Riemannian metric g coincide. It follows now that a submanifold of M is totally geodesic with respect to g if and only if it is totally geodesic with respect to L, i.e., the totally geodesic submanifolds of M, with respect to the Berwald metric L, coincide with the totally geodesic submanifolds with respect to the non-unique Riemannian metric g whose existence is guaranteed by Szabo's results.

The second fundamental form defined in the Section 2.3 has the reference vector non tangent to the submanifold. But this second fundamental form appears naturally in the study of geodesics joining two submanifolds. This shows us an important difference between the Finsler and Riemann cases, and as expected, this difference comes up from the fact that the Cartan connection (and any other connection used in Finsler geometry) has a directional dependence. In the Finsler category, there is no such strong relationship between the asymptotic index and the property of a submanifold to be totally geodesic as in the Riemannnian case.

All the results concerning asymptotic index can be restated for totally geodesic submanifolds N of Berwald manifolds M, with $f : N \to M$ totally isometric immersion and $\nu_f = \dim N$.

Finally we present some problems in Finsler geometry where we expect that these tools can be applied. Grove and Searle ([GS94]) introduced the symmetry rank of a Riemannian manifold (M, g), to be the rank of the isometry group of (M, g). Grove also proposed to classify those manifolds with a large isometry group. One of the aims of this program is to find general obstructions to the existence of Riemannian metrics of positive curvature (taking benefit from the obstructions for manifolds with a large amount of symmetries). We expect that these tools can be applied to the study of manifolds with Finsler metrics of positive flag curvature.

Chapter 3

Compactness theorems in positively curved Finsler manifolds

3.1 Introduction. k- Ricci curvature

The results in this chapter are from [AP14b] and [AKP17] The classical Gauss-Bonnet Theorem opened a series of results that are extracting topological properties of a differentiable manifold from the various properties of certain differential geometric invariants of that manifold. The basic topics in this framework consist of the Hopf-Rinow Theorem, the theory of Jacobi fields and the relationship between geodesics and curvature, the Theorems of Hadamard, Myers, Synge, the Rauch Comparison Theorem, the Morse Index Theorem and others. In the Finslerian setting the most recent account of results of this type is due to D. Bao, S.S. Chern and Z. Shen in [BCS00], Ch. 6-9. For a weakened version of the Myers theorem we refer to [Ana07].

The main differential geometric invariants involved in these results are the flag curvature and the Ricci scalar. Among the many others there exits one denoted by Ric_k and called $k-$ Ricci curvature that interpolates between the flag curvature and the Ricci curvature. In this chapter we consider an $n-$ dimensional, complete Finsler manifold (M, F) , a minimal, compact submanifold P of it and we prove that if the k-Ricci curvature satisfies the condition $\int_0^\infty Ric_k(t) > 0$ along any geodesic $\gamma : [0, \infty) \to M, t \to \gamma(t)$ emanating orthogonally from P or $\int_{-\infty}^0 Ric_k(t) > 0$ along any geodesic $\gamma : (-\infty, 0] \to M, t \to \gamma(t)$ arriving orthogonally to P, then M is compact. For the Riemannian case there are many similar results (see [BT99] and the references therein). By our knowledge our result is the first of this type for general Finsler spaces but the techniques we use here can be adapted to find many others. Some results for Berwald spaces are obtained by Bhinh and Tamassy (see in [BT99]). The differential invariant Ric_k was deeply studied by Z. Shen. In [She01], he proves various results concerning the vanishing of homotopy groups under the assumption that the k-Ricci curvature satisfies $Ric_k \geq k$.

Various differential geometric invariants have strong impact on the topology of differentiable manifolds as illustrated by deep results in the Riemannian geometry like the theorems of Hopf-Rinow, Myers, Rauch, Synge. The area of these results has been extended along years to the Finslerian setting. The most recent and modern account of them is due to D. Bao, S.S. Chern and Z. Shen(see [BCS00], Ch. 6-9). Their book has been followed by many papers in this field. We cite only few [AP14a],[Wu13] as more related to our results.

The main differential geometric invariants involved in the results aiming to establish a topological property are the flag curvature and the Ricci scalar. Among the many others there exits one denoted by Ric_k that interpolates between the flag curvature and the Ricci curvature. It is associated to a $k+1$-dimensional subspace of the tangent space in a point of a manifold in such a way that for $k = 1$ it coincides with the flag curvature and for $k = \dim M - 1$ it is nothing but the Ricci curvature.

In the next sections we prove two results related to Ric_k which are different in their nature. The first one (see Section 3.4) provides a sufficient condition on the average of the k-Ricci curvature in order that the Finsler manifold be compact. The second one (see Section 3.5) says that if Ric_k is positive then two submanifolds of a Finslerian n- dimensional manifold, one with asymptotic index $n-1$ and one minimal must intersect. The proofs of both results are based on the index form written in a special frame along geodesics.

In the next section we prepare all is needed for the detailed proof given in this chapter.

3.2 Preliminaries

We recall some facts about the variation of energy and Morse Index form, mainly from [Pet06].

Let $\sigma : [a, b] \to M$ be a *regular curve* on M. Its length with respect to the Finsler metric $F : TM \to \mathbb{R}^+$ is given by $L(\sigma) = \int_a^b F(\dot{\sigma}(t))dt$ and its energy is given by $E(\sigma) = \int_a^b F^2(\dot{\sigma}(t))dt$.

The Finsler metric induces naturally the (Finslerian) distance by $d(p, q) = \inf_{\sigma \in C(p,q)} L(\sigma)$, where $C(p.q)$ is the set of piecewise smooth curves from p to q. The properties of a distance, except the symmetry hold well. The pair (M, d) is called sometimes a *generalized metric space*. For a non-reversible Finsler metric d is not symmetric, because the length of a curve may not coincide with the length of the reverse curve $\tilde{\sigma}(t) = \sigma(a + b - t) \in M$. The non-reversibility property is also reflected in the notion of Cauchy sequences.

The classical Hopf-Rinow theorem splits into forward and backward versions (see [BCS00], [CJS11]).

The non-reversibility of the distance implies the existence of two open balls, the forward balls

$$B^+(p, r) = \{x \in M | d(p, x) < r\},$$

where $p \in M$ and $r > 0$, and the backward balls,

$$B^-(p, r)) = \{x \in M | d(x, p) < r\}.$$

A symmetrized distance can be defined as

$$d_s(p, q) = \frac{1}{2}(d(p, q) + d(q, p)).$$

The closed balls will be denoted by a bar, i.e. $\overline{B}^+(p,r)$ and $\overline{B}^-(p,r)$. The topologies induced by these two kinds of balls agree with the topology of the manifold. We also denote the associated balls of d_s by $B_s(x,r)$. In [CJS11] (Proposition 2.2) there is proved a Hopf-Rinow theorem for symmetrized closed balls, i.e. the symmetrized distance d_s is complete if $\overline{B}_s(p,r)$ are compact for all $p \in M$ and $r > 0$ (or equivalently $\overline{B}^+(p,r) \cap \overline{B}^-(p,r)$ is compact for all $p \in M$ and $r > 0$). The conditions here are weaker that those in the theorems involving forward or backward completeness. In the same paper [CJS11] there is constructed an example of Randers type with compact symmetrized balls but which fails to be forward or backward complete.

The non-reversibility of the metric also induces two types of geodesic completeness, *forward*, when the domain of the geodesic can be always extended to (a,∞) for some $a \in \mathbb{R}$ and backward when it can be extended to $(-\infty,b)$ for some $b \in \mathbb{R}$.

The critical points of the (length) energy functional are the normal geodesics σ in the Finsler manifold M whenever they are parameterized by arc-length i.e. $F(\dot{\sigma}) = 1$.

Let P be a submanifold of M of dimension $d < n$. We consider the set

$$A = \{(x,v) | x \in P, v \in T_xM\} = \{\widetilde{x} \in \widetilde{M} | \pi(\widetilde{x}) \in P\}.$$

Let $H_{\widetilde{x}}T_xM$ and $H_{\widetilde{x}}T_xP$ be the horizontal lifts of T_xM and T_xP to $\widetilde{x}$ and

$$H_PTM = \bigcup_{\widetilde{x}\in A} H_{\widetilde{x}}T_xM$$

and

$$H_PTP = \bigcup_{\widetilde{x}\in A} H_{\widetilde{x}}T_xP.$$

For horizontal vector fields $X, Y \in H_PTP$ let X^*, Y^* be some prolongations of them to H_PTM. The restriction of $\nabla_{X^*}Y^*$ to $\widetilde{P} = TP \setminus 0$ does not depend of the choice of the prolongations.

Let $P^{\perp}_{\widetilde{x}}$ be the $\langle \cdot,\cdot \rangle_{\widetilde{x}}$ orthogonal complement of $H_{\widetilde{x}}TP$ in $H_{\widetilde{x}}TM$. By the orthogonal decomposition

$$H_{\widetilde{x}}T_xM = H_{\widetilde{x}}T_xP \oplus P_{\widetilde{x}}^{\perp}, \widetilde{x} = (x, v) \in A$$

we obtain that

$$\nabla_{X^*}Y^* = \nabla^*_X Y + \mathbb{I}_v(X, Y).$$

We will call $\mathbb{I}_v(X, Y)$ the second fundamental form at X and Y in the direction of v. Note that for $\tilde{x} = (x, v)$ with $v \in T_xM \setminus T_xP$ we have

$$\langle \nabla_{X^*}Y^*, v^H \rangle_v = \mathbb{I}_v(X, Y). \tag{3.1}$$

Definition 3.1. *Let $P \subset M$ be an p-dimensional submanifold of a Finsler manifold (M, F). The submanifold P is called minimal if for every tangent vector v to M and for any horizontal orthogonal vectors $V_i^H, i = \overline{1, r}$ (i.e. $\langle V_i^H, V_j^H \rangle_v = 0$ for $i \neq j$) we have $\sum_{i=1}^p \mathbb{I}_v(V_i^H, V_i^H) = 0$.*

The condition of minimality is equivalent with the vanishing of the trace of the linear operator A_{v^H}, where A_{v^H} is the linear operator defined by

$$\langle A_{v^H} X^H, Y^H \rangle_v = \langle \mathbb{I}_T(X^H, Y^H), v^H \rangle_v.$$

For details we refer to [Dra86], [She98].

Definition 3.2. *Let $f : N \to M$ be an immersion. The asymptotic index of the immersion f in the direction $v \in T_xM \setminus T_xP$ is defined by*

$$\nu_f = \min_{x \in N} \nu_f(x)$$

where $\nu_f(x)$ is the maximal dimension of a subspace of T_xN on which the second fundamental form vanishes in every direction $v \in T_xM \setminus T_xN$.

Now let $\sigma : [a, b] \to M$ be a normal geodesic in M with $\sigma(a) \in P$ and $\dot{\sigma}^H(a)$ in the normal bundle of P (i.e. $\dot{\sigma}^H(a) \perp (H_{\dot{\sigma}(a)}T_{\sigma(a)}P)$).

Let $\widetilde{\mathfrak{X}}^P = \mathfrak{X}^P[a, b]$ be the vector space of all piecewise smooth vector fields X along σ such that $X^H(a) \in T_{\dot{\sigma}(a)}\widetilde{P}$ and let $\mathfrak{X}^P$ be the subspace of

$\widetilde{\mathfrak{X}}^P$ consisting of these X such that X^H is orthogonal to $\dot\sigma^H$ along the curve σ.

We have that

$$\begin{aligned}\langle \nabla_{T^H} X^H, Y^H\rangle_T &= \langle \nabla_{X^H} T^H + [T^H, X^H] + \theta(T^H, X^h), Y^H\rangle_T \\ &= \langle \nabla_{X^H} T^H, Y^H\rangle_T,\end{aligned} \tag{3.2}$$

because $[T^H, X^H]$ and $\theta(T^H, X^h)$ are vertical vector fields ([AP94]).

And for Y^H orthogonal to T^H we have that

$$0 = X^H\langle T^H, Y^H\rangle_T = \langle \nabla_{X^H} T^H, Y^H\rangle_T + \langle T^H, \nabla_{X^H} Y^H\rangle_T. \tag{3.3}$$

By considering the vector fields X^H, Y^H such that $X^H(a), Y^H(a) \in T_{\dot\sigma(a)}\widetilde{P}$ and taking account of formulas (3.1), (3.2), (3.3) the Morse index form I^P : $\mathfrak{X}^P \times \mathfrak{X}^P \to \mathbb{R}$, becomes

$$I^P(X,Y) = \langle \nabla_{T^H} X^H, Y^H\rangle_T\Big|^b + \langle \mathbb{I}_T(X^H, Y^H), T^H\rangle_T\Big|_a$$

$$- \int_a^b \langle \nabla_{T^H}\nabla_{T^H} X^H + \Omega(T^H, X^H)T^H, Y^H\rangle_T dt.$$

From [Pet06] we know that I^P is symmetric.

Definition 3.3. *[Pet06] Let $P \subset M$ be an d-dimensional submanifold of a Finsler manifold (M, F). A P-Jacobi field J is a Jacobi field which satisfies in addition*

$$J(a) \in T_{\sigma(a)}P$$

and

$$\langle \nabla_{T^H} J^H + A_{T^H} J^H, Y^H\rangle_T\Big|_a = 0$$

for all $Y \in (T_{\sigma(a)}P)^H$.

The last condition means in fact that

$$\nabla_{T^H} J^H + A_{T^H} J^H \in ((T_{\sigma(a)}P)^H)^\perp.$$

The dimension of the vector space of all P-Jacobi fields along σ is equal to the dimension of M and the dimension of the vector space of the P-Jacobi fields satisfying

$$\langle J^H, T^H \rangle = 0$$

is equal to $\dim M - 1$.

If P is a point, then a P-Jacobi field is a Jacobi field J along σ such that $J(a) = 0$.

A point $\sigma(t_0)$, $t_0 \in [a, b]$ is said to be a *P-focal point* along σ if there exists a non-null P-Jacobi field J along σ with $J(t_0) = 0$.

We shall use the following Lemma from [Pet06].

Lemma 3.4. *Let (M, F) be a Finsler manifold and $\sigma : [a, b] \to M$ be a geodesic, and $P \subset M$ be a submanifold of M. Suppose that there is no P-focal point along σ. Let $X \in \widetilde{\mathfrak{X}}^P$ be a vector fields orthogonal to σ and J a P-Jacobi field such that $X(b) = J(b)$. Then*

$$I^P(X, X) \geq I^P(J, J)$$

with equality if and only if $X = J$.

We introduce the k-Ricci curvature Ric_k following [She01]. For a $(k+1)$-dimensional subspace $\mathcal{V} \in T_xM$ the Ricci curvature $Ric_y\mathcal{V}$ on $\mathcal{V}$ is the trace of the Riemann curvature restricted to $\mathcal{V}$, with flagpole y, and is given by:

$$Ric_y(\mathcal{V}) = \sum_{i=1}^{k} \langle R_y(b_i), b_i \rangle_y = \sum_{i=1}^{k} \langle \Omega(y, b_i)y, b_i \rangle_y,$$

where $R_y(b_i) \equiv \Omega(y, b_i)y$ and $y, (b_i)_{i=\overline{1,\ldots,k}}$ is an arbitrary orthonormal basis for $(\mathcal{V}, \langle , \rangle_y)$. $Ric_y(\mathcal{V})$ is well-defined and is positively homogeneous of degree 2 on $\mathcal{V}$,

$$Ric_{\lambda y}(\mathcal{V}) = \lambda^2 Ric_y(\mathcal{V}), \quad \text{for} \quad \lambda > 0, y \in \mathcal{V}.$$

It is clear from the definition that $Ric_y(T_xM)$ is nothing but the Ricci curvature $Ric(y)$ for $y \in T_xM$.

If $\mathcal{V} = \mathcal{P} \subset T_xM$ is a tangent plane, the flag curvature is given by

$$K(\mathcal{P}, y) = \frac{\langle R_y(u), u\rangle_y}{\langle y, y\rangle_y \langle u, u\rangle_y - \langle u, y\rangle_y^2},$$

where $u \in \mathcal{P} \setminus \{0\}$, $\mathrm{span}(y, u) = \mathcal{P}$. This is independent of the choice of $u \in \mathcal{P} \setminus \{0\}$. If u is g_y orthogonal to y and its g_y-norm 1, then it becomes

$$K(\mathcal{P}, y) = \frac{Ric_y \mathcal{P}}{F^2(y)}, \quad y \in P.$$

Consider the following function on M:

$$Ric_k(x) := \inf_{\dim(\mathcal{V})=k+1} \inf_{y \in \mathcal{V}} \frac{Ric_y(\mathcal{V})}{F^2(y)},$$

the infimum being considered over all $(k+1)$-dimensional subspaces $\mathcal{V} \subset T_xM$ and $y \in \mathcal{V} \setminus \{0\}$. From the above definitions it can be seen that

$$Ric_1 \leq \cdots \leq \frac{Ric_k}{k} \leq \cdots \leq \frac{Ric_{n-1}}{n-1},$$

and

$$Ric_1 = \inf_{(P,y)} K(P, y) \;\text{ and }\; Ric_{(n-1)} = \inf_{F(y)=1} Ric(y).$$

We will say that the Finsler manifold (M, F) has positive k-Ricci curvature if $Ric_k > 0$.

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Secondly, we recall a result from the theory of differential equations which will be essential in the proof of our main result.

3.3 Some compactness results.

Theorem 3.5. *([Tip78]) Consider the differential equation*

$$f''(t) + H(t)f(t) = 0 \ , \ t \in [0, \infty)$$

with $H(t)$ continuous. If

$$\int_0^\infty H(t)dt > 0$$

there exists a solution f satisfying the conditions $f(0) = 1$, $f'(0) = 0$ and there exists $t_0 > 0$ for which $f(t_0) = 0$.

Here $\int_0^\infty$ means $\liminf_{l\to\infty} \int_0^l$. The conditions satisfied by the solution f are similar to those meet in the definition of focal points. A differential equation $f''(t) + H(t)f(t) = 0$ admitting such a solution f will be called *focal*. There are several other sufficient conditions for a differential equation $f''(t) + H(t)f(t) = 0$ be focal, [Gal81], [Gal82].

Now we state and prove our main result.

Theorem 3.6. *Let (M, F) be a n-dimensional Finsler manifold which satisfies the condition*

$$B^+(x, r) \cap B^+(x, r) \qquad \textit{is precompact} \ \ \forall x \in M \textit{and } r > 0 \tag{3.4}$$

and P be a p-dimensional compact and minimal submanifold of M. If the k-Ricci curvature satisfies the condition

$$\int_0^\infty \mathbf{Ric}_k(t) > 0$$

along any geodesic $\gamma : [0, \infty) \to M, t \to \gamma(t)$ emanating orthogonally from P and

$$\int_{-\infty}^0 \mathbf{Ric}_k(t) > 0$$

along any geodesic $\gamma : (-\infty, 0] \to M, t \to \gamma(t)$ arriving orthogonally to P then M is compact.

Proof. Suppose, by contrary that, M is not compact.

Then there exists a normal geodesic $\gamma(t)$ emanating from P and orthogonal to P free of focal points, i.e. there exists a sequence of p_i such that the distance $d(p_i, P)$ (or $d(P, p_i)$) tends to infinity, since it is supposed that M is non-compact. Otherwise, because of the fact that P is compact, M would be contained in an intersection $B^+(x_1, r_1) \cap B^-(x_2, r_2)$ with $x_1, x_2 \in M$ and $r_1, r_2 > 0$ and (see [CJM11], Prop 2.2) should be compact.

By the condition 3.4 on M and the compactness of P there exists for each p_i a normal geodesic γ_i which realizes the minimum distance $d(p_i, P)$ (or $d(P, p_i)$) since the Palais-Smale condition for the energy functional is satisfied (see [CJS11] and [CJM10]) so we can apply Morse theory for geodesics (see [CJM11]). Suppose now that the minimum distances $d(p_i, P)$ are realized and $\gamma : [0, \infty) \to M, t \to \gamma(t)$ along any geodesic emanating orthogonally from P (the reverse case is the same via a change of variables in the integral). Denote by x_i the point in P which is joined with p_i by γ_i, $\gamma_i(0) = x_i \in P$, $\gamma(1) = p_i$. It is known that the geodesic γ_i intersects P orthogonally with respect to the inner product $\langle\ ,\ \rangle_{\gamma_i'(0)}$, that is $T_i = \gamma_i'(0)$ is orthogonal to P with respect to $\langle\ ,\ \rangle_{\gamma_i'(0)}$. By the compactness of P there exists an accumulation point $x \in P$ of the sequence x_i and also $T_i \to T$ with $T \perp P$ with respect to $\langle\ ,\ \rangle_T$ and $F(T) = 1$. It follows that the length of the geodesic $\gamma(t)$ with initial data (x, T) is equal to $d(x, \gamma(t))$, so $\gamma(t)$ is P-focal point free.

On the other hand from the conditions in the theorem we will show that $\gamma(t)$ has P-focal points. This contradiction shows that M has to be compact.

The index form along the geodesic γ with variations vector field V is

$$\begin{aligned} I^P(V,V) &= \int_0^l [\langle \nabla_{T^H} V^H, \nabla_{T^H} V^H \rangle_T - \langle \Omega(T^H, V^H) V^H, T^H \rangle_T] dt \quad (3.5) \\ &= \langle \nabla_{T^H} V^H, V^H \rangle_T \Big|^l + \langle \mathbb{I}_T(V^H, V^H), T^H \rangle_T \Big|_0 \\ &\quad - \int_0^l \langle \nabla_{T^H} \nabla_{T^H} V^H + \Omega(T^H, V^H) T^H, V^H \rangle_T dt. \end{aligned}$$

We are going to use the parallel transport with reference vector T. We construct a moving frame $V_i(t)$, $i = \overline{1,p}$ along γ such that

- $V_i(0)$ is an orthogonal basis in $T_{\gamma(0)}P$ and $\langle V_i^H(0), T^H(0) \rangle_{T(0)} = 0$

- $V_i(t)$ are parallel along γ, i.e. $\nabla_{T^H} V_i^H = 0$.

It follows that the $V_i^H(t)$ are orthogonal to each other and to $T^H(t)$ along γ with respect to the inner product $\langle\ ,\ \rangle_T(t)$.

We have, for $i = \overline{1,r}$

$$\begin{aligned} I^P(V_i, V_i) &= \langle \nabla_{T^H} V_i^H, V_i^H \rangle_T \Big|^l + \langle \mathbb{I}_T(V_i^H, V_i^H), T^H \rangle_T \Big|_0 \\ &\quad - \int_0^l \langle \nabla_{T^H} \nabla_{T^H} V_i^H + \Omega(T^H, V_i^H) T^H, V_i^H \rangle_T dt. \end{aligned} \tag{3.6}$$

We summing up from $i = 1$ to p. Since P is minimal we have

$$\sum_{i=1}^{p} \langle \mathbb{I}_T(V_i^H, V_i^H), T^H \rangle_T \Big|_0 = 0$$

and one yields

$$\begin{aligned} \sum_{i=1}^{p} I(V_i, V_i) &= \sum_{i=1}^{p} \langle \nabla_{T^H} V_i^H, V_i^H \rangle_T \Big|^l \\ &\quad - \sum_{i=1}^{p} \int_0^l \langle \nabla_{T^H} \nabla_{T^H} V_i^H + \Omega(T^H, V_i^H) T^H, V_i^H \rangle_T dt\ . \end{aligned}$$

Let us take $X_i(t) = f(t)V_i(t)$ with $f : [0, \infty) \to \mathbb{R}_+$ satisfying $f(0) = 1, f'(0) = 0$. Then

$$X_i(0) = V_i(0), X_i'(t) = f'(t)V_i(t), X''(t) = f''(t)V_i.$$

It follows that

$$\sum_{i=1}^{p} I(X_i, X_i) = r f(t) f'(t) \Big|^l - r \int_0^l (f''(t) + f(t) \frac{1}{r} \mathbf{Ric}_T(\mathcal{V}) f(t) dt, \tag{3.7}$$

where $\mathcal{V}$ is the linear space spanned by $T, V_i, i = 1, 2...p$.

In our hypothesis on $\mathbf{Ric}_k$, setting $rH = \mathbf{Ric}_T(\mathcal{V})$ it comes out that the equation $f''(t) + f(t)H(t) = 0$ is focal. By the Theorem 10, there exists $t_0 > 0$ such that $f(t_0) = 0$. We take $l = t_0$. In the r.h.s. of (7) the first term vanishes because of $f(t_0) = 0$ and the second is null since f is a solution of the focal equation $f"(t) + f(t)H(t) = 0$. Thus (7) reduces to $\sum_{i=1}^{p} I(X_i, X_i) = 0$. It follows that there exists X_i with $I(X_i, X_i) \leq 0$.

Then, the Lemma 3.4 implies that there exists P-focal points on the geodesic γ, which contradicts the assumption that M is not compact. It follows that M has to be compact. □

3.4 A compactness theorem

In this Section we prove

Theorem 3.7. *Let (M, F) be a n-dimensional Finsler manifold which satisfies the condition*

$$B^+(x, r) \cap B^-(x, r) \qquad \text{is precompact for all } x \in M \text{ and } r > 0\ . \tag{3.8}$$

If there exists a point $p \in M$ such that along any geodesic $\sigma : [0, \infty) \to M$ emanating from p and parameterized by arc length t the condition

$$\int_0^\infty t^\alpha Ric_k(t)dt = \infty \tag{3.9}$$

holds at least for a $k = 1, 2, ..., n - 1$ and for some $\alpha \in [0, 1)$, then M is compact.

We divide the proof into three lemmas.

Lemma 3.8. *Let (M, F) be a n-dimensional Finsler manifold which satisfies the condition*

$$B^+(x, r) \cap B^-(x, r) \qquad \text{is precompact for all } x \in M \text{ and } r > 0. \tag{3.10}$$

If there exists a point $p \in M$ such that every geodesic ray emanating from p has a point conjugate to p along the ray, then M is compact.

Proof. Let S_p be the indicatrix in the point $p \in M$. For each $y \in S_p$ issue the unit speed geodesic from p with the initial unity velocity y. Let c_y be the value of t in the first conjugate point of p and i_y the value of t in the cut point of p. By the hypothesis the set of c_y is forwardly bounded from above and since one has $i_y \leq c_y$ it follows that $\sup_{y \in S_p} i_y \leq \sup_{y \in S_p} c_y$ and because the diameter of M is less or equal to $\sup_{y \in S_p} i_y$ it comes out that M is forwardly bounded from the above. As M is closed in its own topology, by the Hopf-Rinow theorem it is compact. □

Before going on we recall that a differential equation

$$x'' + h(t)x = 0, \tag{3.11}$$

where h is continuous function on an interval I is called of *Jacobi type* and it is said to be *conjugate* if there exists a nontrivial solution ϕ which vanishes for at least two values t_1 and t_2 in I.

The equation 3.11 is called oscillatory on $[0, \infty)$ if each solution of it on $[0, \infty)$ has arbitrary large, and hence infinitely many zeros. If (7) is oscillatory then it is conjugate, too.

Lemma 3.9. *Suppose there exists a point $p \in M$ such that along a geodesic $\sigma : [0, \infty) \to M$ emanating from p and parameterized by the arc length t, the Jacobi type equation*

$$x'' + \frac{Ric_k(t)}{k} x = 0, \tag{3.12}$$

is conjugate on $[0, \infty)$. Then p has a conjugate point on σ.

Proof. Since the equation 3.12 is conjugate, there exists a nontrivial solution $\phi : [0, \infty) \to \mathbb{R}$ of it such that $\phi(t_1) = \phi(t_2) = 0$ for $0 \leq t_1 < t_2$. Define a function f that is null on $[0, t_1)$ and coincides with ϕ for $t \in [t_1, t_2]$.

We consider a g_T-orthonormal frame $(e_i(t))$ along σ with each $(e_i(t))$ parallel along σ and $e_n = T$. We set $W_\alpha(t) = f(t)e_\alpha(t), \alpha = 1, 2, \ldots, n-1$. These W_α are C^∞-vector fields on $[0, t_1]$ and $[t_1, t_2]$. We compute the index form $I(W_\alpha, W_\alpha)$ on the interval $[0, t_2)$. Using $D_T W_\alpha = f' e_\alpha$, $D_T D_T W_\alpha =$

$f''e_\alpha$ and the definition of f we get

$$I(W_\alpha, W_\alpha) = -\int_0^{t_2} (f'' + K(e_\alpha \wedge T)f)f\ dt. \tag{3.13}$$

Summing up from 1 to a fixed $k = 1, 2, \ldots, n-1$ one yields

$$\sum_{\alpha=1}^{k} I(W_\alpha, W_\alpha) = -k\int_0^{t_2} ff''dt - \int_0^{t_2} f^2 \sum_{\alpha=1}^{k} K(e_\alpha \wedge T)dt. \tag{3.14}$$

By the definition of Ric_k we get

$$-f^2 \sum_{\alpha=1}^{k} K(e_\alpha \wedge T) \leq -f^2 Ric_k(t) \tag{3.15}$$

and so we obtain

$$\begin{aligned} \sum_{\alpha=1}^{k} I(W_\alpha, W_\alpha) &= -k\int_0^{t_2} ff''dt - \int_0^{t_2} f^2 Ric_k(t)dt = \\ &= -k\int_{t_1}^{t_2} (\phi''(t) + \frac{Ric_k(t)}{k}\phi)\phi(t)dt = 0, \end{aligned} \tag{3.16}$$

because ϕ is a solution of the equation 3.12. Thus there exists at least an α such that $I(W_\alpha, W_\alpha) < 0$. We denote that W_α by W and proceed by contradiction using the Proposition 4. The vector field W satisfies $W(t_1) = W(t_2) = 0$ and it can not be a Jacobi field since is nowhere zero on (t_1, t_2). By the Proposition 4 we have $0 = I(J, J) < I(W, W) < 0$ which is a contradiction.Thus on the geodesic σ there exists some point conjugate to p.

□

Theorem 2 from the paper ([Moo55]) by R.A. Moore, in some particular conditions gives the following

Lemma 3.10. *Let be the equation 3.11 with $t \in [0, \infty)$. If for some α, $0 \leq \alpha < 1$, we have*

$$\int_0^{\infty} t^\alpha h(t)dt = +\infty, \tag{3.17}$$

then the equation (3.11) is oscillatory.

Now let us combine the above three Lemmas. By Lemma 3.10, taking $h(t) = \dfrac{Ric_k(t)}{k}$, the assumptions of Theorem 8 imply that the equation (3.12) is oscillatory, hence conjugate. Thus by Lemma 3.9 there exists a point $p \in M$ such that each geodesic starting from p has a conjugate point of p. By Lemma 3.8 the Finsler manifold (M, F) is compact. Thus Theorem 3.7 is proved. $\square$

The observations in the beginning of subsetion 2.2 and the previous theorem lead to the following

Theorem 3.11. *Let (M, F) be a forward (resp. backward) complete Finsler manifold. If there exists a point $p \in M$ such that along any along each geodesic $\sigma : [0, \infty) \to M$ emanating from p and parameterized by arc length t the condition*

$$\int_0^\infty t^\alpha Ric_k(t)dt = \infty \tag{3.18}$$

holds at least for a $k = 1, 2, ..., n-1$ and some $\alpha \in [0, 1)$, then M is compact.

Remark. If the condition $\int_0^\infty t^\alpha Ric_k(t)dt = \infty$ holds for $k = 1$, that is for $Ric_1 = \inf\limits_{(P,y)} K(P, y)$ then the inequalities on Ric_k from the subsection 2.4 hold for any $k = 2, 3, \ldots, n-1$.

3.5 An intersection theorem

Theorem 3.12. *Let (M, F) be a n-dimensional Finsler manifold of nonnegative k -Ricci curvature which satisfies the condition*

$$B^+(x, r) \cap B^-(x, r) \qquad \textit{is precompact fo rall } x \in M \textit{and } r > 0 \tag{3.19}$$

Let Q be a complete immersed submanifold of $dim\, Q = n-1$ and with asymptotic index $n-1$ and P be a p-dimensional complete, minimal submanifold of M, with $p \geq k$. Suppose that both P and Q are closed, and one of them *is compact. If M has positive k-Ricci curvature either in all points of P or in all points of Q then P and Q must intersect.*

Proof. Suppose, by contrary, that P and Q do not intersect. From [Kob96] there exists a normal geodesic $\gamma : [0, l] \to M$ that which is the minimum of the distance between P and Q (or between Q and P). This geodesic strikes P and Q orthogonally with respect to the inner product $\langle \cdot, \cdot \rangle_{\gamma'}$ and is free of focal points. Consider now a vector v tangent to P. By the parallel transport along γ induced by the Cartan connection it give rise to a vector field along γ and at the endpoint $q = \gamma(l)$ will be a vector tangent to Q, because Q has codimension 1.

Consider now a basis of T_pP, $v_1, \dots v_p$ orthonormal with respect to the inner product $\langle \cdot, \cdot \rangle_{\gamma'(0)}$. The parallel transport induced by the Cartan connection generates the vector fields $V_1, \dots, V_p$ which are orthogonal along γ with respect to the inner product $\langle \cdot, \cdot \rangle_{\gamma'}$.

The Morse index form, along each vector field V_i is

$$\begin{aligned} I^{(P,Q)}(V_i, V_i) &= \langle \mathbb{I}^P_T(V_i, V_i), T^H \rangle_T \Big|_b - \langle \mathbb{I}^Q_T(V_i, V_i), T^H \rangle_T \Big|_a \\ &\quad - \int_0^l \langle \nabla_{T^H} \nabla_{T^H} V_i^H + \Omega(T^H, V_i^H) T^H, V_i^H \rangle_T dt. \end{aligned}$$

The minimality of P implies that

$$\sum_{i=1}^p \langle \mathbb{I}^P_T(V_i, V_i), T^H \rangle_T = 0 \ .$$

From $Ric_k(P) \geq 0$ and $r \geq k$ it follows that, for any subset $\{i_1, \dots, i_k\} \subset \{1, \dots p\}$ we have

$$\sum_{j=1}^k K(E_{j_i}, T) \geq 0$$

and

$$\sum_{i=1}^p K(E_i, T) = \frac{r}{kC_r^k} \sum_{1 \leq i_1 \leq \cdots \leq i_k} \sum_{j=1}^k K(E_{i_j}, T) \geq 0 \ .$$

Further, the fact that M has positive k-Ricci curvature either in all points of P or at all points of Q implies that either

$$\sum_{j=1}^{k} K(E_{i_j}, T)\Big|_0 \geq 0 \text{ or}$$

$$\sum_{j=1}^{k} K(E_{i_j}, T)\Big|_l \geq 0 \ ,$$

for any distinct indices. Hence we have either

$$\sum_{i=1}^{p} K(E_i, T)\Big|_0 > 0 \text{ or}$$

$$\sum_{i=1}^{p} K(E_i, T)\Big|_l > 0 \ .$$

Substituting this in the index form it follows that

$$\sum_{i=1}^{p} I^{(P,Q)}(V_i, V_i) = -\sum_{i=1}^{p} \int_0^l K(E_i, T) dt < 0 \ .$$

Thus $I^{(P,Q)}(V_i, V_i) < 0$ for some index i which contradicts the assumption that γ is of minimal length from P to Q by Lemma 3.4. Hence P and Q must intersect.

□

If the Finsler metric is forward (backward) complete it follows

Theorem 3.13. *Let (M, F) be a forward (resp. backward) n-dimensional Finsler manifold of nonnegative k -Ricci curvature. Let Q be a complete immersed submanifold of $\dim Q = n-1$ and with asymptotic index $n-1$ and P be a p- dimensional complete, minimal submanifold of M, with $r \geq k$. Suppose that both P and Q are closed, and one of them is compact. If M has positive k-Ricci curvature either in all points of P or in all points of Q then P and Q must intersect.*

3.6 Some remarks for Berwald and Riemann manifolds

A Finsler manifold is called of *Berwald type* if the Cartan connection does not depend on the reference vector. In this case the Cartan connection is a linear connection on the manifold M. In this situation the fact that the second fundamental form of a submanifold vanishes is equivalent to the fact that the submanifold is totally geodesic consequently in the Berwald category one has $\nu_f = \dim P$ if and only if P is a totally geodesic submanifold.

It follows that for Berwald manifolds in the Theorems 3.7 and 3.13 we can replace the condition that the asymptotic index of Q is $n-1$ with that the submanifold Q is a totally geodesic submanifold of dimension $n-1$.

The same situation holds for Riemannian metrics and so our results in the above mentioned theorems also generalize the reults of Kenmotsu and Xia in [KX96].

Chapter 4

Injectivity radius and h-parallel Ricci tensor

4.1 Introduction

Let (M, F) be a forward geodesically complete connected Finsler manifold of dimension n. By the Bonnet-Myers Theorem ([BCS00]) if its Ricci scalar Ric has the following uniform positive lower bound $Ric \geq (n-1)a > 0$, then M is compact and its diameter $d(M)$ is finite, at most $\dfrac{\pi}{\sqrt{a}}$. It follows that no geodesic longer then $d(M)$ can remain minimizing and hence any geodesic contains a cut point. Thus the injectivity radius at any point of M is finite. See [Amb61]– [GW14],[Wu13] for variants and generalizations of the Bonnet-Myers theorem.

We will show that conversely, the finiteness of injectivity radius implies a bound for the Ricci scalar. More precisely, let i_x be the injectivity radius of a point $x \in M$. The injectivity radius of M is $i(M) = inf_{x \in M} i_x$. Let us denote by $Ric_{(x,y)}$ the Ricci scalar in $x \in M$ on the direction given by the unitary vector y and let SM be the indicatrix bundle of (M, F). We shall prove in Section 2 that if the injectivity radius $i(M)$ is finite then the following inequality holds

$$\inf_{(x,y)\in SM} Ric_{(x,y)} \leq (n-1)\frac{\pi^2}{i(M)^2}.$$

In Section 1 we recall the necessary results from Finsler geometry, following the books [BCS00], [She01]. In Section 3 we shall assume that the Ricci tensor Ric_{ij} is $h-$ parallel with respect to the Chern-Rund connection, that is satisfies the equation $Ric_{ij|k} = 0$, where $|k$ means the $h-$ derivative with respect to the Chern-Rund connection and we prove that if $i(M) = \infty$, the Ricci scalar is non-positive on M and if $0 < i(M) < \infty$ then

$$\sup_{(x,y)\in SM} Ric_{(x,y)} \leq (n-1)\frac{\pi^2}{i(M)^2}.$$

Then we show that the same inequality holds in the weaker condition $Ric_{ij|k|h} = 0$ but for the reversible Finsler metrics. In the end two examples of Finsler manifolds with $h-$ parallel Ricci tensor are provided. The results are from [Pet17].

4.2 A Ricci Scalar Bound

We shall use the notations, the terminology and results from [BCS00], [She01] without comments.

Let (M, F) be a Finsler manifold. The Finsler structure F is a function $F : TM \to [0, \infty)$, $(x, y) \to F(x, y)$ which is C^∞ on the slit tangent bundle $TM\backslash 0$, positively homogeneous in y and whose Hessian matrix $g_{ij} := \dfrac{1}{2}\dfrac{\partial^2 F^2}{\partial y^i \partial y^i}$ is positive-definite at every point of $TM\backslash 0$.

The Chern-Rund connection of local coefficients $\Gamma^i_{jk}(x, y)$ is a linear connection in the pull-back bundle $\pi^* TM$ over $TM\backslash 0$, where $\pi : TM \to M$ is the natural projection. It is only h-metrical and it has two curvatures $R_j{}^i{}_{kh}$, $P_j{}^i{}_{kh}$.

Let be y a non zero element of T_xM. Then $g_y(x) := g(x, y) = g_{ij}(x, y)dx^i \otimes dx^j$ is an inner product in T_xM which is used to measure lengths and angles.

For a vector field $W(t) := W^i(t)\dfrac{\partial}{\partial x^i}$ along a curve σ, whose tangent vector field is T, the expression,

$$D_T W = \left[\frac{dW^i}{dt} + W^j T^k (\Gamma^i_{jk}(\sigma, T)) \right] \frac{\partial}{\partial x^i} \tag{4.1}$$

is called covariant derivative with reference vector T.

One says that W is parallel long σ if $D_T W = 0$, with reference vector T. Parallel transport (with reference vector T) one defines on the standard way. The parallel transport preserves g_T-lengths and angles.

The constant speed geodesics are solutions of $D_T T = 0$, with reference vector T.

Let $\sigma(t) = exp_x(tT), x \in M, o \leq t \leq L$ be a geodesic of constant speed 1. One abbreviates $g_{(\sigma,T)}$ by g_T.

For two continuous and piecewise C^∞ vector fields V and W along σ the index form is

$$I(V, W) = \int_0^L [g_T(D_T V, D_T W) - g_T(R(V, T)T, W)] dt. \tag{4.2}$$

Here D_T is calculated with reference vector T of length 1 and

$$R(V, T)T := (T^j R^i_{jkh} T^h) V^k \frac{\partial}{\partial x^i}$$

is evaluated at the point (σ, T).

The index form is bilinear and symmetric.

Let be the flag (a plane in $T_x M$) spanned by the flagpole T and by a unit vector V which is orthogonal to the flagpole. The flag curvature in the point $(\sigma(t), T)$ and for the said flag is then given by

$$K(T \wedge V) = g_T(R(V, T)T, V) = V^i (T^j R_{jikh} T^h) V^k =: V^i R_{ik} V^k. \tag{4.3}$$

If W is a continuous piecewise C^∞ vector field such that it is g_T- orthogonal to σ we have

$$I(W,W) = \int_0^L [g_T(D_TW, D_TW) - K(T \wedge W) g_T(W,W)]dt, \tag{4.4}$$

where $K(T \wedge W)$ is the flag curvature of the flag with flagpole T and transverse edge W.

Let $0 =: t_0 < t_1 < ... < t_h := L$ be a partition of $[0, L]$ such that V and W are both C^∞ on each closed subinterval $[t_{s-1}, t_s]$. Using integration by parts, one can rewrite the index form as

$$\begin{aligned} I(V,W) := \quad & g_T(D_TV, W)\Big|_0^L - \sum_{s=1}^{h-1} g_T(D_TV, W)\Big|_{t_s^-}^{t_s^+} - \\ & - \int_0^L g_T(D_TD_TV + R(V,T)T, W)dt. \end{aligned} \tag{4.5}$$

The second term in the right side of the above equality disappears if V is of the class C^∞ along σ. And the first term vanishes if $W(0) = W(L) = 0$. A vector field J along σ is said to be a *Jacobi field* if it satisfies the equation

$$D_TD_TJ + R(J,T)T = 0. \tag{4.6}$$

One says that $q = \sigma(L)$ is conjugate with $p = \sigma(0)$ along σ if there exists a nonzero Jacobi field J along σ which vanishes at p and q i.e. $J(0) = J(L) = 0$.

We recall from [BCS00] p.182 the following

Proposition 4.1. *Let $\sigma(t), 0 \leq t \leq r$ be a geodesic in a Finsler manifold (M, F). Suppose no point $\sigma(t), 0 < t \leq r$ is conjugate to $p := \sigma(0)$. Let W be any piecewise C^∞ vector field along σ and let J denote the unique Jacobi field along σ that has the same boundary values as W. That is, $J(0) = W(0)$ and $J(r) = W(r)$. Then*

$$I(W,W) \geq I(J,J). \tag{4.7}$$

Equality holds if and only if W is actually a Jacobi field, in which case the *said J coincides with W.*

As an application of this result we obtain the following corollaries.

Corollary 4.2. *Let $\sigma(t), 0 \leq t \leq r \leq L$ be a geodesic in a Finsler manifold (M, F). Suppose no point $\sigma(t), 0 < t \leq r$ is conjugate to $p := \sigma(0)$. Let W be a piecewise C^∞ vector field along σ which is nowhere 0 on $(0, r)$ and satisfies $W(0) = W(r) = 0$ and $I(W, W) \leq 0$ on $[0, L]$. Then the geodesic $\sigma(t)$ must contain conjugate points for $L \geq r$.*

Proof. We proceed by contradiction. Suppose that no point $\sigma(t), 0 < t \leq r$ is conjugate to $\sigma(0)$. By the definition of the conjugate points, the unique Jacobi field which vanishes at the endpoints of $\sigma(t), 0 \leq t \leq r$ is identically zero. The vector field W satisfies $W(0) = W(r) = 0$ and it can not be a Jacobi field since is nowhere zero on $(0, r)$. By the Proposition 1.1 we have $0 = I(J, J) < I(W, W) \leq 0$ which is a contradiction. Thus $\sigma(r)$ or an $\sigma(t)$ for $t = L > r$ should be conjugate with $\sigma(0)$. □

Corollary 4.3. *Let $\sigma(t), 0 \leq t \leq r \leq L$ be a geodesic in a Finsler manifold (M, F). Suppose no point $\sigma(t), 0 < t \leq r$ is conjugate to $p := \sigma(0)$. Let W be a piecewise C^∞ vector field along σ which is nowhere 0 on $(0, r)$ and satisfies $W(0) = W(r) = 0$. Then $I(W, W) > 0$ on $[0, r]$.*

Proof. One applies the Proposition 1.1 with $J = 0$. □

We shall recall in the Chern connection setting some notions related to ricci curvature.

Let $\{l = \frac{y}{F(x,y)}, e_\alpha, \alpha = 1, \ldots, n-1\}$ be a g_y-orthonormal basis for the fiber of π^*TM over the point $(x, y) \in TM\backslash 0$. With respect to it one has $K_{(x,y)}(l \wedge e_\alpha) = g_y(R(e_\alpha, l)l, e_\alpha) = R_{\alpha\alpha}$.

The Ricci scalar denoted by $Ric_{(x,y)}$ is

$$Ric_{(x,y)} := \sum_{\alpha=1}^{n-1} K(x, y, l \wedge e_\alpha) = \sum_{\alpha=1}^{n-1} R_{\alpha\alpha}. \tag{4.8}$$

If (M, F) has constant flag curvature c, then

$$Ric_{(x,y)} = (n-1)c. \tag{4.9}$$

We introduce the k-Ricci curvature Ric_k following [She01]. For a $(k+1)$-dimensional subspace $\mathcal{V} \in T_xM$ the Ricci curvature $Ric_y\mathcal{V}$ on $\mathcal{V}$ is defined by

$$Ric_y(\mathcal{V}) = \sum_{i=1}^{k} g_y(R_y(b_i), b_i) = \sum_{i=1}^{k} g_y(R(y, b_i)y, b_i),$$

where $y, (b_i)_{i=\overline{1,\ldots,k}}$ is an arbitrary orthonormal basis for $(\mathcal{V}, g_y)$.

$Ric_y(\mathcal{V})$ is well-defined and positively homogeneous of degree 2 on $\mathcal{V}$,

$$Ric_{\lambda y}(\mathcal{V}) = \lambda^2 Ric_y(\mathcal{V}), \quad \text{for} \quad \lambda > 0, y \in \mathcal{V}.$$

It is clear from the definition that $Ric_y(T_xM)$ is nothing but the Ricci curvature $Ric_{(x,y)}$.

If $\mathcal{V} = \mathcal{P} \subset T_xM$ is a tangent plane, for $y \in \mathcal{V}$ and $u \in \mathcal{P} \setminus \{0\}$ such that $\mathrm{span}(y, u) = \mathcal{P}$ and u is g_y–orthogonal to y and its g_y-norm is 1 then the flag curvature $K(\mathcal{P}, y)$ for the tangent plane $\mathcal{P}$ with the flagpole y is

$$K(\mathcal{P}, y) \equiv K_{(x,y}(y \wedge u) = \frac{Ric_y\mathcal{P}}{F^2(y)}.$$

Consider the following function on M:

$$Ric_k(x) := \inf_{\dim(\mathcal{V})=k+1} \inf_{y \in \mathcal{V}} \frac{Ric_y(\mathcal{V})}{F^2(y)},$$

the inf being considered over all $(k+1)$-dimensional subspaces $\mathcal{V} \subset T_xM$ and $y \in \mathcal{V} \setminus \{0\}$. From the above definitions it can be seen that

$$Ric_1 \leq \cdots \leq \frac{Ric_k}{k} \leq \cdots \leq \frac{Ric_{n-1}}{n-1}, \tag{4.10}$$

where

$$Ric_1 = \inf_{(P,y)} K(P, y), Ric_{n-1} = \inf_{(x,y) \in SM} Ric_{(x,y)}.$$

The Finsler metric above defined is non-reversible and so the distance defined by it is nonsymmetric.Thus we have to distinguish between forward and backward geodesically completeness. Recall that (M, F) is forward geodesically complete if all geodesics, parameterized to have constant Finslerian speed equal to 1, are indefinitely forward extendible.

Let be $x \in M$ and let $\sigma_y(t)$ be the unit speed geodesic that passes through x at $t = 0$ with initial velocity y. Since $F(x, y) = 1$ we have that $y \in S_x M$, where $S_x M$ is the indicatrix of x. We denote by SM the indicatrix bundle over M. Let be $c_y := \sup\{r : \text{no point } \sigma_y(t), 0 \leq t \leq r \text{ is conjugate to } x\}$. If c_y is finite (it could be infinite) the point $\sigma_y(c_y)$ is called the first conjugate point of x along σ_y. The conjugate radius of x is $c_x = \inf_{y \in S_x} c_y$ and the conjugate locus of x is the set of points $\sigma_y(c_y)$ for $y \in S_x$ and $c_y < \infty$. The number $i_y := \sup\{r : \text{the segment } \sigma_y|_{[0,r]} \text{ is globally minimizing}\}$ is called the cut value and if $i_y < \infty$, the point $\sigma_y(i_y)$ is called the cut point of x along σ_y. The injectivity radius in x is defined as $i_x := \inf_{y \in S_x} i_y$. The cut locus Cut_x of x is the set of points $\sigma_y(i_y)$ for $y \in S_x$ and $i_y < \infty$. On proves that $i_y \leq c_y$, hence $i_x \leq c_x$. The injectivity radius of (M, F) is $i(M) = \inf_{x \in M} i_x$.

Theorem 4.4. *Let M be a forward complete Finsler manifold. Assume that $0 < i(M) < \infty$. Then*

$$\inf_{(x,y) \in SM} Ric_{(x,y)} \leq (n-1) \frac{\pi^2}{i(M)^2} \tag{4.11}$$

Proof. Consider $(x, y) \in SM$ (recall that SM is the indicatrix bundle over M) and a real number r such that $0 < r < i_x$. Let σ be a normalized geodesic with initial data $\sigma(0) = x$ and $\dot{\sigma}(0) = y$. Let $y, e_1, \ldots, e_{n-1}$ be a basis of $T_x M$ which is orthonormal with respect to the inner product g_T induced by the Finsler metric (recall that T is the vector field tangent to the geodesic σ; it is parallel along of σ and $T(0) = y$).We consider the parallel transport induced by the Chern- Rund connection and we transport by parallelism along σ each vector of the basis $y, e_1, \ldots, e_{n-1}$ from $T_x M$. Thus we get an orthogonal basis $T, E_1, ..., E_{n-1}$ of parallel vector fields along σ. We consider the vector fields $W_\alpha(t) = f(t) E_\alpha(t), \alpha = 1, ..., n-1$ for some non-null function f of class C^1 on $[0, r]$ with $f(0) = f(r) = 0$.

The Morse Index form along the geodesic σ on interval $[0,r]$ is

$$I(W_\alpha, W_\alpha) = \int_0^r [g(D_T W_\alpha, D_T W_\alpha) - g(W_\alpha, W_\alpha) K(T \wedge W_\alpha)] dt. \tag{4.12}$$

Since in the definition of the flag curvature the dimensions of the flag are irrelevant, that is $K(T,W) = K(T, E_\alpha)$ and because of $D_T W_\alpha = f' E_\alpha$ the Morse Index form reduces to

$$I(W_\alpha, W_\alpha) = \int_0^r \left[(f'(t))^2 - f^2 K(T, E_\alpha) \right] dt. \tag{4.13}$$

Summing up for $\alpha = 1, ..., n-1$ we obtain that

$$\sum_\alpha I(W_\alpha, W_\alpha) = \int_0^r \left[(n-1) (f'(t))^2 - f^2(t) Ric_{(\sigma(t),T)} \right] dt. \tag{4.14}$$

By the choice of r, the geodesic σ is minimizing on the interval $[0,r]$ and so no points on the geodesic segment $\sigma([0,r])$ are conjugate to $\sigma(0)$. Recall that W_α vanishes in 0 and r. Thus by the Corollary 4.3 we have $I(W_\alpha, W_\alpha > 0$ for every α. Hence we have $\sum_\alpha I(W_\alpha, W_\alpha) > 0$ and this inequality implies

$$0 < \int_0^r \left[(n-1) (f'(t))^2 - f^2(t) Ric_{(\sigma(t),T)} \right] dt \leq \int_0^r \left[(n-1) (f'(t))^2 - \rho f^2(t) \right] dt, \tag{4.15}$$

where $\rho := \inf_{(x,y)\in SM} Ric_{(x,y)}$.

The inequality (4.15) holds if

$$\rho \leq (n-1) \frac{\int_0^r \left[(f'(t))^2 \right] dt}{\int_0^r \left[(f(t))^2 \right]} dt.$$

The infimum

$$\inf_{f \in C^1[0,r]} \frac{\int_0^r \left[(f'(t))^2 \right] dt}{\int_0^r \left[(f(t))^2 \right]} dt \tag{4.16}$$

is $\frac{\pi^2}{r^2}$ by the Wirtinger Inequality and we have

$$\inf_{(x,y)\in SM} Ric_{(x,y)} \leq (n-1)\frac{\pi^2}{r^2},$$

for all r chosen as in above. If i_x is infinite for at least an x then for $r \to \infty$ it follows that $\rho \leq 0$ and this inequality is true because of the Bonnet-Myers Theorem. So we may suppose that i_x is finite for all $x \in M$. Recalling that $i(M) = inf_{x\in M} i_x$ it follows that $i(M) \leq r$, hence

$$\inf_{(x,y)\in SM} Ric_{(x,y)} \leq (n-1)\frac{\pi^2}{i(M)^2}.$$

□

Corollary 4.5. *Let M be a forward complete Finsler manifold such that $0 < i(M) < \infty$. Then*

$$Ric_k(x) \leq k\frac{\pi^2}{i(M)^2}. \tag{4.17}$$

Proof. By we (4.10) have $Ric_k(x) \leq \frac{k}{n-1}\rho$ and (4.11) implies (4.17).

□

Corollary 4.6. *Let M be a forward complete Finsler manifold such that $0 < i(M) < \infty$. If in (4.11) we have the equality then M is compact and the diameter of M is $i(M)$.*

Proof. In the case that (4.11) reduces to equality we have that $Ric_{(x,y)} \geq (n-1)\frac{\pi^2}{i(M)^2}$ and by the Bonnet-Myers Theorem M is compact with diameter less then $i(M)$. As for any compact Finsler manifold one has $i(M)$ less then its diameter, the Corollary follows. □

The Chapter II of the book [MPF91] titled "An Inequality ascribed to Wirtinger and related results" contains a large variety of inequalities related in a way or another with the Wirtinger Inequality but no proofs are given. The reader is referred to original papers, hard to be found. Thus for the sake of completeness we give our proof of the inequality that we have used in the above.

Let f be a real function of class C^1 on the interval $[a,b]$ and $f(a)=f(b)=0$. We consider the functional

$$I(f)=\frac{\int_a^b f'^2(x)dx}{\int_a^b f^2(x)dx},$$

for all f as above. This is clearly bounded below by zero and so it has a minimum (inf) $\rho \geq 0$. Let $\tilde{f}(x)=f(x)+\epsilon V(x), \epsilon \in (-\delta,\delta), \delta>0$ be a C^1- variation of f with fixed end points, that is $V(a)=V(b)=0$. A necessary condition in order that f to be an extremal (minimum in our case) of the functional $\mathcal{I}(\epsilon) := I(\tilde{f})$ is that

$$\frac{d\mathcal{I}(\epsilon)}{d\epsilon}|\epsilon = 0.$$

Computing the derivative and taking $\epsilon = 0$ this condition reduces to

$$\int_a^b f'(x)V'(x)dx = \rho \int_a^b f(x)V(x)dx.$$

An integration by parts in the left hand side gives

$$\int_a^b (f"(x)+\rho f(x))V(x)dx = 0$$

and so the extremal has to be solution of the differential equation

$$f" + \rho f = 0.$$

The general solution of this equation is

$$f(x) = A\cos\sqrt{\rho}x + B\sin\sqrt{\rho}x.$$

The boundary conditions $f(a)=f(b)=0$ show that the integration constant A and B are solutions of a linear and homogeneous system of two equations. The determinant of this system should be zero. This condition via some calculation gives $\rho = (\frac{\pi}{b-a})^2$. As $I(f)\geq\rho$, the Wirtinger Inequality follows.

The equality holds for all functions

$$f(x) = c \sin \pi \frac{x-a}{b-a}, c \in \mathbb{R}.$$

For $a = 0$ and $b = r > 0$ the Wirtinger equality becomes

$$\int_0^r f'^2(x)dx \leq \frac{\pi^2}{r^2} \int_0^r f^2(x)dx.$$

The equality holds for all $f(x) = c \sin \frac{\pi}{r} x, c \in \mathbb{R}$.

4.3 h-Parallel Ricci tensor

In the Finslerian setting the Ricci tensor is defined independent of any Finsler connection by

$$Ric_{ij} := \frac{1}{2} \frac{\partial^2 F^2 Ric_{(x,y)}}{\partial y^i \partial y^j},$$

where $Ric_{(x,y)}$ is Ricci scalar. This is clearly covariant and symmetric. The Ricci scalar can be recovered from the Ricci tensor by

$$F^2(x,y) Ric_{(x,y)} = y^i y^j Ric_{ij}.$$

Let $_{|k}$ be the h-covariant derivative with respect to the Chern-Rund connection. Using $F_{|k} = 0$ and $y^i_{|k} = 0$ one easily find that

$$F^2 Ric_{|h} = y^i y^j Ric_{ij|h}, \quad F^2 Ric_{|r|s} = y^i y^j Ric_{ij|r|s}. \tag{4.18}$$

From (4.18) it follows that if the Ricci tensor is h-parallel then the Ricci scalar is h-constant, that is $\delta_k Ric = 0$, where $\delta_k = \frac{\partial}{\partial x^k} - N^i_k(x,y)\frac{\partial}{\partial y^i}$. Here N^i_k are the local coefficients of the nonlinear connection defined by the given Finsler structure.

Let now be a curve $\sigma : t \to \sigma(t) = ((x(t))$ on M and $\dot\sigma(t) = (\dot x(t))$ be its tangent vector field. A real function $L(x.y)$ on TM defines a function $L(t) = L(x(t), \dot x(t))$ and a short computation yields $\frac{dL}{dt} = (\delta_i L)\dot x^i + (\ddot x^j + N^j_i \dot x^i)\frac{\partial L}{\partial y^j}$. In particular, if σ is a geodesic parameterized such that $F(x(t), \dot x(t)) = 1$

it follows that $\frac{dL}{dt} = (\delta_i L)\dot{x}^i$. Applying these formulas to the Ricci scalar $Ric(t) = Ric_{x(t),\dot{x}(t)}$ along the geodesic σ we get

$$\frac{dRic}{dt} = \dot{x}^r \dot{x}^s Ric_{rs|i}\dot{x}^i, \quad \frac{d^2Ric}{d^2t} = \dot{x}^r \dot{x}^s Ric_{rs|i|j}\dot{x}^i\dot{x}^j. \tag{4.19}$$

Let us assume that $Ric_{rs|i|j} = 0$ and let σ be the normalized geodesic with initial data $\sigma(0) = x$ and $\dot{\sigma}(0) = y$. By (4.19) it follows that $\frac{d^2Ric}{d^2t} = 0$ and so we have

$$Ric(t) = at + b, \tag{4.20}$$

where a and b are constants depending on the geodesic σ with $b = Ric_{(x,y)}$ and $a = \frac{dRic}{dt}(0) = y^r y^s Ric_{rs|i}(x,y)y^i$.

Now we consider (4.14) with $Ric(t)$ given by (4.20)and $f(t) = \sin\frac{\pi}{r}t$. Computing the required integral one gets

$$-4r\sum_{\alpha} I(W_\alpha, W_\alpha) = ar^3 + 2br^2 - 2(n-1)\pi^2 =: h(r). \tag{4.21}$$

It follows that the sign of $\sum_\alpha I(W_\alpha, W_\alpha)$ is contrary to the sign of the function h depending on $r > 0$.

We analyze first the case $Ric_{rs|i} = 0$, that is the case when the Ricci tensor is h- parallel. It follows that $Ric_{rs|i|j} = 0$ and $a = 0$ for any geodesic like σ. Conversely, if $a = 0$ for the normalized geodesics with the initial condition any pair $(x, y) \in SM$, then the Ricci tensor is $h-$parallel.

If $a = 0$, then $\frac{h(r)}{2} = br^2 - (n-1)\pi^2$. Therefore, we have $\sum_\alpha I(W_\alpha, W_\alpha) \leq 0$ if $b > 0$ and $r \geq \pi\sqrt{\frac{n-1}{b}}$. If it happens that the Ricci scalar is uniformly bounded below by $(n-1)\lambda$ we get $\sum_\alpha I(W_\alpha, W_\alpha) \leq 0$ for $r \geq \frac{\pi}{\sqrt{\lambda}}$. Hence there exists at least a W_α, let denote it by W such that $I(W, W) \leq 0$. By the Corollary 4.2, every geodesic with length $\frac{\pi}{\sqrt{\lambda}}$ or longer must contain conjugate points. The conclusions of the Bonnet-Myers Theorem, in the particular case when the Ricci tensor is parallel, follow.

Now we prove the following theorem.

Theorem 4.7. *Let M be a forward complete Finsler manifold with $h-$ parallel Ricci tensor. Then we have*

(1) *If* $i(M)=\infty$*, the Ricci scalar is non-positive on* M*,*

(2) *If* $0<i(M)<\infty$ *then*

$$\sup_{(x,y)\in SM} Ric_{(x,y)} \leq (n-1)\frac{\pi^2}{i(M)^2}. \tag{4.22}$$

Proof. Continuing to analyse the case $a=0$, assume that $0<r\leq i_x$. Then the segment of geodesic $\sigma([0,r])$ is minimizing and so it does not contain conjugate points. By the Corollary 4.3 for any piecewise C^∞ vector field W along σ which is nowhere 0 on $(0,r)$ and satisfies $W(0)=W(r)=0$ we have $I(W,W)>0$ on $[0,r]$. Hence $\sum_\alpha I(W_\alpha,W_\alpha)>0$ on $[0,r]$ holds. By (4.21) with $a=0$ and $b>0$ necessarily follows $r\leq\sqrt{\frac{n-1}{b}}\pi$ or equivalently

$$Ric_{(x,y)} \leq (n-1)\frac{\pi^2}{r^2}. \tag{4.23}$$

1. If $i(M)=\infty$ then $i_x=\infty$ and from (4.23) with $r\to\infty$ it follows that $Ric_{(x,y)}\leq 0$ for all $(x,y)\in SM$ and since the Ricci scalar is homogenous of degree zero, the inequality holds on TM.

2. From $r<i_x$ and $i(M)=inf_{x\in M}i_x$ it follows that $i(M)\leq r$ and (4.23)implies that $Ric_{(x,y)}\leq(n-1)\frac{\pi^2}{i(M)^2}$. Therefore, (4.22) holds □

In the paper [LS] some consequences of the weaker condition $Ric_{rs|i|j}=0$ are found but only for Riemannian manifolds. The author proves that if $i(M)\neq 0$ this condition is equivalent to the condition $Ric_{rs|i}=0$. In the Finslerian setting there are some reason to think that this fact is no longer true. However, several results valid for the Riemannian manifolds can be extended to any Finsler manifold or at least to those Finsler manifolds having reversible Finsler metric.

Theorem 4.8. *Let* (M,F) *be a forward complete Finsler manifold satisfying* $Ric_{rs|i|j}=0$*. If the normalized geodesic* $\sigma:[0,\infty)\to M$ *with the initial condition* $(x,y)\in SM$ *is a ray, that is each segment of it is minimizing, then* a *and* b *from (4.20) satisfy* $a<0, b\leq 0$ *, equivalently the Ricci scalar* $Ric(t)$ *is decreasing on* σ*.*

Proof. If σ is a ray, it is minimizing for all $r > 0$ and so necessarily $\sum_\alpha I(W_\alpha, W_\alpha) > 0$ for all $r > 0$. A study of the sign of the function h from (4.22) shows that it has constant negative sign on $[0, \infty)$, and hence $\sum_\alpha I(W_\alpha, W_\alpha) > 0$ only when $a > 0, b \leq 0$. In this case (4.20) shows that $Ric(t)$ is decreasing on σ. □

Recall that the Finsler metric F is reversible if $F(x, -y) = F(x, y)$ for all $(x, y) \in TM$. In this case the forward and backward completeness coincide. The condition of reversibility is quite strong. It excludes the Randers spaces. Happily it holds for the Riemannian spaces. The following generalization of the Theorem 4.7 holds.

Theorem 4.9. *Let F be a reversible Finsler metric and (M, F) be a complete Finsler manifold with $Ric_{rs|i|j} = 0$. If the injectivity radius $i(M)$ satisfies $0 < i(M) < \infty$ then*

$$\sup_{(x,y)\in SM} Ric_{(x,y)} \leq (n-1)\frac{\pi^2}{i(M)^2}.$$

Proof. Let us replace σ with the geodesic having the initial condition $(x, -y) \in SM$. It follows that in 4.21 the constant a should be replaced by $-a$ and the function $h(r)$ becomes $\tilde{h}(r) = -ar^3 + 2br^2 - 2(n-1)\pi^2$. For $r < i_x < \infty$ the function h and $\tilde{h}$ should have the same negative sign. Taking their sum one gets

$$br^2 - (n-1)\pi^2 < 0. \tag{4.24}$$

Since σ is not a ray, by the Theorem 4.7 we have $b > 0$ and (4.24) implies $Ric_{(x,y)} \leq (n-1)\frac{\pi^2}{r^2}$. Then based on the definition of the injectivity radius the conclusion follows. □

For reversible Finsler metrics we have also the theorem

Theorem 4.10. *Let F be a reversible Finsler metric and (M, F) be a complete Finsler manifold with $Ric_{rs|i|j} = 0$. If the normalized geodesic $\sigma : (-\infty, +\infty) \to M$ is a line, that is each segment of it is minimizing, then $\frac{dRic}{dt}(0) = 0$ and $Ric_{(x,y)} \leq 0$.*

Proof. If the geodesic σ is a line, the function h must have constant negative sign for all $r \in \mathbf{R}$. This is possible only if in (4.20) we have $a = 0$ and $b \leq 0$ and the Theorem follows. $\square$

In the end we will give two examples of Finsler manifolds having $h-$ parallel Ricci tensor.

Example 1. Any Finsler manifold of constant flag curvature has $h-$ parallel Ricci tensor.

Indeed, if the Finsler manifold (M, F) is of constant flag curvature c, the Ricci scalar is $c(n-1)$ and the Ricci tensor is $Ric_{ij} = c(n-1)g_{ij}$. Since the Chern-Rund connection is $h-$ metrical, it follows that $Ric_{ij|k} = 0$.

Example 2. In Ch. 7 from [She01] a Finsler metric F is called Einstein if there exists a constant λ such that $Ric_{(x,y)} = \lambda F(x,y)$. The Finsler manifold (M, F) with F an Einstein Finsler metric has $h-$ parallel Ricci tensor. Indeed, by a direct computation one finds that the Ricci tensor is given by $Ric_{ij} = \frac{\lambda}{2}F(g_{ij} + \frac{y_i y_j}{F^2})$ and using $F_{|k} = 0, g_{ij|k} = 0, y_{i|k} = 0$ one easily finds $Ric_{ij|k} = 0$.

Example 3. A large class of examples are provided by affine symmetric Berwald spaces (see Theorem 5.2 in [Den]).

Chapter 5

Weinstein's theorem for Finsler manifolds

5.1 Introduction

The results in this chapter are from [KP06]. We prove the generalization of Weinstein's theorem for Finsler manifolds: an isometry of a compact oriented Finsler manifold of positive flag curvature has a fixed point supposed that it preserves the orientation of the manifold if its dimension is even, or reverses it if odd.

Alan Weinstein proved in 1968 [Wei68] that a conformal diffeomorphism of a compact oriented Riemannian manifold of positive sectional curvature has a fixed point supposed that it preserves the orientation of the manifold if its dimension is even, or reverses it if odd. Especially, it is true for isometries. It is not known whether the theorem is still true for any diffeomorphism. If yes, this would imply that $S^2 \times S^2$ does not carry a metric of positive curvature, since the map which is the antipodal map on each factor preserves the orientation, and does not have a fixed point.

Weinstein's theorem implies Synge's earlier theorem [Syn36] stating that a compact manifold M with positive sectional curvature is simply connected if M is orientable and its dimension is even. Synge's theorem has been generalized for Finsler manifold by Auslander [Aus55], see also in [BCS00,

p. 221]. The odd dimensional case remained open there.

An isometry of a Finsler manifold is a diffeomorphism $f : M \to M$ which preserves the distance, or equivalently, preserves the Finsler norm of the tangent vectors. This equivalence, the generalization of Myers-Steenrod theorem of Riemannian geometry was proved in [Den]. Few other results are known about isometries of Finsler manifolds. In [Sza81a] Szabó determines all the non-Riemannian Finsler spaces having a group of motions of the largest order, and further shows interesting and important results on the problems of Berwald spaces, scalar curvature and projective flatness.

Our aim is now to prove the following

Theorem 5.1. WEINSTEIN'S THEOREM FOR FINSLER MANIFOLDS: *Let f be an isometry of a compact oriented positively homogeneous Finsler manifold M of dimension n. If M has positive flag curvature and f preserves the orientation of M for n even and reverses the orientation of M for n odd, then f has a fixed point.*

As immediate consequence we obtain Synge's theorem for both even and odd dimensions. The proof of the first part is different from that given in [Aus55; BCS00], and the second assertion was not covered there. Further consequences of the generalized Weinstein theorem will be described in a forthcoming paper.

A *Finsler manifold* is a manifold equipped with a Banach norm $F(x,\cdot)$ at each tangent spaces T_xM, called a Finsler fundamental function if

1. $F(x,y) > 0 \ \forall x \in M, y \in TM, \ \ y \neq 0$
2. $F(x,\lambda y) = \lambda F(x,y) \ \ \forall \lambda \in \mathbb{R}^+, y \in TM$
3. F is smooth except on the zero section
4. $g_{ij}(x,y) = \dfrac{\partial^2(\frac{1}{2}F^2)}{\partial y^i \partial y^j}(x,y)$ is positive definite for any $(x,y) \neq 0$.

We remark that in condition (3) the exclusion of the zero section ensures that the homogeneity does not imply linearity. The last condition implies that the indicatrix body is strongly convex, and conversely. Notice that in condition (2) the homogeneity is supposed for positive λ only, therefore we deal with positively homogeneous, *non-reversible Finsler metrics.*

The arc length of a curve $\gamma : [a,b] \to M$ in a Finsler manifold (M,F) is defined as
$$s = \int_a^b F(\gamma(t), \dot{\gamma}(t))\, dt.$$

From the recent flourishing literature of Finsler geometry, we refer the reader the books . In the proof we shall utilize the Chern connection. See for details [BCS00] or [She98]. Specially, we need the Riemann curvature $R_y(u)$ and the flag curvature $K(P,y)$ for any $y \in T_xM, u \in T_xM, x \in M$ with $P = \mathrm{span}\{y,u\}$. Finsler manifolds with positive flag curvature have been extensively studied recently. Bryant in [Bry02] and Shen in [She01] constructed fine examples for such spaces.

5.2 The proof

We follow the line of Weinstein, carefully adapted for Finsler setting.

Suppose that the isometry f has no fixed points:$f(x) \neq x$ for all $x \in M$. Since the manifold M is compact, the function $h : M \to \mathbb{R}$ given by $h(x) = d(x, f(x))$ attains its minimum at a point $x \in M$, so $h(x) > 0$ for all $x \in M$.

The completeness of the manifold M implies that there exists a minimizing normalized forward geodesic $\sigma : [0,\ell] \to M$ joining x and $f(x)$. We show that the curve formed by σ and $f \circ \sigma$ gives a forward geodesic. Consider the forward geodesic $f \circ \sigma$ which joins $f(x)$ to $f^2(x)$, and a point $y = \sigma(t), \;\; t \in (0,\ell)$ on σ between x and $f(x)$. Since f is an isometry, $d(x,y) = d(f(x), f(y))$. By the triangle inequality it follows that:

$$\begin{aligned} d(y, f(y)) &\leq d(y, f(x)) + d(f(x), f(y)) \\ &= d(y, f(x)) + d(x,y) \\ &= d(x, f(x)). \end{aligned}$$

Since x is a minimum for the function h, we have

$$d(y, f(y)) = d(y, f(x)) + d(f(x), f(y)),$$

so, the curve formed by σ and $f \circ \sigma$ is a forward geodesic and this implies that it is smooth, that is

$$(f \circ \sigma)^{\cdot}(0) = \dot{\sigma}(\ell).$$

Clearly, if a map f is an isometry of (M, F): $F(x, u) = F(f(x), df_x(u))$ for $x \in M$ and $u \in T_xM$, i.e. f is an isometry between the Minkowski spaces $(T_xM, F(x, \cdot))$ and $(T_f(x)M, F(f(x), df_x(\cdot))$ (cf. [BCS00]), then, by the chain rule, we obtain that $g_{ij}(x, y)(v, w) = g_{ij}(f(x), df_x(y))(df_x(v), df_x(w))$, i.e. the isometry of a Finsler space gives rise to an isometry of the fibers over (x, y) and $(f(x), df_x(y))$ in π^*TM.

Along a forward geodesic the Chern connection is metric compatible (see [BCS00], p. 122). This implies that for any forward geodesic $\sigma(t), t \in [0, \ell]$, the linearly parallel transport P_σ (see [CS05, p.73]) induced by the Chern connection along the forward geodesic σ, preserves the inner products $g_{\dot{\sigma}}$ along σ, that is

$$g_{\dot{\sigma}(t)}(P_{c(t)}(u), P_{c(t)}(v)) = g_{\dot{\sigma}(0)}(u, v), \text{ for } \ u, v \in T_{\sigma(0)}M.$$

See also [She01], p. 89. This formula means that we have an isometry between the tangent spaces $T_{\sigma(t)}M$ along σ with inner products $g_{\dot{\sigma}}$, induced by the linearly parallel transport.

Denote shortly by P the linearly parallel transport along the forward geodesic σ between the tangent spaces T_xM and $T_{f(x)}M$. We can consider its inverse $G = P^{-1} : T_{f(x)}M \to T_xM$, which is an isometry, again. We consider now the map $B = G \circ df_x : T_xM \to T_xM$. By the above observations B is an isometry.

We have the following relations

$$B(\dot\sigma(0)) = G \circ df_x(\dot\sigma(0)) = G((f\circ\sigma)^{\cdot}(0)) = G(\dot\sigma(\ell)) = \dot\sigma(0).$$

This means that B leaves $\dot\sigma(0)$ fixed. Let A be the restriction of B to the $g_{\dot\sigma(0)}$–orthogonal complement of $\dot\sigma(0)$. A is an isometry and since P is an isometry which preserves the orientation it follows that

$$\det A = \det B = \det(G \circ df_x) = (-1)^n,$$

because of the hypothesis on f and the fact that G preserves the orientation. By the Lemma from [Car92] p. 203, A leaves a vector invariant. Let $E(t)$ be a unit linearly parallel vector field along σ such that, $E(0)$ belongs to the $g_{\dot\sigma(0)}$–orthogonal complement of $\dot\sigma(0)$ and $E(0)$ is invariant by A: $A(E(0)) = E(0)$.

Next, take the forward geodesic $\alpha(s), s \in (-\epsilon, \epsilon)$, such that $\alpha(0) = x$, and $\dot\alpha(0) = E(0)$. We have $df_x(E(0)) = E(\ell)$ because $G \circ df_x(E(0)) = E(0)$, i.e., the forward geodesic $f \circ \alpha$ has the property that $(f\circ\alpha)(0) = f(x)$ and $(f \circ \alpha)^{\cdot}(0) = E(\ell)$.

Consider now the variation of σ given by

$$h : (-\epsilon, \epsilon) \times [0, \ell] \to M$$
$$h(s,t) = \exp_{\sigma(t)}(sE(t)),\ s \in (-\epsilon, \epsilon),\ t \in [0, \ell].$$

Clearly $h(s, 0) = \alpha(s)$, moreover, we have

$$h(s, \ell) = \exp_{f(x)}(sE(\ell)) = (f \circ \alpha)(s),$$

for $(f \circ \alpha)^{\cdot}(0) = E(\ell)$. It follows then

$$\frac{\partial}{\partial s}\exp_{\sigma(t)}(sE(t))|_{s=0} = E(t),$$

so the transversal vector of the variation h is linearly parallel transported along σ.

The second variation formula of the arc-length has the following form

([She01, p.161])

$$\begin{aligned} L''(0) &= \int_0^\ell \{g_{\dot\sigma}(\nabla_{\dot\sigma}E, \nabla_{\dot\sigma}E) - g_{\dot\sigma}(R_{\dot\sigma}(E), E)\}dt \\ &\quad + g_{\dot\sigma(\ell)}(\kappa_\ell(0), \dot\sigma(\ell)) - g_{\dot\sigma(0)}(\kappa_0(0), \dot\sigma(0)) \\ &\quad + \mathbf{T}_{\dot\sigma(0)}(E(0)) - \mathbf{T}_{\dot\sigma(\ell)}(E(\ell)) \end{aligned}$$

Here the quantities $\kappa_\ell(0)$ and $\kappa_0(0)$ are the geodesic curvatures of the transversal curves $\alpha(s)$ for $s = 0$. Being the transversal curves forward geodesics, the geodesic curvatures are zero. Furthermore, $\mathbf{T}$ represents the T-curvature (see [She01, p. 153]), which depends on the Finsler metric, only. The points $\sigma(0)$ and $\sigma(\ell)$ are coupled by the isometry f, moreover, $df_x(E(0)) = E(\ell)$ holds, therefore $\mathbf{T}_{\dot\sigma(0)}(E(0)) = \mathbf{T}_{\dot\sigma(\ell)}(E(\ell))$. Finally in the first term $\nabla_{\dot\sigma}E$ is zero along the forward geodesic σ, since E is linearly parallel transported along σ. By the above observations the second variation formula reduces to

$$L''(0) = -\int_0^\ell g_{\dot\sigma}(R_{\dot\sigma}(E), E)dt = -\int_0^\ell K(P, \dot\sigma)g_{\dot\sigma}(E, E)dt = -\int_0^\ell K(P, \dot\sigma)dt$$

so the second variation is negative because the flag curvature is positive. But this contradicts the minimality of σ, the curve which joins x and $f(x)$. Therefore $d(x, f(x)) > 0$ is impossible. □

5.3 Synge's theorem for Finsler manifolds

In this section we prove the Synge theorem in the Finslerian context, using our main result.

Theorem 5.2. *Let (M, F) be a compact Finsler manifold of positive flag curvature of dimension n.*

1. *If M is orientable and n is even, then M is simply connected.*

2. *If n is odd, then M is orientable.*

Proof. (1) Consider the universal covering $\pi : \widetilde{M} \to M$ and the covering metric on $\widetilde{M}$. We can choose an orientation on $\widetilde{M}$ such that the covering map π preserves the orientation. Because M is compact, the flag curvature is strictly positive, that is, there exists $\delta > 0$ such that the flag curvature is greater or equal to δ. The same bound on the curvature holds on $\widetilde{M}$ because π is a local isometry (see [BCS00], p. 197). Consider now a covering transformation $\tau : \widetilde{M} \to \widetilde{M}$: $\pi \circ \tau = \pi$. τ is an isometry of $\widetilde{M}$ which preserves the orientation. Because n is even, due to our main theorem τ has a fixed point, so τ is the identity (because a covering transformation which has a fixed point is the identity). This implies that the group of covering transformations reduces to the identity, and therefore M is simply connected.

(2) Suppose that M is not orientable and consider the orientable double cover $\widetilde{M}$ of M (see [Car92], p. 34), with the covering metric and denote the covering map by $p : \widetilde{M} \to M$. $\widetilde{M}$ is compact because it is the double cover of a compact manifold. Consider a deck transformation τ of $\widetilde{M}$, $\tau \neq \mathrm{id}$ with the covering metric, that is $p \circ \tau = p$. From the unique lifting property of the covering space and the fact that M is connected, so $\widetilde{M}$ is, it follows that the deck transformation is completely determined by the image of a point, particularly only the identity transformation of $\widetilde{M}$ has fixed points.

Denote by $\widetilde{F} = p^*F$ the covering metric, that is $\widetilde{F}(\widetilde{x}, \widetilde{v}) = F(x, v)$ where $p(\widetilde{x}) = x$, $dp(\widetilde{v}) = v$. Because $p \circ \tau = p$ and the Finsler metric on $\widetilde{M}$ is the pull back of the metric of M, τ is an isometry of $\widetilde{M}$ which reverses the orientation of $\widetilde{M}$ because M is not orientable. Consequently, τ has a fixed point, for n is odd, which gives a contradiction. □

Remark. It is clear that the assumptions (1) of this theorem are really necessary. Namely, the case of real projective space of dimension 2 shows the necessity of orientability, and the real projective space of dimension 3 gives a counterexample for odd dimension. See also [BCS00], page 224.

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Part II

Hardy type inequalities in Minkowski spaces.

Chapter 6

Hardy inequality in Minkowski spaces

6.1 Introduction

There is a very rich literature of different types of Hardy inequalities, see the very recent books of A.A. Balinsky, W.D. Evans, R.T. Lewis [BEL] and N. Ghoussoub, A. Moradifam [GM13] and their references. These books contain very recent results concerning generalizations of the Hardy inequalities with singularities in the interior of the domain and on the boundary and for general elliptic operators [GM13]. In the book [BEL12] are presented the Hardy inequalities using the distance function from points in the domain to the boundary. These inequalities are proved in Euclidean cases. In many such results the mean convexity of the domain plays an important role. The well known Hardy inequality, in the case of domains with boundaries, can be fomulated by using the distance function from points in the domain to its boundary. In [LLL12] several Hardy type inequalities are proved in a domain $\Omega \subset \mathbb{R}^n$ by using the Euclidean metric.

In the case $n \geq 2$ most of the Hardy-type results assume that the domain is convex. There are also some results that show that the Hardy inequality should hold for non-convex domains too. In [FMT07c] is proved that a Hardy-Sobolev inequality holds in a small enough tubular neighborhood of the boundary of a bounded domain. Moreover, in [FMT07a] the authors proved a Hardy-Sobolev inequaltity on a bounded C^2 domain $\Omega \subset \mathbb{R}^{n+1}, n \geq 2$ provided that

$$-\boldsymbol{\Delta}\delta(x) \geq 0, \tag{6.1}$$

i.e. the superharmonicity condition holds (Theorem 1.1 and condition (C) in [FMT07a]).

Moreover, an improved Hardy inequality is demonstrated by Filippas, Moschini and Tertikas in [FMT07c] for a class of domains satisfying

$$-\mathrm{div}(|x|^{n-1}\boldsymbol{\nabla}\delta(x)) \geq 0 \text{ a.e. in } \Omega. \tag{6.2}$$

Both conditions (6.1) and (6.2) are global conditions, they depend on the properties of the whole domain, so generally they are hard to be verified. The only examples stated in [FMT07c] satisfying condition (6.2) are balls B_R. Well–known examples which satisfy (6.1) are convex domains. A non–convex case is the ring torus, as it is studied in Armitage and Kuran [AK85]. In the same example the superharmonicity has been shown to hold a.e. (i.e. out of a set of measure zero) by Balinsky, Evans and Lewis [BEL12]. Other non–convex domains on which Hardy-type inequalities hold are small tubular neighborhoods of a surface [FMT04] and convex domains with punctured balls [AL].

Next we describe shortly the structure of this chapter see [PV17]. In Section 6.2 we briefly introduce some basic facts and notions on Minkowski spaces. The next section, Section 6.3, introduces the very basic geometry of hypersurfaces in Minkowski spaces. In Section 6.4 we study the distance function to the boundary in Mikowski spaces. The tools which we develop here are widely used in the next sections. Among other results we derive the form of the Hessian of the distance function in principal coordinates. Section 6.5 contains one of the important results of this chapter, the superharmonicity of the distance function in Minkowski spaces. In Section 6.6 we prove the main result of the chapter, the improved Hardy-Brezis-Marcus Theorem in Minkowski spaces, and its relations with mean curvature. Also we give an explicit way to obtain classes of inequalities of the same type. The results obtained in this section are twofold extensions of some results obtained earlier. One side is that we obtain Hardy inequalities in Minkowski spaces, which generalize main results from [LLL12]. On the other side we explicitly extend the results from [DBG16], where the strong convexity of the distance function is assumed. Our results make the bridge between the geometry of the boundary of the domain (its mean curvature) and the Hardy inequality. In the last section, in Section 6.7, we apply the Hardy-Brezis-Marcus Theorem to a minimization problem.

6.2 Preliminary notions and results from Minkowski geometry

Denote by $\langle \cdot, \cdot \rangle$ the usual innner product on $\mathbb{R}^n$.

Definition 6.1. *A function $F : \mathbb{R}^n \to [0, +\infty)$ is called Minkowski norm if it satisfies the following properties:*

(M1) $F \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\})$*;*

(M2) F *is positively homogeneus of degree one, i.e.*

$$F(tv) = tF(v) \text{ for any } t \geq 0 \text{ and } v \in \mathbb{R}^n;$$

(M3) For any $x \in \mathbb{R}^n \setminus \{0\}$ *the symmetric bilinear form* $g_x : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *defined by*

$$g_x(u, v) = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} [F^2(x + su + tv)] \Big|_{s=t=0} = \langle \mathrm{Hess}^2 \left(\frac{F^2}{2} \right) (x) x, x \rangle$$

is positive definite, where $\mathrm{Hess}^2 \left(\frac{F^2}{2} \right)(x)$ *denotes the Hessian of* $\frac{F^2}{2}$ *in* x*.*

The Minkowski norm F is absolutely homogeneous if, in addition,

(M'2) F is positively homogeneus of degree one, i.e.

$$F(tv) = |t|F(v) \text{ for any } t \in \mathbb{R} \text{ and } v \in \mathbb{R}^n.$$

The pair $(\mathbb{R}^n, F)$ is a Minkowski space (see [BC00]), which is the simplest not necessarily reversible Finsler manifold whose flag curvature is identically zero, the geodesics are straight lines, and the intrinsic forward distance between two points $x, y \in \mathbb{R}^n$ is given by

$$d_F(x, y) = F(y - x). \tag{6.3}$$

One clearly has that $d_F(x, y) = 0$if and only if $x = y$, and d_F verifies the triangle inequality. The open *forward* and *backward metric balls* with center in $x_0 \in M$ and radius $\rho > 0$ are defined by $B^+(x_0, \rho) = \{x \in \mathbb{R}^n : d_F(x_0, x) < \rho\}$, respectively $B^-(x_0, \rho) = \{x \in \mathbb{R}^n : d_F(x, x_0) < \rho\}$. In fact $(\mathbb{R}^n, d_F)$ is a quasi-metric space and in general $d_F(x, y) \neq d_F(y, x)$.

Let

$$r_F = \sup_{x\in\mathbb{R}^n} \{F(-x) : F(x) = 1\} = \sup_{x\in\mathbb{R}^n} \frac{F(x)}{F(-x)} \tag{6.4}$$

be the reversibility constant of F, see Rademacher [Rad09]. Another quantity used is

$$l_F = \inf_{y\in\mathbb{R}^n} l_F(y) \quad \text{where} \quad l_F(y) = \inf_{y,v,w\in\mathbb{R}^n\setminus\{0\}} \frac{g_v(y, y)}{g_w(y, y)},$$

which is the *uniformity constant of* F measuring how far F and F^* are from being inner product structures, see Egloff [Egl97]. All along the chapter $\mathbb{R}^{n\star}$ will denote the dual space of $\mathbb{R}^n$.

Definition 6.2. *The dual norm $F^\star$ on $\mathbb{R}^{n\star}$ is defined by*

$$F^\star(\xi) = \sup_{y\in\mathbb{R}^n\setminus\{0\}} \frac{\xi(y)}{F(y)}. \tag{6.5}$$

Remark 6.3. *It can be proved that the dual norm $F^\star$ has the following* *properties:*

(DM1) $F^\star(t\xi) = tF^\star(\xi)$ *for every* $t \geq 0$ *and* $\xi \in R^{n\star}$*;*

(DM2) $F^\star$ *is* C^∞ *on* $\mathbb{R}^{n\star} \setminus \{0\}$*;*

(DM3) For every non-zero $\xi \in \mathbb{R}^{n\star}$ *the induced quadratic form* $g^{\star\eta}$ *in* $\mathbb{R}^{n\star}$ *is an inner product, where*

$$g^{\star\eta}(\xi,\zeta) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}(F^{\star 2})(\eta + s\xi + t\zeta))_{s=t=0} \,.$$

The inner products g_y *and* $g^{\star\xi}$ *defined by* F *and* $F^\star$*, respectively, are related in the following way: If*

$$\xi(y) = F^\star(\xi)F(y),$$

then

$$\xi(x) = Cg_y(y,x) \;\; \textit{for all } x \in \mathbb{R}^n, \tag{6.6}$$

$$\eta(y) = \frac{1}{C}g^{\star\xi}(\xi,\eta) \;\; \textit{for all } \eta \in \mathbb{R}^{n\star}, \tag{6.7}$$

where $C = \frac{F^\star(\xi)}{F(y)}$.

It can be shown that $l_F \leq 1$, and $l_F = 1$if and only if F is an inner product, see Ohta [Oht09] (in fact it is proved that for a Finsler manifold the above equality implies that the Finsler metric is actually a Riemannian one). In the same manner, we can define the constant l_{F^*} for F^*, and it follows that $l_{F^*} = l_F$. The definition of l_F in turn shows that

$$F^{*2}(t\alpha + (1-t)\beta) \leq tF^{*2}(\alpha) + (1-t)F^{*2}(\beta) - l_F t(1-t)F^{*2}(\beta - \alpha) \tag{6.8}$$

for all $x \in \mathbb{R}^n$, $\alpha, \beta \in \mathbb{R}^{n\star}$ and $t \in [0,1]$.

A specific non-reversible Finsler structure is provided by the *Randers metric*, which will serve to us as a model case. To be more precise, we introduce on $\mathbb{R}^n$ the Minkowski structure $F : \mathbb{R}^n \to [0,\infty)$ defined by

$$F(y) = \sqrt{h(y,y)} + \beta(y), \;\; y \in \mathbb{R}^n, \tag{6.9}$$

where h is an inner product on $\mathbb{R}^n$, β is an 1-form on $\mathbb{R}^n$, and we assume that

$$\|\beta\|_h = \sqrt{h^*(\beta,\beta)} < 1.$$

Here, the co-metric h^* can be identified by h^{-1}, the inverse of the symmetric, positive definite matrix h. Clearly, the Randers metric in (6.9) is symmetricif and only if $\beta = 0$. For the Randers metric (6.9) a direct computation gives that

$$r_F = \frac{1+\|\beta\|_h}{1-\|\beta\|_h} \quad \text{and} \quad l_F = \left(\frac{1-\|\beta\|_h}{1+\|\beta\|_h}\right)^2, \tag{6.10}$$

see also Yuan and Zhao [YZ13].

Remark 6.4. *Let $F : \mathbb{R}^n \to [0,+\infty)$ be a positively homogeneous Minkowski norm. Then we have $r_F \geq 1, l_F \in (0,1]$ and $0 < l_F r_F^2 \leq 1$.*

Remark 6.5. *Minkowski metrics can be constructed on products $\mathbb{R}^n \times \mathbb{R}^m$ as pointed in [CS05]. Let*

$$F(x,y) = \begin{cases} F_1(y_1) & \text{for } y = y_1 \oplus 0 \in \mathbb{R}^n \\ F_2(y_2) & \text{for } y = 0 \oplus y_2 \in \mathbb{R}^m, \end{cases}$$

where $(\mathbb{R}^n, F_1)$ and $(\mathbb{R}^m, F_2)$ are inner product spaces defined in the following manner: Take a C^∞ positive and positively homogeneous function $f : [0,\infty) \times [0,\infty) \to [0,\infty)$, i.e. $f(\lambda s, \lambda t) = \lambda f(s,t)$ for all $\lambda > 0$ and $f(s,t) \neq 0$ for all $(s,t) \neq (0,0)$. Define $F : \mathbb{R}^n \times \mathbb{R}^m \to [0,\infty)$ by $F(y_1 \oplus y_2) = \sqrt{f(F_1^2(y_1), F_2^2(y_2))}$. It is proved in [CS05, p. 14] that by using some simple conditions involving the partial derivatives of f one obtains by this construction a Minkowski metric.

The Minkowski space $(\mathbb{R}^n, F)$ is at the same time a Finsler manifold. This can be done by assigning the same F to every tangent space of $\mathbb{R}^n$, obtaining in this way a locally Minkowskian space. All the geodesics are straight lines, so these spaces are geodesically both forward and backward complete (surprinsingly even if F is only positively homogeneous, see [BC00]). We briefly recall the procedure here: A Finsler metric on $\mathbb{R}^n$ is called *locally Minkowskian*, if at every point $x \in \mathbb{R}^n$ the metric in the tangent space $T_x\mathbb{R}^n$

depends only on $y \in \mathbb{R}^n$ (i.e it does not depend on $x \in \mathbb{R}^n$). The method used is as follows. First start with a Minkowski norm on $\mathbb{R}^n$. Regard $\mathbb{R}^n$ as a manifold (in fact a linear one). For a tangent vector v based at $x \in \mathbb{R}^n$ we translate it parallel along x until it is moved in $0 \in \mathbb{R}^n$. More precisely,

$$F(x,v) = F\left(x, v^i \frac{\partial}{\partial x^i}\right) := F\left(v^i \frac{\partial}{\partial x^i}\right). \tag{6.11}$$

We consider the *polar transform* (or, co-metric) of F, defined for every $(x,\alpha) \in \mathbb{R}^{n\star}$ by

$$F^*(\alpha) = \sup_{y \in \mathbb{R}^n \setminus \{0\}} \frac{\alpha(y)}{F(y)}. \tag{6.12}$$

Note, that for every $x \in \mathbb{R}^n$ the function $F^*(\cdot)$ is a Minkowski norm on $\mathbb{R}^{n\star}$. Furthermore, because $F^{*2}(\cdot)$ is twice differentiable on $\mathbb{R}^{n\star} \setminus \mathbb{R}^{n\star}\{0\}$, as for F we consider $g^\star_{ij}(\alpha) := [\frac{1}{2}F^{\star 2}(\alpha)]_{\alpha^i\alpha^j}$ for every $\alpha = \sum_{i=1}^{n} \alpha^i \mathrm{d}x^i \in \mathbb{R}^{n\star} \setminus \{0\}$. In particular, if F is a Randers metric of the form (6.9), then

$$F^*(\alpha) = \frac{\sqrt{h^{\star 2}(\alpha,\beta) + (1 - \|\beta\|_h^2)\|\alpha\|_h^2} - h^\star(\alpha,\beta)}{1 - \|\beta\|_h^2}, \ \alpha \in \mathbb{R}^{n\star}, \tag{6.13}$$

where h^* denotes the co-inner product acting on $\mathbb{R}^{n\star}$ associated to the inner product h. Moreover, the symmetrized Minkowski metric and its polar transform associated with the Randers metric (6.9) is

$$F_s(y) = \sqrt{h(y,y) + \beta^2(y)}, \ F_s^*(\alpha) = \sqrt{\|\alpha\|_h^2 - \frac{h^{*2}(\alpha,\beta)}{1 + \|\beta\|_h^2}}. \tag{6.14}$$

The *Legendre transform* $J^* : \mathbb{R}^{n\star} \to \mathbb{R}^n$ associates to each element $\alpha \in \mathbb{R}^{n\star}$ the unique maximizer on $\mathbb{R}^n$ of the map $y \mapsto \alpha(y) - \frac{1}{2}F^2(y)$. This element can also be interpreted as the unique vector $y \in \mathbb{R}^n$ with the properties

$$F(y) = F^*(\alpha) \text{ and } \alpha(y) = F(y)F^*(\alpha). \tag{6.15}$$

In particular, if $\alpha = \sum_{i=1}^n \alpha^i \mathrm{d}x^i \in \mathbb{R}^{n\star}$, one has

$$J^*(\alpha) = \sum_{i=1}^n \frac{\partial}{\partial \alpha_i}\left(\frac{1}{2}F^{*2}(\alpha)\right)\frac{\partial}{\partial x^i}. \tag{6.16}$$

Next we introduce the Finsler-Laplace operator and the volume form. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function in the distributional sense. The *gradient* of u is defined by

$$\boldsymbol{\nabla} u(x) = J^*(Du(x)), \tag{6.17}$$

where $Du(x) \in \mathbb{R}^{n\star}$ denotes the (distributional) *derivative* of u at $x \in \mathbb{R}^n$. In local coordinates one has

$$Du(x) = \sum_{i=1}^n \frac{\partial u}{\partial x^i}(x)\mathrm{d}x^i, \tag{6.18}$$

$$\boldsymbol{\nabla} u(x) = \sum_{i,j=1}^n g^*_{ij}(Du(x))\frac{\partial u}{\partial x^i}(x)\frac{\partial}{\partial x^j}.$$

In general, $u \mapsto \boldsymbol{\nabla} u$ is not linear. If $x_0 \in \mathbb{R}^n$ is fixed, then due to Ohta and Sturm [OS09], one has

$$F^*(Dd_F(x_0,x)) = F(\boldsymbol{\nabla} d_F(x_0,x)) = Dd_F(x_0,x)(\boldsymbol{\nabla} d_F(x_0,x)) = 1 \text{ for a.e. } x \in M. \tag{6.19}$$

The Busemann or Hausdorff volume is the multiple of the Lebesgue measure for which the volume of the unit ball equals to the volume of the Euclidean unit ball of dimension n. Let $\{e^i\}_{i=1,\dots,n}$ be a basis for $\mathbb{R}^n$ and $\{\mathrm{d}x^i\}_{i=1,\dots,n}$ be its dual basis for $\mathbb{R}^{\star n}$ Let $B(1) = \{y = (y^i) : F(y^i e^i) < 1\} \subset \mathbb{R}^n$. The *Hausdorff volume form* $\mathsf{dm} = \mathrm{d}V_F$ on $\mathbb{R}^n$ is defined by

$$\mathsf{dm}(x) = \mathrm{d}V_F(x) = \sigma_F(x)\mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^n, \tag{6.20}$$

where $\sigma_F = \frac{\omega_n}{\mathrm{Vol}(B(1))}$. $\mathrm{Vol}(S)$ and ω_n are the Euclidean volume of the set $S \subset \mathbb{R}^n$ and the $n-$dimensional unit ball, respectively. The *Minkowski-volume* of an open set $S \subset \mathbb{R}^n$ is $\mathrm{Vol}_F(S) = \int_S \mathsf{dm}(x)$.

Buseman proved that this volume coincides with the Hausdorff measure on the metric space $(\mathbb{R}^n, d_F)$. For a Minkowski space $(\mathbb{R}^n, F)$ by (6.20), $\mathrm{Vol}_F(B^+(x,\rho)) = \omega_n \rho^n$ for every $\rho > 0$ and $x \in \mathbb{R}^n$, and $\sigma_F(x)$ =constant. If F is the Randers metric of the form (6.9) on a manifold $\mathbb{R}^n$, then

$$\mathrm{d}V_F = \left(1 - \|\beta\|_h^2\right)^{\frac{n+1}{2}} \mathrm{d}V_h, \tag{6.21}$$

where $\mathrm{d}V_h$ denotes the canonical volume form of the inner product h on $\mathbb{R}^n$. Let

$$W^{1,2}(\mathbb{R}^n, F, \mathsf{m}) = \left\{u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} F^{*2}(x, Du(x))\mathrm{d}\mathsf{m}(x) < +\infty\right\}$$

and let $W_0^{1,2}(M, F, \mathsf{m})$ be the closure of $C_0^\infty(M)$ with respect to the (asymmetric) norm

$$\|u\|_F = \left(\int_M F^{*2}(x, Du(x))\mathrm{d}\mathsf{m}(x) + \int_M u^2(x)\mathrm{d}\mathsf{m}(x)\right)^{1/2}. \tag{6.22}$$

The symmetrized distance associated to a metric F is

$$F_s(x,y) = \left(\frac{F^2(x,y) + F^2(x,-y)}{2}\right)^{1/2}, \ (x,y) \in TM.$$

The metric F_s is reversible, and F is reversibleif and only if$F = F_s$. Another point is that the symmetrized Finsler metric associated with F^* may be different from F_s^*, i.e., in general $2F_s^{*2}(x,\alpha) \neq F^{*2}(x,\alpha) + F^{*2}(x,-\alpha)$; such a concrete case is shown for Randers metrics (see [FKV16]).

In this part we will use the following theorem (see [FKV16]):

Theorem 6.6. *Let (M, F) be a complete, $n-$dimensional Finsler manifold such that $r_F < +\infty$. Then $(W_0^{1,2}(M, F, \mathsf{m}), \|\cdot\|_{F_s})$ is a reflexive Banach space, while the norm $\|\cdot\|_{F_s}$ and the asymmetric norm $\|\cdot\|_F$ are equivalent.*

In particular,

$$\left(\frac{1+r_F^2}{2}\right)^{-1/2}\|u\|_F \leq \|u\|_{F_s} \leq \left(\frac{1+r_F^{-2}}{2}\right)^{-1/2}\|u\|_F \quad \textit{for all } u \in W_0^{1,2}(\mathbb{R}^n, F, \mathsf{m}). \tag{6.23}$$

In the above iequality the norm $\|\cdot\|_{F_s}$ is considered also with respect to the Hausdorff measure $\mathrm{d}\mathsf{m} = \mathrm{d}V_F$ (and not with $\mathrm{d}V_{F_s}$), i.e.,

$$\|u\|_{F_s} = \left(\int_{\mathbb{R}^n} F_s^{*2}(Du(x))\mathrm{d}\mathsf{m}(x) + \int_{\mathbb{R}^n} u^2(x)\mathrm{d}\mathsf{m}(x)\right)^{1/2}. \tag{6.24}$$

Let

$$L^2(\mathbb{R}^n, \mathsf{m}) = \left\{u : \mathbb{R}^n \to \mathbb{R} : u \text{ is measurable, } \|u\|_{L^2(\mathbb{R}^n,\mathsf{m})} < \infty\right\},$$

where

$$\|u\|_{L^2} = \|u\|_{L^2(\mathbb{R}^n,\mathsf{m})} = \left(\int_{\mathbb{R}^n} u^2(x)\mathrm{d}\mathsf{m}(x)\right)^{1/2}.$$

It is standard that $(L^2(\mathbb{R}^n, \mathsf{m}), \|\cdot\|_{L^2(\mathbb{R}^n,\mathsf{m})})$ is a Hilbert space. Since F^{*2} is a (strictly) convex function, so is F_s^{*2} too, and one can prove that $(W_0^{1,2}(\mathbb{R}^n, F, \mathsf{m}), \|\cdot\|_{F_s})$ is a closed subspace of the Hilbert space $L^2(\mathbb{R}^n, \mathsf{m})$.

Let X be a vector field on $\mathbb{R}^n$. In a local coordinate system (x^i), on account of (6.20), the *divergence* is defined by $\mathrm{div}(X) = \frac{1}{\sigma_F}\frac{\partial}{\partial x^i}(\sigma_F X^i)$. The *Minkowski-Laplace operator*

$$\boldsymbol{\Delta} u = \mathrm{div}(\boldsymbol{\nabla} u)$$

acts on $W_{\mathrm{loc}}^{1,2}(\mathbb{R}^n)$ and for every $v \in C_0^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} v\boldsymbol{\Delta} u\mathrm{d}\mathsf{m}(x) = -\int_{\mathbb{R}^n} Dv(\boldsymbol{\nabla} u)\mathrm{d}\mathsf{m}(x), \tag{6.25}$$

see Ohta and Sturm [OS09] and Shen [She]. Note that in general $\boldsymbol{\Delta}(-u) \neq -\boldsymbol{\Delta} u$, unless F is reversible.

By using (6.15) one has that in $(\mathbb{R}^n, F)$

$$\boldsymbol{\Delta} u = \Delta_F u = \operatorname{div}(F^*(Du)\nabla F^*(Du)) = \operatorname{div}(F(\nabla u)\nabla F(\nabla u))$$

is precisely the Finsler-Laplace operator considered by Cianchi and Salani [CS09], Ferone and Kawohl [FK09], Wang and Xia [WX11; WX13], see also their references.

6.3 Hypersurfaces in Minkowski spaces

A Minkwoski space $(\mathbb{R}^n, F)$ is the most common example of a Finsler space. It inherits the Chern Finsler connection as follows.

For a vector field $X = X^i \frac{\partial}{\partial x^i}$ the *covariant derivative* of X along $v = v^i \frac{\partial}{\partial x^i}$ with respect to $w \in TM \setminus \{0\}$ is

$$D_v^w X(x) = \left\{ v^j \frac{\partial X^i}{\partial x^j}(x) + \Gamma^i_{jk}(x, w) v^j X^k(x) \right\} \frac{\partial}{\partial x^i},$$

where $\Gamma^i_{jk}(x, w)$ are the Christofell coefficients of the Chern connection. We present some facts about hypersurfaces in Minkowski spaces (see also [HS16]).

Let N be a hypersurface in a Minkowski space $\mathbb{R}^n$ of class C^2. For any $x \in N$ one has that $T_x N = \ker \nu$ for some one form $\nu \in T_x^* M$. The image of this form by the Legendre map is a vector in $T_x \mathbb{R}^n \setminus T_x N$ and by considering it of norm one we obtain exactly two *normal vectors* $\boldsymbol{n}_\pm$ such that

$$T_x N = \{X \in T_x M | g_{\boldsymbol{n}_\pm}(\boldsymbol{n}_\pm, X) = 0, \quad g_{\boldsymbol{n}_\pm}(\boldsymbol{n}_\pm, \boldsymbol{n}_\pm) = 1\}.$$

As we already mentioned $\boldsymbol{n}_\pm$ are exactly $J^*(\pm\nu)$, where ν is a unit one form. For reversible case one has $\boldsymbol{n}_- = -\boldsymbol{n}_+$. As pointed in [Xia13, p. 103] this is the anisotropic normal (that is the normal with respect to $g_{\boldsymbol{n}}$).

The volume form $\mathrm{d}\mathsf{m}(x) = \mathrm{d}V_F(x) = \sigma_F(x)\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$ induces a volume form dA on N by

$$\begin{aligned}\mathrm{d}A &= \sigma_{\boldsymbol{n}}(u)\mathrm{d}u^1 \wedge ... \wedge \mathrm{d}u^{n-1} \\ &= \sigma_f(u) i_{\boldsymbol{n}}(\mathrm{d}x^1 \wedge ... \wedge \mathrm{d}x^n), \quad u \in N.\end{aligned}$$

For any tangent vector $y \in T_xN$ at $x \in N$ there exists a unique geodesic $\overline{\gamma}$ in $(N, \overline{F})$ such that the curve $\gamma(t) = \phi \circ \overline{\gamma}$ with $\gamma(0) = x$ and $\dot{\gamma}(0) = y$. The normal curvature $\Lambda_{\boldsymbol{n}}(y)$ with respect to $\boldsymbol{n}$ is defined by

$$\Lambda_{\boldsymbol{n}}(y) = g_{\boldsymbol{n}}(\boldsymbol{n}, D_{\dot{\gamma}}^{\dot{\gamma}}\dot{\gamma}(0)).$$

One has

$$\Lambda_{\boldsymbol{n}}(y) = F^2(y) g_{\boldsymbol{n}}(\boldsymbol{n}, h(y)), \quad y \in T_xN.$$

Clearly, $(N, \overline{F})$ is a totally geodesicif and only if $\Lambda_{\boldsymbol{n}}(y) = 0$ for all $y \in T_XN$, where $\overline{F}$ is the restriction of F to T_xN by the identification procedure.

The *Weingarten formula* with respect to $g_{\boldsymbol{n}}$ is given by

$$D_X^{\boldsymbol{n}}\boldsymbol{n} = \hat{\nabla}_X^{\perp}\boldsymbol{n} - \hat{A}_{\boldsymbol{n}}(X),$$

where $X \in \Gamma(TN)$ and $\hat{\nabla}^{\perp}$ is the *induced normal* connection. It follows easily that

$$\hat{\nabla}^{\perp}\boldsymbol{n} = g_{\boldsymbol{n}}(D_X^{\boldsymbol{n}}\boldsymbol{n}, \boldsymbol{n})\boldsymbol{n} = \frac{1}{2}[X g_{\boldsymbol{n}}(\boldsymbol{n}, \boldsymbol{n})]\boldsymbol{n} = 0.$$

The $g_{\boldsymbol{n}}$ *second fundamental* and the *Weingarten* or the *shape* operator are defined by

$$\hat{h}(X, Y) = g_{\boldsymbol{n}}(\boldsymbol{n}, D_X^{\boldsymbol{n}}Y) = \hat{g}(\hat{A}_{\boldsymbol{n}}(X), Y), \quad X, Y \in \Gamma(TN), \tag{6.26}$$

where $\hat{g}$ is the pulled back metric (the induced metric by $g_{\boldsymbol{n}}$), $\hat{A}$ is linear and $\hat{h}$ is bilinear (for the last it can be easily checked in the local coordinates in N).

The symmetry of $\hat{h}$ implies that $\hat{A}$ is a self-adjoint operator with respect to $\hat{g}$. The eigenvalues $k_1, \ldots, k_{n-1}$ of $\hat{A}$ are called the *principal curvatures* of N with respect to $\boldsymbol{n}$. Furthermore the basis for which the operator $\hat{A}$ is diagonalizable consists of principal directions. The inverses of principal curvatures are the principal radii. We remark that in the operator is hidden the reference vector $\boldsymbol{n}$. In [Xia13, p. 104] these are called the anisotropic curvatures and their inverses the anisotropic principal radii.

A torsion free connection on N can be defined by

$$\hat{\nabla}_X Y = (D^{\boldsymbol{n}}_X Y)^\top = D^{\boldsymbol{n}}_X Y - \hat{h}(X,Y)\boldsymbol{n}, \quad X, Y \in \Gamma(TN). \tag{6.27}$$

It follws that $\hat{\nabla}$ is a torsion free linear connection on N and it is not the Levi Civita connection on the manifold $(N, \hat{g})$, since it satisfies

$$\begin{aligned} \hat{\nabla}_X \hat{g}(Y,Z) &= X\hat{g}(Y,Z) - \hat{g}(\hat{\Delta}_X Y, Z) - \hat{g}(Y, \hat{\Delta}_X Z) \\ &= -2C_{\boldsymbol{n}}(\hat{A}(X), Y, Z), \quad X, Y, Z \in \Delta(TN). \end{aligned}$$

The *Gauss-Weingarten* formulas still hold:

$$D^{\boldsymbol{n}}_X Y = \hat{\nabla}_X Y = (D^{\boldsymbol{n}}_X Y)^\top + \hat{h}(X,Y)\boldsymbol{n}\,, \tag{6.28}$$

$$D^{\boldsymbol{n}}_X \boldsymbol{n} = -\hat{A}_{\boldsymbol{n}}(X), \quad X, Y \in \Gamma(TN). \tag{6.29}$$

In an orthonormal frame $\{e_i\}_{i=\overline{1,n}}$ with respect to $g_{\boldsymbol{n}}$ such that $e_n = \boldsymbol{n}$ define

$$H := \sum_{a=1}^{n-1} \hat{h}(e_a, e_a). \tag{6.30}$$

H is independent of the choice of the local frame field $\{e_a\}_{a=\overline{1,n-1}}$, $\hat{h}$ and $\hat{H}$ are called the *second fundamental form* and the *mean curvature* of N in $(\mathbb{R}, F)$. From (6.26) one has

$$H = \sum_{a=1}^{n-1} k_a,$$

where k_a are the principal curvatures of N.

Lemma 6.7. *[She98] Let (M, F) be a Berwald manifold. Then we have*

$$\Lambda_{\boldsymbol{n}}(X) = \hat{h}(X, X) = g_{\boldsymbol{n}}(\boldsymbol{n}, D_X^{\boldsymbol{n}} X) = \hat{g}(\hat{A}_{\boldsymbol{n}}(X), X), \quad X \in \Gamma(TN).$$

The next observation will be one of the main ingredients in working with the distance function in Minkowski spaces. In Minkowski spaces the geodesics are segments. Consider a hypersurface N of class C^2 in the Minkowski space $(\mathbb{R}^n, F)$ and a point $x \in \mathbb{R}^n \setminus N$, and $y \in N$. The distance between x and y is $d(x, y) = F(y - x)$. Suppose that there exists $z \in N$ such that

$$d_F(x, z) = d_F(x, N) = \inf_{y \in N} d_F(x, y) = \inf_{y \in N} F(y - x) = F(z - x). \tag{6.31}$$

Because in Minkowski spaces the geodesics are segments it follows that the vector $z - x$ is orthogonal to the tangent space $T_z S$ with respect to the inner product generated by $g_{(z-x)}$. This means in fact that the normal is collinear to $z - x$, and due to the fact that the coefficients of the metric inner product $g_{(z-x)}$ are zero-homogeneous, it is precisely the inner product $g_{\boldsymbol{n}}$.

Setting $M_f = \{x \in M | df(x) \neq 0\}$ one can define $\nabla^2 f(x) \in T_x^* M \otimes T_x M$ for $x \in M_f$ by using the covariant derivative

$$\nabla^2 f(v) := D_v^{\boldsymbol{\nabla} f} \boldsymbol{\nabla} f(x) \in T_x M, \; v \in T_x M.$$

For

$$D^2 f(X, Y) := g_{\boldsymbol{\nabla} f}(\nabla^2 f(x), Y) = g_{\boldsymbol{\nabla} f}(D_X^{\boldsymbol{\nabla} f}(\boldsymbol{\nabla} f), Y).$$

It follows that we have the symmetry

$$g_{\boldsymbol{\nabla} f}(D_X^{\boldsymbol{\nabla} f}(\boldsymbol{\nabla} f), Y) = D^2 f(X, Y) = D^2 f(Y, X) = g_{\boldsymbol{\nabla} f}(D_Y^{\boldsymbol{\nabla} f}(\boldsymbol{\nabla} f), X),$$

for all $X, Y \in T_x M$.

In the next we specialized for the case where N is a level set. Let $f : U \subset M \to \mathbb{R}$ be a C^2 function such that

$$\begin{cases} N \cap U = \{x \in U | f(x) = c\}; \\ \quad df(x) \neq 0, \quad x \in N \cup U, \end{cases}$$

where U is a neighborhood of $x_0 \in N$. Then we have $0 = Yf = g_{\nabla f}(\nabla f, Y)$ for any $y \in \Gamma(T(N \cap U))$.

Therefore, $\boldsymbol{n} = \frac{\nabla f}{F(\nabla f)}|_{N \cap U}$ is a normal vector of $N \cap U$ and from the previous section we have

$$\begin{aligned} \hat{h}(X,Y) &= g_{\nabla f}(\frac{\nabla f}{F(\nabla f)}, D_X^{\nabla f} Y) = g_{\nabla f}(\hat{A}_{\frac{\nabla f}{F(\nabla f)}}(X), Y) && (6.32) \\ &= -g_{\nabla f}(D_X^{\nabla f} \frac{\nabla f}{F(\nabla f)}, Y) = -\frac{1}{F(\nabla f)} g_{\nabla f}(D_X^{\nabla f} \nabla f, Y) && (6.33) \\ &= -\frac{1}{F(\nabla f)} D^2 f(X,Y), \quad X, Y \in \Gamma(TN). && (6.34) \end{aligned}$$

Lemma 6.8. *Let $\{e_i\}_{i=\overline{1,n}}$ be an orthonormal $g_{\nabla f}$ basis such that $e_n = \boldsymbol{n} = \frac{\nabla f}{F(\nabla f)}$. Then one has*

$$F(\nabla f) H = -\sum_{a=1}^{n-1} D^2 f(e_a, e_a) = -\sum_{a=1}^{n-1} f_{aa},$$

where $f_{aa} = D^2 f(e_a, e_a)$.

From [She] we have the following result, which shows that the mean curvature is the Laplacian of the level set equation in the case of Minkowski metrics:

Lemma 6.9. *Let $(\mathbb{R}^n, F)$ be an Minkowski space and $f : M \to \mathbb{R}$ be a smooth function. Then on M_f we have*

$$\boldsymbol{\Delta} f = tr_{g_{\nabla f}}(D^2 f) = \sum_i f_{ii}.$$

6.4 Distance function and Minkowski boundary geometry

Let F be a Minkowski norm on $\mathbb{R}^n$ and $\Omega \subseteq \mathbb{R}^n$ be a domain with C^2 boundary $\partial\Omega$. We say that Ω (or $\partial\Omega$) is (strictly) *mean convex* if the mean curvature $H(y) > 0$ for all $y \in \partial\Omega$; and *weakly mean convex* if $H(y) \geq 0$ for all $y \in \partial\Omega$. See also [CM07] for an approach to the distance to the boundary in Minkowski spaces.

Definition 6.10. *Let* $\delta(x) = \inf_{y \in \mathbb{R}^n \setminus \Omega} \operatorname{dist}(x, y)$ *denote the distance from* $x \in \Omega$ *to* $\partial\Omega$*. For* $x \in \Omega$ *let* $N_{\partial\Omega}(x) = \{y \in \partial\Omega : \delta(x) = \operatorname{dist}(x, y)\}$*. If* $N_{\partial\Omega}(x)$ *is a point, it will be denoted by* $N(x)$*.*

The distance function has been studied by many authors in the Finslerian context (see [LN05; TS16; Win14]).

Let G be the largest open subset of Ω such that for every $x \in G$ there is a unique closest point $y \in \partial\Omega$ to x. The length of curves in $\overline{\Omega}$ going from $\partial\Omega$ to x are measured in the Minkowski metric.

From [LN05] we have that the distance function from $\partial\Omega$ to x is in $C^{1,1}(G \cup \partial\Omega)$.

Definition 6.11. *Let* $G \subset \Omega$ *be the open subset of* Ω *such that for every* $x \in G$ *there exists a unique nearest point on* $\partial\Omega$ *to* x*. We will call it the* ***good set****. The complement of the good set is called a* ***singular set*** *and is denoted by* $S = \Omega \setminus G$*.*

Theorem 6.12. *[LN05] From every point* $y \in \partial\Omega$ *move along the inner normal until first hitting a point* $m(y) \in S$*. The length* $\overline{s}(y)$ *of the resulting segment is Lipschitz continuous in* y*.*

S is in fact the cut locus of $\partial\Omega$. It is obtained as follows. From $y \in \partial\Omega$ we consider a geodesic segment going into Ω in a normal direction until it hits a point $x = m(y) \in S$. The point $m(y)$ is called the *cut point* of $y \in \partial\Omega$. It means that if we go beyond x on the geodesic to any point x', then x' has a closer point in $\partial\Omega$.

The cut point of $y \in \partial\Omega$ can be alternatively defined as follows: Consider the geodesics from y going into Ω where the initial tangent vector is the normal direction with respect to the Minkowski metric with unit speed. Denote it by $\xi(y, s)$, where s is the parameter of the geodesic. The set of $s > 0$ satisfying

$$\operatorname{dist}(\partial\Omega \textit{ to } \xi(y, s)) = s$$

is either $(0,\infty)$ or $(0,\tilde{s}(y)]$ for some $0<\tilde{s}(y)<\infty$. In the latter case the point $\tilde{m}(y):=\xi(y,\tilde{s}(y))$ is the cut point of $y\in\partial\Omega$, and the collection of all $\tilde{m}(y)$ for all $y\in\partial\Omega$ is called the cut locus of $y\in\partial\Omega$ and it is denoted by $\tilde{S}$. In [LN05] it is proved that $m(y)=\tilde{m}(y)$ for all $y\in\partial\Omega$ and $\tilde{S}=S$.

Lemma 6.13. *Let $\Omega\subset\mathbb{R}^n$. Suppose that $x\in G$ and let $y=N(x)$ be the nearest point of x to the boundary at y with respect to the outward unit normal, then*

$$1-\delta(x)k_i(y)>0,$$

for all $x\in G$ and for all $i=\overline{1,n-1}$.

Proof. For $x\in G$ consider the Minkowski ball centered at x with radius δ (i.e. satisfying $\overline{B}_\delta(x)\cap(\mathbb{R}^n\setminus\Omega)=\{y\}$). Assume $k_i>0$, otherwise the statement is trivial. The principal radius is the reciprocal of the principal curvature $r_i:=\frac{1}{k_i}$. It is clear that $\delta(x)\le r_i$. Otherwise the disc of center x and radius $\delta(x)$ in the plane generated by $[x,y]$ and the i-th principal direction in $y\in\partial\Omega$ intersects $\partial\Omega$ in more than one point in contradiction with the fact that $N(x)$ is unique. This means that $1-\delta k_i\ge 0$.

By Corollary 4.11 in [LN05] there exists $\epsilon>0$ such that

$$x_t:=N(x)+[\delta(x)+t]\eta(N(x))\in G,\ 0<t\le\epsilon,$$

for $\eta(N(x))=-\nu(N(x))$ the inward normal at $N(x)$ and

$$\delta(x_t)=\delta(x)+t.$$

It means that $B(x_t,\epsilon)\subset G$, which implies that $1-\delta(x)k_i>1-[\delta(x)+\epsilon]k_i\ge 0$. □

Lemma 6.14. *Let $\Omega\subset\mathbb{R}^n$. Then the distance function δ is in $C^2(G)$.*

Proof. Let $y\in\partial\Omega$. Consider $\boldsymbol{n}(y)$ and $T_y(\partial\Omega)$ to be the outer normal and the tangent hyperplane to $\partial\Omega$ at y. By a rotation of coordinates we can assume that the x_n coordinate axis lies in the direction $-\boldsymbol{n}(y_0)$. There exists a neighborhood $\mathcal{N}$ of y_0, such that in it $\partial\Omega$ is given by $x_n=\phi(x')$, where $x'=(x_1,\dots x_{n-1})$, $\phi\in C^2(T_{y_0}(\partial\Omega)\cap\mathcal{N})$ and $D\varphi(y_0')=0$.

Consider now the Minkowski metric in this local coordinate system. From (6.26) in Section 6.3, because the Weingarten operator is selfadjoint we can choose an orthonormal frame with respect to the inner product induced by the Minkowski norm with reference vector $-\boldsymbol{n}(y_0)$ which diagonalise by unit eigenvectors the second fundamental form. The eigenvalues will be the principal curvatures k_i. By a further change of coordinates given by $\boldsymbol{n}(y_0)$ and the eigenvectors of the second fundamental form (which are in $T(y_0)$), in this new coordinate system we have

$$D^2\varphi(y_0') = \operatorname{diag}[k_1, \dots k_{n-1}].$$

This coordinate system is called the *principal coordinate system* at y_0. At a point $y = (y', \phi(y')) \in \partial\Omega \cap \mathcal{N}$ the unit normal vector has the form

$$\boldsymbol{n}_i = \frac{D_i\phi(y')}{\sqrt{1 + |D\phi(y')|^2}},\ i = \overline{1, n-1},\ \boldsymbol{n}_n = \frac{-1}{\sqrt{1 + |D\phi(y')|^2}}.$$

Any point $x \in G$ is related to its unique closest point$y = N(x)$ by

$$x = y - \delta\boldsymbol{n}(y),$$

where $\delta(x) = d(x, y) = F(y - x)$, because in a Minkowski space the geodesics are lines.

Consider a point $x_0 \in G$, and let $y_0 = N(x_0)$ and consider the principal coordinate system at y_0. Let $\boldsymbol{n}_{y_0}$ and $T_y(\partial\Omega)$ be the unit outward normal and the tangent hyperplane to $\partial\Omega$ at y_0. We can choose a local chart around y_0 such that the x_n coordinate axis lies in the direction $-\boldsymbol{n}(y_0)$. In a neighborhood $\mathcal{V}$ of y_0, $\partial\Omega$ is given by $x_{n+1} = \varphi(x')$, with $x' = (x_1 \dots x_{n-1})$, $\phi \in C^2(T_y(\partial\Omega) \cap \mathcal{V})$ and $D\varphi(y_0) = 0$. We can choose the local chart to be the principal one around y_0, so, in this coordinates

$$D^2\varphi(y_0') == \operatorname{diag}[k_1, \dots k_{n-1}]. \tag{6.35}$$

From the expressions of the unit outward normal vector at the point

$y = (y', \varphi(y')) \in \mathcal{V} \cap \partial\Omega$, in the principal coordinate system at y_0 it follows that

$$D_j \boldsymbol{n}_i(y'_0) = k_i \delta_{ij}, \quad i, j = \overline{1, n-1}. \tag{6.36}$$

For a point $x \in G$ there exists a unique point $y = y(x) \in \partial G$ such that $d(x, y) = F(y - x) = \delta(x)$. As we previously observed we have that

$$x = y - \delta(x) y. \tag{6.37}$$

Next we will show that the above equation defines y and δ as C^1 functions of x.

Let $x_0 \in G$ and $y_0 = y(x_0)$ and choose a principal coordinate system at y_0. Consider further the mapping

$$\boldsymbol{g} = (g^1, \ldots g^{n-1}) : \mathcal{U} := \mathcal{V}(y_0) \cap T_{y_0}(\partial\Omega) \times \mathbb{R} \to \mathbb{R}^n \tag{6.38}$$

given by

$$\boldsymbol{g}(y', \delta) = y - \boldsymbol{n}(y)\delta, \quad y = (y', \varphi(y')).$$

We have $\boldsymbol{g} \in C^1(\mathcal{U})$. The Jacobian matrix of $\boldsymbol{n}$ at $(y', \varphi(y'))$ is given by

$$\det[D\boldsymbol{g}] = (1 - k_1 \delta(x_0)) \cdots (1 - k_{n-1} \delta(x_0)) > 0. \tag{6.39}$$

For $x \in G$, it follows from the local inverse mapping theorem that for some neighborhood $\mathcal{M} = \mathcal{M}(x_0)$ of x_0, the mapping y' is of class $C^1(\mathcal{M})$. From (6.37) we have that $D\delta(x) = -\boldsymbol{n}(y(x)) = --\boldsymbol{n}(y'(x)) \in C^1(\mathcal{M})$ for $x \in \mathcal{M}$, therefore $\delta \in C^2(G)$.

□

In fact the new coordinate system is adapted to the unit eigenvectors of the second fundamental form.

Lemma 6.15. *Let $\Omega \subset \mathbb{R}^n$. Suppose that $x \in G$ and let $y = N(x)$ be the nearest point on the boundary. Let $k_i(y)$, $i = \overline{1, \ldots n-1}$, be the principal curvatures of the boundary at y, in terms of a principal coordinate system at*

y. For all $x \in G$ we have

$$[D^2\delta(x)] = diag\left[\frac{-k_1}{1-\delta k_1}, \dots, \frac{-k_n}{1-\delta k_{n-1}}, 0\right],$$

where $[D^2\delta(x)]$ is the Hessian matrix of the distance function and the right–hand part is a diagonal matrix.

Proof. The proof of Lemma 14.17 in [GT01] runs without change. □

By using some algebraic results (Lemma 2.5 and Proposition 2.6 in [LLL12]) we obtain the following proposition:

Proposition 6.16. *Let $k = (k_1, \dots, k_{n-1}) \in \mathbb{R}^{n-1}$ be the principal curvatures and let H be the mean curvature of the boundary at a point $\partial\Omega \in C^2$. Then*

$$\sum_{i=1}^{n-1} \frac{k_i}{1-\delta k_i} \geq \frac{nH}{n-\delta H},$$

where $1 - \delta k_i > 0$ for all $i = 1, \dots, n-1$, with equalityif and only if $k_1 = \dots = k_{n-1}$.

From Lemma 6.15 and Proposition 6.16 follows the next result:

Corollary 6.17. *Let $\Omega \in \mathbb{R}^n$. Then for any $x \in G$,*

$$-\boldsymbol{\Delta}\delta(x) \geq \frac{nH(y)}{n-\delta H(y)}, \tag{6.40}$$

where $\delta(x) := \inf_{y\in\mathbb{R}^n\setminus\Omega} \operatorname{dist}(x,y)$ and $H(y)$ is the mean curvature at the nearest point $y = N(x) \in \partial\Omega$ of x.

6.5 superharmonicity of the distance function

The main result of this section is the next theorem, which is the main ingredient for Farfy-Brezis-Marcus type results:

Theorem 6.18. *Let $n \geq 1, \Omega \in \mathbb{R}^n$ and $\delta(x) := \inf_{y\in\mathbb{R}^n\setminus\Omega} dist(x,y)$. Then*

$$-\boldsymbol{\Delta}\delta(x) \geq \frac{nH(y)}{n-\delta H(y)}$$

in distributional sense: for any $f \in C_0^\infty(\Omega)$, $\phi \geq 0$, we have

$$\int_\Omega df(\nabla\delta)\,\mathsf{dm}(x)(x)\phi \geq \int_\Omega f\frac{nH\circ N}{n-\delta H\circ N}\,\mathsf{dm}(x)$$

where $H(y)$ is the curvature at the nearest point $y = N(x) \in \partial\Omega$ for points $x \in G$.

The function $(H \circ N)(x)$ is well defined on G, so it is a well defined L^∞-function due to the fact that $\Omega \setminus G$ has zero Lebesgue measure. The proof is divided in two parts. First we prove Theorem 6.18 for $C^{2,1}$ domains, then for C^2 domains.

Proof of Theorem 6.18 when $\partial\boldsymbol{\Omega} \in \mathbf{C^{2,1}}$. Let $z \in \partial\Omega$ and

$$\overline{\rho}(z) := \sup\{t|z + t\eta(z) \in G\},$$

where $\eta = -\boldsymbol{n}$ is the inward unit normal. From a point $z \in \partial\Omega$ move along the inner normal until first hitting a singular point in S (in fact the cut point of z). This point will be denoted by $m(z)$. From [LN05] we know that

$$m(z) := z + \overline{\rho}(z)\eta(z)\ .$$

Theorem 6.19. (*see [IT01; LN05]*) *The map m and the function $\overline{\rho}$ in the above relation are in $C^{0,1}_{loc}(\partial\Omega)$.*

Corollary 6.20. (*see [LN05]*) *Let $\Omega \subset \mathbb{R}^n$ and $S \subset \Omega$ be the singular set defined in Definition 6.11. The Hausdorff measure of the singular set satisfies $H^{n-1}(S) < \infty$.*

Let $x \in G = \Omega \setminus S$ and $N(x)$ be the unique point on $\partial\Omega$ such that

$$\delta(x) = d(x, N(x))\ ,$$

i.e. $N(x)$ is the nearest point on $\partial\Omega$, then $\delta(x) \in C^2(G)$. Consider

$$h(x) := \frac{\delta(x)}{\Lambda(x)},$$

where $\Lambda(x) = \overline{\rho}(N(x))$, $N(x)$ is the closest nearest point of x to $\partial\Omega$ and $\overline{\rho}$ is the function in Theorem 6.19. The functions $\Lambda(x)$ and $h(x)$ are defined originally on G, but they can be extended by continuity to $\Omega = G \cup S$ by $h(x) = 1$ for $x \in S$. So Λ and h are in $C^{0,1}_{loc}(\overline{\Omega} \setminus S) \cap C^0(\overline{\Omega})$. These facts are consequences of the following lemma:

Lemma 6.21. *For all $\overline{x} \in S$ one has*

$$\lim_{x \to \overline{x}, x \in G} h(x) = 1.$$

Proof. Let $\overline{x} \in S$. There exists $\overline{z} \in \partial\Omega$ such that:

$$\overline{x} = m(\overline{z}) = \overline{z} + \overline{t}\eta(\overline{z})),$$

where η is the unit inner normal of Ω at $\overline{z}$, and $d(\overline{z}, m(\overline{z})) = d(\overline{x}, m(\overline{z})) = \overline{t}$.

Consider now a sequence $x_i \in G$, $x_i \to \overline{x}$. For every $i, \exists!\ z_i = N(x_i) \in \partial\Omega$ such that $d(x_i, z_i) = \delta(x_i)$. From Corrolary 4.11 of [LN05] it follows that

$$\Lambda(x_i) \geq d(x_i, z_i) \Longleftrightarrow \liminf_{i \to \infty} \Lambda(x_i) \geq \delta(\overline{x}).$$

Because $m(z_i) = z_i + t_i\eta(z_i)$ follows that $\Lambda(x_i) = t_i$.

Now we are going to prove by contradiction the following inequality

$$\limsup_{i \to \infty} \Lambda(x_i) \leq \delta(\overline{x}).$$

Suppose that the above inequality does not hold. There exists $\alpha > 0$ with $\Lambda(x_i) > \delta(\overline{x}) + \alpha$ for i large enough. So we may assume (by passing to a subsequence) that

$$z_i \to \hat{z} \in \partial\Omega$$
$$\Lambda(x_i) = t_i \to \hat{t} \geq \delta(\overline{x}) + \alpha\,.$$

By the continuity of $m(z)$ (see [LN05]) we have

$$m(z_i) = z_i + t_i\eta(z_i) \to m(\hat{z})$$

and $m(\hat{z}) = \hat{z} + \hat{t}\eta(\hat{z})$. One has that $x_i \to \overline{x}, x_i = z_i + \tilde{t}\eta(z_i)$, and $d(z_i, x-i) = \overline{t}_i, \overline{t}_i \to \overline{t}$, and finally

$$\overline{x} = \hat{z} + t\eta(\hat{z}).$$

It follows

$$\tilde{t}_i = d(x_i, z_i) = \operatorname{dist}(x_i\partial\Omega) \leq \operatorname{dist}(x_i, \overline{(}x)) + \operatorname{dist}(\overline{x}, \partial\Omega) = \operatorname{dist}(x_i, \overline{x}) + \delta(\overline{x}),$$

where $\operatorname{dist}(x_i\overline{x}) \to 0$. It implies that$\tilde{t}\delta(\overline{x}) \leq \hat{t} - \alpha < \hat{t}$. By Corollary 4.11 of [LN05]

$$\overline{x} = \tilde{z} + \tilde{t}\eta(\hat{z}) \in G$$

(from $m(\hat{z}) = \hat{z} + \hat{t}\eta(\hat{z})$), which gives a contradiction.

Thus it is proved that

$$\lim_{i\to\infty} \Lambda(x_i) = \delta(\overline{x}),$$

and

$$\lim_{x\to\overline{x}} \Lambda(x) = \delta(\overline{x})$$

so the lemma is proved.

□

Lemma 6.22. *The normalized distance function* $h \in C^{0,1}_{loc}(\overline{\Omega} \setminus S) \cap C^0(\overline{\Omega})$ *and*

$$f(x) = \begin{cases} 0 & x \in \overline{\Omega} \\ 1 & x \in S, \end{cases}$$

and $0 < h(x) < 1$ *otherwise.*

Let
$$h_\epsilon(x) = \int_{B(0,\epsilon)} h(x-y)\phi_\epsilon(y)\mathsf{d}\mathsf{m}(x),$$
where $\phi_\epsilon(x) = \epsilon^{-n}\phi(\frac{x}{\epsilon})$ is the standard molifier with compact support in an ϵ–neighborhood of x and
$$h_\epsilon \to h \text{ in } C^0_{loc}(\overline{\Omega} \setminus S).$$

Let $\mu \in (0,1)$ be fixed, and let λ_μ be such that they are regular values of h_ϵ, $\lambda_\mu \to 1-\mu$ for $\epsilon \to 0$. It means that for small ϵ (which depends on μ) the level sets
$$\Sigma_\epsilon := \{h_\epsilon = \lambda_\epsilon\}$$
are regular smooth hypersurfaces.

Lemma 6.23. *Let $\Omega \in \mathbb{R}^n$. The hypersurface Σ_ϵ defined above satisfies*
$$\lim_{\epsilon \to 0} \operatorname{dist}(\Sigma_\epsilon, \{x, h(x) = 1-\mu\}) = 0. \tag{6.41}$$

Proof. Suppose, by contrary, that there exist $\alpha > 0$ and $x_\epsilon \in \Sigma_\epsilon$ such that
$$\operatorname{dist}(x_\epsilon, \{h = 1-\mu\}) \geq \alpha, \tag{6.42}$$
for a sequence $\epsilon \to 0$. There exists a convergent subsequence of x_ϵ, denoted further again by x_ϵ and an $\overline{x} \in \mathbb{R}^{n+1}$ such that $x_\epsilon \to \overline{x}$. The continuity of h implies that
$$h(\overline{x}) = \lim_{\epsilon \to)} h_\epsilon(x_\epsilon) = \lim_{\epsilon \to 0} \lambda_\epsilon = 1-\mu.$$

From (6.42) it follows that $|x_\epsilon - \overline{x}| \geq \alpha > 0$, which contradicts the convergence of x_ϵ to $\overline{x}$. □

For $\mu \in (0, \frac{1}{8})$ there exists (from Lemma 6.23) a positive $\epsilon(\mu) > 0$ such that
$$\Sigma_\epsilon \subset \left\{1 - \frac{5\mu}{4} \leq h \leq 1 - \frac{3\mu}{4}\right\}.$$

Lemma 6.24. *For any fixed* $0 < \mu < 1/8$ *and* $0 \leq \epsilon_1(\mu)$*, there exists* $C(\mu) > 0$ *such that for* $\epsilon > 0$ *small enough*

$$g_{\boldsymbol{n}_\epsilon}(\boldsymbol{n}_\epsilon, \boldsymbol{\nabla}\delta) \geq C(\mu) > 0$$

on all points Σ_ϵ*.*

Remark 6.25. *In a principal orthonormal frame adapted in a point of* Σ_ϵ *the above inequality takes the form*

$$\boldsymbol{n}_\epsilon \cdot \boldsymbol{\nabla}\delta \geq C(\mu) > 0.$$

Proof. In a principal coordinate system the proof reduces to the proof of Lemma 3.6 in [LLL12]. □

Proof of Theorem 6.18 for $\mathbf{C^{2,1}}$ domains. By Lemma 6.23 there exists a sequence of $\lambda \to 1^-$ such that $\Sigma_\epsilon := \{x \in \Omega, h_\epsilon(x) - \lambda_\epsilon\}$ has C^∞ boundary.

The corresponding sets

$$\Omega_\epsilon := \{x \in \Omega, h(x) < \lambda_\epsilon\}$$

satisfy

$$\bigcup_{\epsilon > 0} \Omega_\epsilon = G$$

and $g_{\boldsymbol{n}_\epsilon}(\boldsymbol{n}_\epsilon, \boldsymbol{\nabla}\delta) \geq C(\mu) > 0$ on $\partial\Sigma_\epsilon$, where $\boldsymbol{n}_\epsilon$ is the outer normal to the boundary of Ω_ϵ. The Stokes theorem and the Finslerian version of the divergence theorem (see [She00]) applied in our situation gives

$$\int_\Omega \operatorname{div} X \mathsf{d}\mathsf{m}(x) = \int_{\partial\Omega} g_{\boldsymbol{n}}(\boldsymbol{n}, X) dA\,.$$

Using that $\operatorname{div}(fX) = f\operatorname{div}X + df(X)$ and applying Stokes' theorem for $X = f\boldsymbol{\nabla\delta}$, using Green's formula (since $\delta(x)$ is C^2 on $\overline{\Omega}_\epsilon \subset G \cup \partial\Omega$) we obtain:

$$\begin{aligned}\int_{\partial\Omega_\epsilon} g_{\boldsymbol{n}}(\boldsymbol{n}, f\boldsymbol{\nabla}\delta)dA &= \int_{\Omega_\epsilon} \mathrm{div}(f\boldsymbol{\nabla}\delta)\mathsf{dm}(x)\\ &= \int_{\Omega_\epsilon} f\mathrm{div}(\boldsymbol{\nabla}\delta)\mathsf{dm}(x) + \int_{\Omega_\epsilon} df(\boldsymbol{\nabla}\delta)\mathsf{dm}(x)\\ &= \int_{\Omega_\epsilon} f\boldsymbol{\Delta}\delta\mathsf{dm}(x) + \int_{\Omega_\epsilon} df(\boldsymbol{\nabla}\delta)\mathsf{dm}(x)\end{aligned}$$

and by Lemma 6.24

$$\begin{aligned}\int_{\Omega_\epsilon} df(\boldsymbol{\nabla}\delta)\mathsf{dm}(x) &= \int_{\partial\Omega_\epsilon} g_{\boldsymbol{n}}(\boldsymbol{n}, f\boldsymbol{\nabla}\delta)dA - \int_{\Omega_\epsilon} f\boldsymbol{\Delta}\delta\mathsf{dm}(x)\\ &\geq -\int_{\Omega_\epsilon} f\boldsymbol{\Delta}\delta\mathsf{dm}(x) \geq \int_{\Omega_\epsilon} f\frac{nH\circ N}{n-\delta H\circ N}\mathsf{dm}(x).\end{aligned}$$

The proof is obtained by $\epsilon \to 0$ in the above inequality. □

Proof of Theorem 6.18 for $\mathbf{C^2}$ domains. Let $z \in \partial\Omega$ and $\tilde{m}(z) = z + \tilde{\rho}(z)\boldsymbol{n}$, where $\boldsymbol{n}$ is the unit inner normal at $z \in \partial\Omega$ amd $\tilde{\rho}(z)$ is the largest number so that

$$\mathrm{dist}(z + t\boldsymbol{n}(z), \partial\Omega) = t \text{ for all } t \in (0, \tilde{\rho}(z)).$$

The C^2 regularity of $\partial\Omega$ ensures by Lemma 4.2 in [LN05] that $\tilde{\rho}(z) \geq \overline{\rho}(z)$, and thus

$$B(m(z), \overline{z}) \subset \Omega, \quad z \in \partial B(m(z), \overline{z}), \quad \forall z \in \partial\Omega.$$

Lemma 6.26. *For every $h \in C^2(\partial\Omega)$ satisying*

$$0 < h(z) < \overline{\rho}(z)$$

let

$$\Sigma := \{z + h(z)\boldsymbol{n}(z) | z \in \partial\Omega\}.$$

Then Σ is a C^1 hypersurface with

$$g_{\boldsymbol{n}^{\Sigma}(z)}(\boldsymbol{n}^{\Sigma}(z)\boldsymbol{\nabla}(x)) > 0 \text{ for all } x \in \Sigma,$$

where $\boldsymbol{n}^{\Sigma}(z)$ denotes the outer normal of the boundary of

$$\{z + th(z)\boldsymbol{n}(z) | z \in \partial\Omega, t \in (0,1)\}.$$

Proof. Following [LLL12], for a point $z \in \partial\Omega$, one can suppose that $\overline{\rho}(z) = 1$. After a translation and rotation with respect to $g_{\boldsymbol{n}}$ we may assume further that $z = 0$ and the boundary near 0 is given by

$$x_n = g(x'), \quad x' = (x^1, \ldots, x_{n-1}),$$

where g is a local chart C^2 function near $0' \in \mathbb{R}^{n-1}$ induced by the principal coordinate system with $g(0') = 0$. In this chart $e_n = \boldsymbol{n}$ and the coordinates are orthonormal with respect to $g_{\boldsymbol{n}}$. From this special coordinate system the conclusion follows as in Lemma 3.8 from [LLL12].

□

Proposition 6.27. *Let $\Omega \subset \mathbb{R}^n, n \geq 1$ and let G be the good set of Ω. Then*

$$\inf_{x \in \partial G}(-\boldsymbol{\Delta}\delta(x)) = \inf_{y \in \partial\Omega} H(y),$$

where $H(y)$ is the mean curvature of the boundary at y.

Proof. From Lemma 6.15 we have

$$-\boldsymbol{\Delta}\delta(x) = \sum_{i=1}^{n-1} \frac{k_i(N(x))}{1 - \delta k_i(N(x))}. \tag{6.43}$$

As a function of δ, $\sum_{i=1}^{n-1} \frac{k_i}{1 - \delta k_i}$ is non-decreasing (if $1 - \delta k_i > 0$ for all i), we have from (6.43) that

$$-\boldsymbol{\Delta}\delta(x) \geq \sum_{i=1}^{n-1} k_i(N(x)) = H(N(x)) \geq \inf_{y\in\partial\Omega} H(y), \quad x \in G.$$

It follows that

$$\inf_{x\in\partial G}(-\boldsymbol{\Delta}\delta(x)) \geq \inf_{y\in\partial\Omega} H(y).$$

On the other hand, for $y \in \partial\Omega$ and for $t > 0$ small one has that $x_t = y + t\boldsymbol{n}(y) \in G$ and from (6.43) that

$$\inf_{x\in\partial G}(-\boldsymbol{\Delta}\delta(x)) \leq \inf_{t\to 0^+} -\boldsymbol{\Delta}\delta(x_t) = H(y),$$

which implies

$$\inf_{x\in\partial G}(-\boldsymbol{\Delta}\delta(x)) \leq \inf_{y\in\partial\Omega} H(y)$$

and the proposition is proved. □

Theorem 6.28. (*Equivalence Theorem*) *Let $\Omega \subset \mathbb{R}^n$ and δ be the distance function to the boundary. Then δ is superharmonic function on Ω of the singular set Sif and only if$\partial\Omega$ is weakly mean convex.*

Proof. If δ is superharmonic, then $-\boldsymbol{\Delta}\delta(x) \geq 0$ for all $x \in G$ and if Ω is weakly mean convex, then $H(y) \geq 0$ for all $y \in \partial\Omega$. With this observations the proof is a simple consequence of the previous proposition. □

6.6 Hardy-Brezis-Marcus inequality in Minkowsky spaces

Let $u \in C_0^\infty(\Omega)$, $\frac{u^2}{\delta}$ is a Lipschitz function with compact support in Ω. We have

$$\int_\Omega D\left(\frac{u^2}{2\delta}\right)(\boldsymbol{\nabla}\delta)\mathsf{dm}(x) = -\int_\Omega \frac{u^2}{2\delta^2}D\delta(\boldsymbol{\nabla}\delta)\mathsf{dm}(x) + \int_\Omega \frac{u}{\delta}Du(\boldsymbol{\nabla}\delta)\mathsf{dm}(x) \tag{6.44}$$

$$= -\int_\Omega \frac{u^2}{2\delta^2}\mathsf{dm}(x) + \int_\Omega \frac{u}{\delta^2}Du(\boldsymbol{\nabla}\delta)\mathsf{dm}(x),$$

the last eqality comes from $F(\boldsymbol{\nabla}\delta) = 1$.

Let $F^\star$ be the dual of the norm defined on $\mathbb{R}^{n\star}$ and $\alpha,\ \beta \in \mathbb{R}^{n\star}$. The inequality $(F^\star(\beta) - F^\star(\alpha))^2 \geq 0$ is equivalent to

$$F^{\star 2}(\beta) - F^{\star 2}(\alpha) \geq 2F^\star(\alpha)F^\star(\beta) - 2F^{\star 2}(\alpha) \geq 2g^{\star\alpha}(\alpha, \beta) - 2F^{\star 2}(\alpha). \tag{6.45}$$

In the inequality (6.45) we consider $\beta = Du$ and $\alpha = \frac{u}{2\delta}D\delta$ and we obtain

$$F^{\star 2}(Du) - \frac{u^2}{4\delta^2}F^{\star 2}(D\delta) \geq \frac{u}{\delta}F^\star(Du)F^\star(D\delta) - \frac{u^2}{2\delta^2}F^{\star 2}(D\delta) \geq \frac{u}{\delta}(Du)(\boldsymbol{\nabla}\delta) - \frac{u^2}{2\delta^2}F^{\star 2}(D\delta)$$

which is equivalent to

$$F^{\star 2}(Du) - \frac{u^2}{4\delta^2} \geq \frac{u}{\delta}(Du)(\boldsymbol{\nabla}\delta) - \frac{u^2}{2\delta^2}$$

by using the fundamental inequality (see [BC00, p. 8-9]). Integrating the extreme terms and taking into account that $F^{\star 2}(d\delta) = 1$ we obtain

$$\int_\Omega F^{\star 2}(Du)\mathsf{dm}(x) - \int_\Omega \frac{u^2}{4\delta^2}F^{\star 2}(D\delta)\mathsf{dm}(x) \geq \int_\Omega \frac{u}{\delta}(Du)(\boldsymbol{\nabla}\delta)\mathsf{dm}(x) - \int_\Omega \frac{u^2}{2\delta^2}F^{\star 2}(D\delta)\mathsf{dm}(x).$$

Using (6.44) follows that

$$\int_\Omega F^{\star 2}(Du)\mathsf{dm}(x) - \int_\Omega \frac{u^2}{4\delta^2}F^{\star 2}(D\delta)\mathsf{dm}(x) \geq \int_\Omega d\left(\frac{u^2}{2\delta}\right)(\boldsymbol{\nabla}(\delta))\mathsf{dm}(x). \tag{6.46}$$

We are ready now to prove the following theorem:

Theorem 6.29. *(Improved Hardy-Brezis-Marcus inequality for Minkowski spaces) Suppose $\Omega \subset \mathbb{R}^n$ is weakly mean convex and assume that $H_0 := \inf_{x\in\partial\Omega} H(x) \geq 0$, then for any $f \in C_0^\infty(\Omega)$*

$$\int_\Omega F^{\star 2}(Du)\,\mathsf{dm}(x) - \frac{1}{4}\int_\Omega \frac{u^2}{\delta^2}\mathsf{dm}(x) \geq \lambda(n, \Omega)\int_\Omega u^2\,\mathsf{dm}(x),$$

where $\lambda(n,\Omega) := \inf_{x\in\Omega} \dfrac{-\boldsymbol{\Delta}\delta(x)}{2\delta(x)} \geq \dfrac{2}{n}H_0^2$.

Proof. We apply (6.46) in Theorem 6.18 and we obtain

$$\int_\Omega F^{\star 2}(Du)\mathsf{dm}(x) - \int_\Omega \frac{u^2}{4\delta^2}\mathsf{dm}(x) \geq \int_\Omega \frac{nH}{n-\delta H}\frac{u^2}{2\delta}\mathsf{dm}(x).$$

If $H_0 = 0$ the theorem holds. Suppose that $H_0 > 0$. For $a \in \mathbb{R}$ consider the function $f : \mathbb{R} \setminus \{0, a\} \to \mathbb{R}$ $f(t) = \frac{1}{at-t^2}$. For $t \in (0, a)$ it follows that

$$f(t) \geq \frac{4}{a^2}, \qquad \text{for all } t \in (0,a).$$

For $a = \frac{n}{H}$ and $t = \delta$ from Lemma 6.13 it follows that $t < a$ for $x \in \Omega \setminus S$. Hence for $x \in G$ we have $\frac{H}{(n-\delta H)\delta} \geq \frac{4H^2}{n^2}$ and this implies

$$\int_\Omega \frac{nH}{n-\delta H}\frac{u^2}{2\delta}\mathsf{dm}(x) \geq \int_\Omega \frac{2}{n}H^2u^2\mathsf{dm}(x) \tag{6.47}$$

using that $\Omega \setminus S$ has zero measure. Using (6.47) in (6.48) we obtain

$$\int_\Omega F^{\star 2}(Du)\mathsf{dm}(x) - \frac{1}{4}\int_\Omega \frac{u^2}{\delta^2}\mathsf{dm}(x) \geq \frac{2}{n}H_0^2\int_\Omega u^2\mathsf{dm}(x), \tag{6.48}$$

which proves the Hardy inequality with $\lambda(n,\Omega) \geq \frac{2}{n}H_0^2$. □

In the last part of this section we present a method to obtain a class of Hardy type inequalities.

Remark 6.30. *We start again with the inequality* (6.45)*,*

$$F^{\star 2}(\beta) - F^{\star 2}(\alpha) \geq 2g^{\star\alpha}(\alpha,\beta) - 2F^{\star 2}(\alpha).$$

For $u \in C_0^\infty(\Omega)$ *take* $\beta = Du$ *and* $\alpha = v(u,\delta)D\delta$ *for a positive function* $v \in C^1(\mathbb{R}^2)$*. The above inequality writes now (we write* v *for* $v(u,\delta)$*)*

$$\begin{aligned} F^{\star 2}(Du) - v^2F^{\star 2}(D\delta) &\geq 2vg^{\star\alpha}(Du,\delta) - 2v^2F^{\star 2}(D\delta) \\ &= (2vDu - 2v^2D\delta)(\boldsymbol{\nabla}(\delta)). \end{aligned}$$

The integrability condition in the last therm implies that for $v(u,\delta)$ implicitly defined by $\varphi(\delta-\frac{u}{2v},v)=0$, where $\varphi\in C^1(\mathbb{R}^2)$, the last therm is the differential of some function $\Psi(u,\delta)$ (i.e. $D\Psi=2vDu-2v^2D\delta$), so the last inequality becomes

$$F^{\star 2}(Du)-v^2F^{\star 2}(D\delta)\geq D\Psi(\boldsymbol{\nabla}(\delta)).$$

Integrating on Ω we obtain

$$\int_\Omega F^{\star 2}(Du)\,\mathsf{dm}(x)-\int_\Omega v^2F^{\star 2}(D\delta)\,\mathsf{dm}(x)\geq\int_\Omega D\Psi(\boldsymbol{\nabla}(\delta))\,\mathsf{dm}(x).$$

By applying now Theorem 6.18 we obtain

$$\int_\Omega F^{\star 2}(Du)\,\mathsf{dm}(x)-\int_\Omega v^2F^{\star 2}(D\delta)\,\mathsf{dm}(x)\geq\int_\Omega\frac{nH}{n-\delta H}\Psi\,\mathsf{dm}(x).$$

6.7 Applications to a minimization problem

In this section we apply Hardy-Brezis-Marcus inequality to prove a singular Laplace problem.

For $\mu\in\mathbb{R}$, on the Sobole space $W_0^{1,2}(\Omega,F,\mathsf{m})$ we define the *singular Finsler-Laplace operator*

$$\mathcal{L}_F^\mu u=-\boldsymbol{\Delta}(u)-\frac{1}{2}\frac{u}{\delta^2}-\mu u.$$

We consider the singular Poisson problem

$$\begin{cases}\mathcal{L}_F^\mu u=\kappa(x) & \text{in}\quad\Omega;\\ u=0 & \text{on}\quad\partial\Omega,\end{cases}\tag{6.49}$$

where $\Omega\subset\mathbb{R}^n$ is an open, bounded domain.

The singular energy functional $\mathscr{K}_\mu:W_0^{1,2}(\Omega,F,\mathsf{m})\to\mathbb{R}$ associated to the singular Finsler-Laplace operator is given by

$$\mathscr{K}_\mu(u)=\int_\Omega F^{*2}(x,Du(x))\mathsf{dm}(x)-\frac{1}{4}\int_\Omega\frac{u^2}{\delta^2}\mathsf{dm}(x)-\mu\int_\Omega u^2\mathsf{dm}(x).\tag{6.50}$$

First we shall prove that the energy functional associated to the singular Finsler-Laplace operator is strictly convex.

Theorem 6.31. *Let F be a Minkowski metric on $\mathbb{R}^n$ with $l_F > 0$. The functional $\mathscr{K}_\mu : W_0^{1,2}(\Omega, F, \mathsf{m}) \to \mathbb{R}$ defined by*

$$\mathscr{K}_\mu(u) = \int_\Omega F^{*2}(x, Du(x))\mathrm{d}\mathsf{m}(x) - \frac{1}{4}\int_\Omega \frac{u^2}{\delta^2}\, d\mathsf{m}(x) - \mu\int_\Omega u^2\, d\mathsf{m}(x) \tag{6.51}$$

is positive for $u \neq 0$ and strictly convex whenever $0 \leq \mu < l_F\lambda(n, \Omega)$.

Proof. Let $0 \leq \mu < l_F\lambda(n, \Omega)$. By (6.10), one has $0 \leq l_F^{-1} \leq 1$. The positivity of $\mathcal{K}_\mu$ follows by Theorem 6.29. Let $0 < t < 1$ and $u, v \in W_0^{1,2}(\Omega, F, \mathsf{m})$, $u \neq v$ be fixed. Then, by (6.8) it follows that

$$\begin{aligned}
\mathscr{K}_\mu\,(tu + (1-t)v) &= \int_\Omega F^{*2}(tDu(x) + (1-t)Dv(x))\mathrm{d}\mathsf{m}(x) - \frac{1}{4}\int_\Omega \frac{(tu + (1-t)v)^2}{\delta^2}\mathrm{d}\mathsf{m}(x) \\
&\quad - \mu\int_\Omega (tu + (1-t)v)^2\mathrm{d}\mathsf{m}(x) \\
&\leq t\int_\Omega F^{*2}(Du(x))\mathrm{d}\mathsf{m}(x) + (1-t)\int_\Omega F^{*2}(Dv(x))\mathrm{d}\mathsf{m}(x) \\
&\quad - l_F t(1-t)\int_\Omega F^{*2}(D(v-u)(x))\mathrm{d}\mathsf{m}(x) \\
&\quad \frac{1}{4}\int_\Omega \frac{(tu + (1-t)v)^2}{\delta^2}\mathrm{d}\mathsf{m}(x) - \mu\int_\Omega (tu + (1-t)v)^2\mathrm{d}\mathsf{m}(x) \\
&= t\mathscr{K}_\mu(u) + (1-t)\mathscr{K}_\mu(v) - l_F t(1-t)\int_\Omega F^{*2}(D(v-u)(x))\mathrm{d}\mathsf{m}(x) \\
&\quad + \frac{1}{4}t(1-t)\int_\Omega \frac{(v-u)^2}{\delta^2}\mathrm{d}\mathsf{m}(x) + \mu t(1-t)\int_\Omega (v-u)^2\mathrm{d}\mathsf{m}(x) \\
&= t\mathscr{K}_\mu(u) + (1-t)\mathscr{K}_\mu(v) - l_F t(1-t)\int_\Omega [F^{*2}(D(v-u)(x) \\
&\quad - l_F^{-1}\frac{1}{4}\frac{(v-u)^2}{\delta^2} - \mu l_F^{-1}(v-u)^2]\mathrm{d}\mathsf{m}(x) \\
&< t\mathscr{K}_\mu\,(u) + (1-t)\mathscr{K}_\mu\,(v)
\end{aligned}$$

the last inequality comes from $l_F \geq 1$ which is implied by (6.10).

$\square$

Theorem 6.32. *Let F be a Minkowski metric on $\mathbb{R}^n$. Let $\Omega \subset \mathbb{R}^n$ be an open, bounded domain and $\kappa \in L^\infty(\Omega)$. Then the problem 6.49 has a unique* *non-negative weak solution for every* $0 \leq \mu < l_F \lambda(n, \Omega)$.

Proof. Let $0 \leq \mu < l_F \lambda(n, \Omega)$ and consider the energy functional associated to the Poisson problem 6.49

$\mathcal{E}_\mu : W_0^{1,2}(\Omega, F, \mathsf{m}) \to \mathbb{R}$ given by

$$\mathcal{E}_\mu(u) = \frac{1}{2}\mathscr{K}_\mu(u) - \int_\Omega \kappa(x)u(x)\mathrm{d}\mathsf{m}(x). \tag{6.52}$$

One has that $\mathcal{E}_\mu \in C^1(W_0^{1,2}(\Omega, F, \mathsf{m}), \mathbb{R}^n)$, and its critical points are exactly the weak solution of problem 6.49. Consider now a ball which contains Ω ($\exists R > 0 x_0 \in \mathbb{R}^n$ with $\Omega \subset B^+(x_0, R)$).

Using the same line as in [FKV16] $\mathcal{E}_\mu$ is bounded from below and coercive on the reflexive Banach space $W_0^{1,2}(\Omega, F, \mathsf{m})$ (with the norm induced by F_s). By the Theorem 6.31, and al these implies that $\mathcal{E}_\mu$ has a unique (global) minimum point $u_\mu \in W_0^{1,2}(\Omega, F, \mathsf{m})$ of $\mathcal{E}_\mu$ which is also the unique critical point of $\mathcal{E}_\mu$. The comparison principle in [FKV16] (Prop 5.1 in [FKV16] still holds) that is $\kappa \geq 0$ implies $u_\mu \geq 0$. □

Remark 6.33. *One of the most important problems related to Hardy-Brezis-Marcus inequalities is the so called "best constant" problem, see [AFT09; AW12; BM98; FMT07b; MS00]. In [AFT09] it is obtained the sharp constant for the Hardy-Sobolev inequality involving the distance to the origin in an Euclidean space. In [FMT07b] it is shown that the* L^p-*Hardy inequality involving the distance to the boundary of a convex domain, can be improved by adding an* L^q *norm* $q \geq p$*, with a constant depending on the interior diameter of* Ω*. In the next period the problem of the best constant in a Minkowski spaces will be considered. Also it will be of interest to obtain such types of results in complete Finsler manifolds.*

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Part III

Perspectives and future work

Chapter 7

Perspectives and future research work

The research in global differential geometry of both Finsler and Riemann spaces will continue, the problems which are presented are modern and important. Also a lot of further directions can be seen from here.

A direction will be the study of manifolds of positive sectional curvature. Namely we are very interested if we can change a Riemann (Finsler) metric by some special diffeomorphisms in such a way that the new metric has same sign of curvature as the original one.

We are interested to continue the connectedness problems in the case of some special immersions (for example minimal) and to extend the results to Kähler-Finsler manifolds.

The compactness criteria will be further studied in order to obtain some more general and weak conditions which ensure that the manifold is compact.

Related to Hardy-Brezis inequalities we are interested in the "best constant" problem. On the other hand we try to prove the inequality presented here in the Riemann and Finsler context, and to find others inequalities which involve curvature properties, both in Riemann and Finsler case.

Another direction in research is given by the papers [MPP15], [MPP16]. Here we intend to obtain some generalized convexity properties in infinite dimensional case.

În ultimii ani am participat la mai multe proiect interne si inernaţionale ca si membru in echipa (în CV este lista ultimilor ani), fiind responsabil cu partea matematică. Acest fapt demonstreaza disponibilitatea de a lucra in echipă si capacitatea de a conduce/propune teme actuale. Menţionez ca am fost referent ştiintific de lucrari de diploma si am articole scrise in colaborare cu studenţi de la master si doctorat de la Facultatea de Automatică şi calculatore.

Beside the presented work the research activity has an impportant part in applied mathematics in real-life problems (see the publication list).

In the last years I am involved in several research projects (both national and international) as team member (see CV). My team role is related to construct mathematical approaches.

This fact proves that I am a team-worker and that I am able to propose/conduct mathematical approaches. I was scientific advisor to some Master Thesis and I have several papers with master and PhD students.

We intend to continue the collaboration with my colleagues from Computer Science department in directions as computer vision, neuroscience etc, and to initiate a new research group in medical applications.

Many of the papers published in these projects have citations in good journals([DPN10], [NPM11], [NPM12], [NNP14][NNP15].

Five years ago I started a joint work with a research team from Memorial Sloan Kettering Cancer Center. In this joint research we wrote some papers in important journals, one of them being Nature ([Rod+16], [Jha+14]) and we had some talks in major conferences. This is an important project, strongly interdisciplinary (experts from chemistry, physics, medical science, mathematics, informatics are in the team).

The mathematics plays a central place in cancer research and (one can see for example the strong groups in mathematical oncology at Harvard, MIT, Viena, etc.).

We are trying to start such a group in Cluj, we need a lot of maths and math related topics in this field (mathematics, artificial intelligence, computer vision, machine learning and pattern recognition, to mention only some of them).

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