Habilitation thesis

Classes of analytic functions, integral operators and related research

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Contents

Abstract

In this habilitation thesis we have described the significant results achieved by us after obtaining the PhD degree in Mathematics from Babeş-Bolyai University of Cluj-Napoca, in 2004. We would like to mention that the PhD thesis was dedicated to a different subject from mathematical analysis, namely, Spline Based Numerical Methods Applied in Statistics, a subject related to functional analysis, statistics and numerical calculus. The research results presented here are concerned with the *theory of the functions of one complex variable*, a classic topic of mathematical analysis which still remains an attractive research area for many mathematicians from all over the world.

The theory of functions of one complex variable was established in the middle of the past century, as one of the mathematical fields, a subdomain of complex analysis. An important area of the theory of functions of one complex variable is the geometric theory of analytic functions, called also *geometric function theory* in which the goal is to give geometrical meaning to some analytically expressed conditions, such that the correctness of the analytic judgement is tightly linked with the intuitive one. Hence, one can take advantage from this duality, by combining analytic proof with geometric intuition in order to study various classes of functions.

Among the properties that have been studied for the functions of one complex variable, there is the so called, univalence. An analytical (holomorphic) and injective function on the domain $\mathcal{U}$, $\mathcal{U} \subset \mathbb{C}$, $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ is called a univalent function on $\mathcal{U}$. The condition $f'(z) \neq 0, \forall z \in \mathcal{U}$, is just a necessary univalence condition, ensuring a local univalence only. The goal is to find some supplimentary conditions, which can enforce the univalence of a function f, on the domain $\mathcal{U}$. Hence, it is suitable to obtain both necessary and sufficient conditions for univalence. Taking into account also the geometric point of view, the univalence can be linked with conformal mappings.

Being one of the main concept in geometric function theory, the univalent function (holomorphic and injective), was studied for the first time by P. Koebe, in 1907. Within a century, the theory of univalent functions has been developed considerably by many mathematicians. Necessary and sufficient conditions for univalence were obtained for the first time in 1931 by Gh. Calugareanu, after that being studied by many authors, among

we recall here Z. Nehari, C. Pomerenke, A.W. Goodman, G. M. Goluzin, etc. ([42], [65])) and some classical works as: Z. Nehari, *Conformal mappings*, 1952, L.V. Ahlfors, *Conformal invariants, Topics in Geometric Function Theory*, 1973, Ch. Pommerenke, *Univalent functions*, 1975, A.W. Goodman, *Univalent functions*, 1984, S.S. Miller, P.T. Mocanu, *Differential Subordinations, Theory and Applications*, 2000. There is also a well organized romanian research group founded by P.T. Mocanu, having an important impact on the international research group, more recent research directions being based on the theory of differential subordinations introduced by P.T. Mocanu and S.S. Miller ([63]).

Even if the geometric function theory is considered more like a theoretical domain, some practical applications were also derived from the theoretical studies, for example, in fluid mechanics, electrotechnics, nuclear physics and others.

An important field in geometric function theory is given by the study of integral operators on spaces of analytic functions ([7], [81]), the first mathematician who introduced an integral operator on a class of univalent functions being J.W. Alexander, in 1915. From the last century the integral operators have been studied by a lot of mathematicians, among which we mention here R. Libera, S. Bernardi, S.S. Miller, P.T. Mocanu, M.O. Reade, R. Singh, N.N. Pascu and many others. Nowadays new frontiers of integral operators are designed to stimulate interest among the young researchers in the field of geometric function theory.

Our contribution to this subject began in 2000, by working together with D. Breaz to extend some of well known classical integral operators and to prove their properties on various classes of analytic functions. The first representative papers were published in 2002 ([9] and [10]), the integral operators introduced in those papers being cited in more than 100 scientific articles written by mathematicians from the country and abroad. Over the years we published a series of papers and one book (see [5], [8]-[39], [43], [82]-[86], [96], [100], this being only a selective list). The book is related to recent studies on univalent integral operators and it came as a result of joint work together with Daniel Breaz from "1 Decembrie 1918" University of Alba Iulia and Maslina Darus from Kebangsaan University, Kuala Lumpur, Malaysia (see[27]).

We would like to outline that also some of our scientific results published and presented here were obtained as joint work with researchers from Japan, Egypt, Canada, Turkey and Romania, while taking part in various scientific events as for example, different editions of Geometric Function Theory and Applications Symposium, an itinerary conference on the aimed domain but also during the scientific seminars attended as visiting professor in Kinki University, Osaka, Japan (2010) and Kebangsaan University, Kuala Lumpur, Malaysia, 2012 and 2013. Coming from a different subject considered during PhD studies, the support received over the years from all our collaborators was both needed and appreciated. From this point of view, we would like to thank them all gratefully, begining with D. Breaz, V. Pescar and S. Owa with whom we have published most of our joint papers and also continuing with R. El-Ashwah, M, Darus, H. Srivastava, Y. Polatoglu, J. Nishiwaki, M. Acu, M.K. Aouf, N. Ularu and others.

Among the results that we got in this field, from which most of them are presented in this thesis, we mention the following:

- introduction of new integral operators as an extension of already known operators that can be recovered as particular cases from our operators,
- the study of geometric properties as univalence, convexity, starlikeness for some integral operators,
- the study of preserving class properties for some integral operators,
- extension of some Becker type univalence criteria for integral operators,
- the study of some classes of analytic functions taking into account various aspects as for example the behaviour of Hadamard product on those classes,
- coefficients estimates for some classes of analytic functions,
- distorsion theorems for some classes of analytic functions.

These results and also some new and not published yet ideas are presented here, in the habilitation thesis, in the main chapter but also in the chapter dedicated to our future research plans.

The thesis is structured into three chapters, the main one being the second chapter which contains the published work conducted by the candidate, in this field, a chapter which is supported by the first one, giving the preliminaries and continued by the last one, related to future plans.

Chapter 1 comprises preliminary instruments that will be further used for deriving our results, as the definition of some well known classes of analytic functions. More precisely, the univalent, starlike, convex and some other type of functions are recalled here together with some of their properties. A set of integral operators used to obtain our operators is also presented, together with some related results. We would like to mention that the chapter is focused strictly on those classical results which are most used in the next chapters.

Chapter 2 is dedicated to the contribution of the authors in the field of geometric function theory and is divided into eighteen sections. At the beginning of each section we mention the papers where the results are contained. Also, we would like to mention that some of the proofs and some of the secondary results (as some of the corollaries) are omited but all of them can be found in the papers that are already published.

In the Section 2.1, we present the first four of the general integral operators that we have introduced over the years, having as basis some of the classical well known operators, but using more than one function in the construction of them. Our operators cover some classical operators as those of Kim and Merkes, respectively, Pfaltzgraff. For these operators, we obtained univalence criteria which generalize the univalence criteria given by V. Pescar, respectiely V. Pescar and S. Owa. The paper in which we published the result, [9], has over 100 citations, being a reference for other new integral operators that in the meantime were defined by us or by other mathematicians.

Within the Section 2.2, we disscused some starlikeness conditions obtained by us for the Bernardi operator and for another general integral operator which covers both Bernardi and Alexander operators.

Two univalence criteria are proved in the Section 2.3, for a general integral operator defined as a generalization for n functions of an operator given by V. Pescar. Both ours and Pescar operators are particular cases of Mocanu-Miller-Reade operator. The operator was studied on a class of univalent functions, introduced by Ozaki and Nunokawa.

Convexity properties for a general integral operator of Kim-Merkes type were presented in the Section 2.4, by considering three special classes of univalent functions, given by Stankiewicz and Wisniowska and two, respectively by Ronning.

In the Section 2.5, a univalence criterion is studied for a general integral operator introduced by Senivasagan and Breaz, on the subclass of univalent functions, defined by Ozaki and Nunokawa.

For a general integral operator of Pfaltzgraff type, some class preserving properties are given in the Section 2.6, taking into account the following type of functions: univalent, starlike, convex, convex of a given order and respectively, uniformly convex functions.

In the Section 2.7, we studied the behaviour of two general integral operators, of Kim-Merkes and Pfaltzgraff type, on some classes of analytic functions of complex order and real type, given by B. Frasin.

We found the convexity order and some coefficient estimates for two general integral operators, on some class of convex functions related to a hyperbola, in the Section 2.8.

In the Section 2.9, we presented a result regarding the univalence for a general integral operator introduced by us, for which the number of functions that compose the operator depends on a complex number.

The behaviour of the general integral operator introduced by Senivasagan and Breaz is studied with respect to the univalence, on the subclasses of univalent functions $S(\alpha)$ and $T_{2,\mu}$, in the Section 2.10.

In the Section 2.11, we studied the convexity of two of our general integral operators, on the classes of some special analytic functions.

Three univalence criteria for a general integral operator built on two sets of functions, respectively regular and Caratheodory functions are presented in the Section 2.12.

Kudriasov type sufficient univalence conditions for two of our general integral operators (Kim-Merkes type and Pfaltzgraff type) were presented in the Section 2.13.

Section 2.14 contains coefficients estimates and modified Hadamard product properties for some analytic functions, p-valent, with negative coefficients. These extend previous results obtained by us on classes of starlike, respectively convex functions of order α, p-valent, with negative coefficients, defined with a differential operator.

In the Section 2.15, we obtain new conditions of univalence for two general integral operators, T_n and B_n, by applying the improvement of Becker univalence criterion, obtained by Pascu in the paper [76]. Also, a lemma given by Mocanu and Şerb in the paper [66], will be used to get some parts of the results.

Section 2.16 presents a new class of analytic and p-valent functions involving higher-order derivatives. For this p-valent function class, we derive several interesting properties including coefficient inequalities, distortion theorems, extreme points, and the radii of close-to-convexity, starlikeness and convexity. Several applications involving an integral operator are also considered. Finally, we obtain some results for the modified Hadamard product of the functions belonging to the p-valent functions class introduced here.

Using the Möbius transformations, some properties and examples of fractional calculus are presented in the Section 2.17.

In the Section 2.18, applying the extremal function for the subclass of analytic functions, $\mathcal{S}^*(\alpha)$, new classes $\mathcal{P}^*(\alpha)$ and $\mathcal{Q}^*(\alpha)$ are considered using certain subordinations. The object of this section is to present some interesting properties for $f(z)$ belonging to these classes.

Chapter 3 includes a perspective plan for the present and future projects in scientific research and profesional career of the author. We will continue our research in the field of geometric function theory, both on the study of the integral operators and the study of some classes of analytic functions. At the same time, we intend to maintain focus on the certain applications of spline functions in statistics which was the other field, aimed in our scientific work, during PhD research studies. We also have in view to organize some scientific seminars for PhD students that are interested in the field of geometric function theory and to write a scientific monograph related to our contributions in this domain.

Regarding the research goals, motivated by the recent results in the field of geometric function theory and willing to extend our previous work described in the Chapter 2, we will focuse on three general research directions, namely:

- study of new geometric properties for the operators considered in this thesis with respect to their univalence (research direction A),

- construction of new integral operators that cover the already known operators as particular cases (research direction B),

- construction of the classes of analytic functions having interesting geometric properties (research directions C).

Within the research direction A, we aim to extend the results that we have already obtained for the integral operators J_1 - J_8, most of them on univalence (see Chapter 2, where various univalence conditions were obtained), by investigating other properties of the operators, as convexity and starlikeness for example. In order to approach the study of these operators with respect to other properties, we will consider some particular classes of analytic functions. For the research direction B, we have in mind to investigate the existence of each new integral operator (to be well defined), to find other motivation of the operators, besides their generality, taking into account possible geometric properties and some particular interesting examples and finally to investigate geometric properties of the operators. Related to the research direction C, for each of the new introduced classes, we have in view to study at least the following lines: finding examples of functions that prove the nontriviality, study of Hadamard product on those classes (or some modified version of Hadamard product), characterization of the classes by finding coefficients estimates, and respectively, finding of class preserving properties for some integral operators. All of these research directions are briefly presented in the Section 3.1, by mentioning some problems to solve, particular examples of study and the approach methods that will be considered.

Some of the research items which are part of our current work are given in the last two sections of the chapter. Thus, on the short term, we aim to continue the joint project started with V. Pescar and D. Breaz, on the subject of the applications of the univalence of some integral operators in the field of fluid mechanics. Some results concerning the univalence of the inverse boundary problem solution, obtained together with V. Pescar are described in the Section 3.2. Also, we are about to finish the research project started together with S. Owa, J. Nishiwaki and D. Breaz, related to the study of some new classes of analytic functions using methods based on differential subordinations. These classes are defined starting from the classical definitions of starlike and respectively, convex functions and some part of the results are already accepted for publication, being described in the Section 3.3.

Other lines of work that we aim to follow are related to: extending of other type of univalence criteria from functions to integral operator, in the same manner as we have worked far now with Pascu criterion in the results presented in Chapter 2, study of some integro-differential operators, the analysis of already obtained results through the extremal function issue, finding some applications for the theoretical results obtained (as it is started in the Section 3.2) and using of specialized software to outline the geometric properties of some integral operators mapping.

Rezumat

În prezenta teză de abilitare, sunt descrise rezultatele semnificative obţinute de către autor, după obţinerea titlului de doctor în matematică, la Universitatea Babeş-Bolyai, Cluj-Napoca, în anul 2004. Dorim să precizăm că teza de doctorat a fost dedicată unui subiect de cercetare diferit, din cadrul analizei matematice, şi anume, *Metode numerice bazate pe funcţii spline, aplicate în statistică*, subiect bazat pe analiză funcţională şi calcul numeric. Rezultatele de cercetare prezentate aici se referă la *teoria funcţiilor de o variabilă complexă*, un subiect clasic din analiza matematică, încă atractiv pentru mulţi matematicieni din ţară şi străinătate.

Teoria funcţiilor de o variabilă complexă a debutat la mijlocul secolului trecut ca un subdomeniu al matematicii, respectiv al analizei complexe. Un subiect important este *teoria geometrică a funcţiilor analitice*, în care unul dintre obiective este acela de a da interpretări geometrice unor condiţii exprimate analitic, astfel că rigoarea raţionamentului analitic este strâns legată cu intuiţia. Profitând de această dualitate, se poate combina demonstraţia analitică şi intuiţia geometrică, pentru a studia diferite clase de funcţii.

Printre proprietăţile care au fost studiate pentru funcţiile de o variabilă complexă, se numără aşa numita proprietate de univalenţă. O funcţie analitică (olomorfă) şi injectivă pe domeniul $\mathcal{U}$, $\mathcal{U} \subset \mathbb{C}$, $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ se numeşte funcţie univalentă în $\mathcal{U}$. Condiţia $f'(z) \neq 0, \forall z \in \mathcal{U}$, este doar o condiţie necesară de univalenţă, asigurând o univalenţă locală. Obiectivul este acela de a găsi condiţii suplimentare care să asigure univalenţă funcţiei f, în $\mathcal{U}$. Prin urmare, este de dorit să obţinem condiţii necesare şi suficiente de univalenţă. Din punct de vedere geometric, funcţiile univalente sunt legate de transformările conforme.

Fiind unul dintre conceptele de bază în teoria geometrică a funcţiilor, funcţia univalentă (olomorfă şi injectivă) a fost studiată, încă din 1907, de către P. Koebe. Timp de un secol, teoria univalentă a funcţiilor s-a dezvoltat considerabil. Condiţii necesare şi suficente au fost obţinute pentru prima dată, în 1931 de către Gh. Călugareanu, după care univalenţa a fost studiată de mulţi alţi matematicieni, printre care îi amintim aici pe Z. Nehari, C. Pomerenke, A.W. Goodman, G. M. Goluzin, etc. ([42], [65])). Amintim de asemenea câteva lucrări clasice în domeniu, cum ar fi *Conformal mappings*, 1952, L.V. Ahlfors, *Conformal invariants, Topics in Geometric funcţion Theory*, 1973, Ch. Pommerenke, *Univalent funcţions*, 1975, A.W. Goodman, *Univalent functions*, 1984, S.S. Miller, P.T. Mocanu, *Differential Subordinations, Theory and Applications*, 2000. În România există o importantă şcoală de cercetare în domeniu, fondată de P.T. Mocanu, având un impact important asupra şcolii internaţionale de cercetare. Printre direcţiile relativ recente de cercetare o amintim pe cea bazată pe subordonări diferenţiale, introduse de P.T. Mocanu şi S.S. Miller ([63]).

Cu toate că teoria geometrică a funcţiilor este considerată un domeniu teoretic, există şi câteva aplicaţii practice ale acesteia în domenii precum mecanica fluidelor, electrotehnica, fizica nucleară şi altele.

Un subiect important de cercetare în acest domeniu este cel dat de studiul operatorilor integrali pe spaţii de funcţii analitice ([7], [81]). Primul matematician care a introdus un operator integral pe clase de funcţii univalente a fost J.W. Alexander, în 1915. În ultimul secol, operatorii integrali au fost studiaţi de mai mulţi matematicieni, printre care îi amintim aici pe R. Libera, S. Bernardi, S.S. Miller, P.T. Mocanu, M.O. Reade, R. Singh, N.N. Pascu şi alţii. În prezent, noi aspecte ale operatorilor integrali stimulează interesul tinerilor cercetători din domeniul teoriei geometrice a funcţiilor.

Contribuţia noastră la acest domeniu a început în 2000, prin colaborarea cu D. Breaz împreună cu care am lucrat la extinderea unor operatori integrali cunoscuţi şi am demonstrat proprietăţile noilor operatori introduşi, pe diferite clase de funcţii analitice. Primele rezultate semnificative le-am obţinut în 2002 ([9] şi [10]), operatorii integrali introduşi în acele lucrări fiind citaţi în peste 100 de articole ştiinţifice scrise de matematicieni din ţară şi străinătate. De-a lungul anilor am publicat în acest domeniu o serie de lucrări şi o carte (see [5], [8]-[39], [43], [82]-[86], [96], [100], aceasta fiind doar o listă selectivă). Cartea conţine studii recente pe operatori integrali univalenţi, şi a fost scrisă ca urmare a unei colaborări cu Daniel Breaz de la Universitatea "1 Decembrie 1918" din Alba Iulia şi Maslina Darus de la Universitatea Kebangsaan, din Kuala Lumpur, Malaezia ([27]).

Dorim să subliniem că o parte din rezultatele ştiinţifice publicate şi prezentate în această teză au fost obţinute de asemenea, în urma unor colaborări cu cercetători din Japonia, Egipt, Canada, Turcia şi România, în timpul diverselor ediţii ale conferinţei itinerante *Geometric function Theory and Applications Symposium*, dar şi în timpul seminariilor ştiinţifice la care am luat parte în cadrul vizitelor la Universitatea Kinki, din Osaka, Japonia (2010) respectiv, la Universitatea Kebangsaan, Kuala Lumpur, Malaezia, 2012 şi 2013. Având în vedere că am obţinut doctoratul pe o altă temă de cercetare, sprijinul primit de-a lungul anilor de la colaboratori a fost foarte important. În acest sens, amintim aici numele colaboratorilor care ne-au sprijinit în cercetările din acest domeniu, începând cu D. Breaz, V. Pescar şi S. Owa, cu care am publicat majoritatea lucrărilor şi continuând cu R. El-Ashwah, M, Darus, H. Srivastava, Y. Polatoglu, J. Nishiwaki, M. Acu, M.K. Aouf, N. Ularu şi alţii.

Printre rezultatele pe care le-am obţinut în acest domeniu, majoritatea prezentate în această teză, menţionăm următoarele:

- introducerea unor operatori integrali ca extensie a unor operatori deja cunoscuţi care se regăsesc ca şi cazuri particulare,
- studiul proprietăţilor geometrice precum univalenţă, convexitate, stelaritate pentru anumiţi operatori integrali,
- studiul unor proprietăţi de conservare a unor clase de către operatorii integrali,
- extinderea criteriului de univalenţă de tip Becker la operatori integrali,
- studiul unor clase de funcţii analitice luând în considerare diverse aspecte ca spre exemplu, produsul Hadamard pe aceste clase,
- estimări de coeficienţi pentru anumite clase de funcţii analitice,
- teoreme de distorsiune pentru anumite clase de funcţii analitice.

Aceste rezultate precum şi alte câteva idei încă nepublicate sunt prezentate aici în teza de abilitare, în capitolul principal dar şi în capitolul dedicat planului de cercetare.

Teza este structurată pe trei capitole, capitolul principal fiind capitolul doi care conţine contribuţia autorului la acest domeniu, susţinut de primul capitol în care sunt date câteva noţiuni şi rezultate suport şi urmat de cel de-al treilea, legat de planurile de cercetare.

Capitolul 1 conţine câteva concepte şi rezultate preliminare necesare în susţinerea prezentării rezultatelor proprii, mai precis, definiţiile unor clase de funcţii analitice deja cunoscute, ca: funcţii univalente, stelate, convexe şi altele, precum şi câteva proprietăti ale acestora. Sunt amintiţi de asemenea, principalii operatori integrali utilizaţi în obţinerea operatorilor propuşi de noi, împreună cu câteva rezultate de bază. Dorim să menţionam că acest capitol este orientat strict pe acele rezultate clasice care au fost cel mai des utilizate în Capitolul 2.

Capitolul 2 este dedicat contribuţiilor aduse de către autor în domeniul teoriei geometrice a funcţiilor şi este structurat pe optsprezece secţiuni. La începutul fiecărei secţiuni sunt menţionate lucrările în care rezultatele descrise au fost publicate. Au fost omise anumite demonstraţii şi rezultate secundare dar acestea pot fi găsite în lucrările publicate, menţionate în lista bibliografică.

În Secţiunea 2.1, prezentăm patru operatori integrali generali, având ca punct de pornire câţiva operatori integrali cunoscuţi dar fiind construiţi pe mai mult de o funcţie. Operatorii introduşi de către noi acoperă operatori clasici precum cei daţi de Kim şi Merkes, respectiv de Pfaltzgraff. Pentru aceşti operatori, am obţinut criterii de univalenţă care generalizează criteriul de univalenţă dat de V. Pescar. Articolul în care aceste rezultate au fost publicate, [9], a înregistrat peste 100 citari, fiind un articol de referinţă în domeniul operatorilor integrali generali, introduşi în ultimii ani de diverşi matematicieni.

În Secţiunea 2.2, am discutat câteva condiţii de stelaritate pentru operatorul lui Bernardi şi pentru un alt operator integral general care acoperă operatorii clasici daţi de Bernardi şi Alexander.

Două criterii de univalenţă sunt demonstrate în Secţiunea 2.3, pentru un operator integral general, definit ca o generalizare pentru n funcţii, a unui operator dat de V. Pescar. Atât operatorul introdus de noi cât şi cel dat de Pescar sunt operatori de tipul Mocanu-Miller-Reade. Operatorul a fost studiat pe o clasă de funcţii univalente introdusă de Ozaki şi Nunokawa.

Câteva proprietăţi de convexitate pentru un operator integral general de tip Kim-Merkes sunt prezentate în Secţiunea 2.4, considerând trei clase de funcţii univalente, introduse de Stankiewicz şi Wisniowska, respectiv de Ronning.

În Secţiunea 2.5, se studiază un criteriu de univalenţă pentru un operator integral general introdus de Senivasagan şi Breaz, pe clasa de funcţii univalente introdusă de Ozaki şi Nunokawa.

Câteva proprietăţi de conservare a clasei pentru un operator integral general de tip Pfaltzgraff sunt date în Secţiunea 2.6, considerând următoarele clase de funcţii: univalente, stelate, convexe, convexe de un anumit ordin, respectiv uniform convexe.

În Secţiunea 2.7, am studiat comportamentul a doi operatori integrali generali de tip Kim-Merkes şi Pfaltzgraff pe clase de funcţii analitice de ordin complex şi tip real, introduse de B. Frasin.

În Secţiunea 2.8 am găsit ordinul de convexitate şi estimări pentru coeficienţii a doi operatori integrali generali, pe o anumită clasă de funcţii convexe, definite în conexiune cu o hiperbolă.

În Secţiunea 2.9, am prezentat un rezultat referitor la univalenţa unui operator integral general introdus de noi, pentru care numărul de funcţii aflate în componenţă este definit prin intermediul unui număr complex.

Comportamentul operatorului integral general introdus de Senivasagan şi Breaz este studiat în raport cu univalenţa pe subclasele de funcţii univalente $S(\alpha)$ si $T_{2,\mu}$, în Secţiunea 2.10.

În Secţiunea 2.11, am studiat convexitatea a doi operatori integrali generali, pe anumite clase de funcţii analitice.

Trei criterii de univalenţă au fost obţinute pentru un operator integral general construit pe două tipuri de funcţii, regulare respectiv de tip Caratheodory, rezultatele fiind prezentate in Secţiunea 2.12.

Condiţii suficiente de univalenţă de tip Kudriasov pentru doi dintre operatorii integrali generali introdusi de noi, unul de tip Kim-Merkes şi celălalt de tip Pfaltzgraff au fost prezentate în Secţiunea 2.13.

Secţiunea 2.14 conţine estimări de coeficienţi şi proprietăţi legate de produsul Hadamard modificat pentru un anumit tip de funcţii analitice, p-valente, cu coeficienţi negativi. Acestea extind alte rezultate introduse de noi pentru clase de funcţii stelate, respectiv convexe de ordin α, p-valente, cu coeficienţi negativi, definite cu ajutorul unui operator diferenţial.

În Secţiunea 2.15, obţinem noi condiţii de univalenţă pentru doi operatori integrali generali, T_n si B_n, aplicând o versiune a criteriului de univalenţă al lui Becker, dată de Pascu în [76]. De asemenea, este folosit şi rezultatul dat de Mocanu şi Şerb în [66].

Secţiunea 2.16 prezintă o nouă clasă de funcţii analitice şi p-valente bazate pe derivate de ordin multiplu. Pentru această clasă de funcţii p-valente obţinem câteva proprietăţi interesante incluzând inegalităţi pentru coeficienţi, teoreme de distorsiune, puncte de extrem, raze de aproape convexitate, stelaritate şi convexitate. De asemenea, pe această clasă, am prezentat şi câteva aplicaţii ale unui operator integral. În final, obţinem şi câteva rezultate referitoare la produsul Hadamard modificat pe clasa de funcţii p-valente propusă.

Folosind transformări Möbius obţinem câteva proprietăţi şi exemple pe calcul fracţional, în Secţiunea 2.17.

În Secţiunea 2.18, aplicând funcţia extremală a subclasei de funcţii analitice, $\mathcal{S}^*(\alpha)$, şi folosind definiţia subordonărilor diferenţiale, sunt introduse noi clase, $\mathcal{P}^*(\alpha)$ şi $\mathcal{Q}^*(\alpha)$, pentru care prezentăm câteva proprietăţi.

Capitolul 3 include un plan pentru cercetările curente şi viitoare, în domeniul ştiinţific şi profesional. Ne propunem să continuăm cercetările în domeniul teoriei geometrice a funcţiilor, atât pe operatori integrali, cât şi pe diverse clase de funcţii analitice. În acelaşi timp, vom continua să acordăm interes şi aplicaţiilor bazate pe funcţii spline în statistică, acesta fiind cel de-al doilea domeniu de interes, pe care a fost elaborată teza de doctorat. Avem în vedere şi organizarea unor seminarii ştiintifice dedicate doctoranzilor din domeniul teoriei geometrice a funcţiilor, precum şi elaborarea unei monografii în domeniu, care să conţină rezultatele proprii.

În ce priveşte obiectivele de cercetare, motivaţi de rezultatele recente din domeniul teoriei geometrice a funcţiilor şi dorind să continuăm propriile cercetări, descrise în Capitolul 2, ne vom orienta asupra a trei direcţii generale de cercetare şi anume:

- studiul unor noi proprietăţi geometrice pentru operatorii consideraţi în această teză, în raport cu univalenţa (direcţia de cercetare A),

- construcţia unor noi operatori integrali care acoperă operatorii deja cunoscuţi ca şi cazuri particulare (direcţia de cercetare B),

- construcţia de clase de funcţii analitice având proprietăţi geometrice interesante (direcţia de cercetare C).

În cadrul primei direcţii de cercetare, A, urmărim să extindem rezultatele pe care le-am obţinut pentru operatorii integrali J_1 - J_8, majoritatea pe univalenţă (în Capitolul 2), investigând şi alte proprietăţi ale operatorilor ca de exemplu, convexitatea şi stelaritatea. Ca abordare, vom considera anumite clase de funcţii analitice pe care vom studia comportamentul operatorilor respectivi. Pentru direcţia de cercetare B, intenţionăm să analizăm existenţa fiecărui operator introdus, în sensul bine definirii acestuia şi să găsim şi alte motivaţii dincolo de generalitatea lor, luând în considerare posibile aplicaţii şi câteva exemple particulare, iar în cele din urmă să investigăm proprietăţile acestora. Legat de direcţia de cercetare C, pentru fiecare clasă de funcţii introdusă, avem în vedere cel puţin următoarele linii de studiu: găsirea unor exemple de funcţii care să dovedească netrivialitatea, studiul produsului Hadamard (sau versiuni modificate) pe aceste clase, caracterizarea prin estimări de coeficienţi, şi respectiv, găsirea unor proprietăţi de conservare a claselor pentru anumiţi operatori integrali. Toate aceste direcţii de cercetare sunt prezentate pe scurt în Secţiunea 3.1, menţionând problemele generale care necesitâ a fi rezolvate, exemple concrete de probleme precum şi metodele de rezolvare care urmează a fi abordate.

Anumite rezultate care fac parte din munca de cercetare curentă sunt prezentate în ultimele două secţiuni ale capitolului. Astfel, pe termen scurt, ne propunem să finalizăm proiectul de cercetare început cu V. Pescar şi D. Breaz, în direcţia găsirii de aplicaţii pentru operatori integrali univalenţi în domeniul mecanicii fluidelor. Câteva rezultate referitoare la univalenţa soluţiei problemei inverse pe frontieră, obţinute împreună cu V. Pescar, sunt descrise în Secţiunea 3.2. De asemenea, ne propunem să finalizăm proiectul de cercetare început cu S. Owa, J. Nishiwaki si D. Breaz, legat de studiul unor clase de funcţii analitice utilizând subordonări diferenţiale. Aceste clase sunt construite pornind de la definiţia stelarităţii respectiv a convexităţii, anumite rezultate deja acceptate spre publicare, fiind descrise în Secţiunea 3.3.

Alte direcţii de cercetare pe care le avem în vedere sunt legate de următoarele aspecte: extinderea altor criterii de univalenţă de la funcţii la operatori integrali, în aceeaşi manieră în care am lucrat cu criteriul de univalenţă dat de Pascu, în cadrul rezultatelor prezentate în Capitolul 2, studiul unor operatori integro-diferenţiali, analiza rezultatelor deja obţinute prin prisma găsirii funcţiilor extremale, găsirea unor aplicaţii pentru rezultatele teoretice obţinute (ca spre exemplu, cea din Secţiunea 3.2) şi utilizarea unor soft-uri specializate, în vederea evidenţierii unor proprietăţi geometrice ale imaginilor operatorilor integrali.

Chapter 1

Basic concepts

1.1 Classes of analytic functions

In this section we recall definitions and properties of those classes of analytic functions that will be of interest in the whole thesis. (see Mocanu et al, [65])

Definition 1.1.1. *We consider and denote by* $\mathcal{A}$, **the class of analytic functions** *(holomorphic) in the open unit disk,* $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$, *as the class of functions having the form*

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1.1}$$

normalized by the conditions $f(0) = f'(0) - 1 = 0$ *(regular function).*

One of the main concept in geometric function theory is **the univalent function** (holomorphic and injective), concept that was studied for the first time by P. Koebe, in 1907).

Definition 1.1.2. *An analytic (holomorphic) and injective function on the domain* $\mathcal{U}$, $\mathcal{U} \subset \mathbb{C}$, $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ *is called an univalent function on* $\mathcal{U}$. *We denote by* $\mathcal{S}$, $\mathcal{S} \subset \mathcal{A}$, **the class of univalent functions** *in the open unit disk.*

Remark 1.1.3. (see [65]) The necessary condition for a function to be univalent is to have not nule derivative, $f'(z) \neq 0, \forall z \in \mathcal{U}$.

Necessary and sufficient conditions for univalence were obtained for the first time in 1931 by Gh. Calugareanu, after that being studied by many authors, among we recall here

Z. Nehari, C. Pomerenke, A.W. Goodman, G. M. Goluzin, etc. ([42], [65])), more recent research directions being based on the theory of differential subordinations introduced by P.T. Mocanu and S.S. Miller ([63]).

Definition 1.1.4. *Let f, g be holomorphic functions. We say that f is subordinated to g, $f \prec g$ if there exists a holomorphic function w, with $w(0) = 0$ and $|w(z)| < 1$, $z \in U$, such that $f(z) = g[w(z)]$, $z \in U$.*

Remark 1.1.5. (see [65]) For f, g holomorphic functions and g univalent function, it holds: $f \prec g \Leftrightarrow f(0) = 0$ and $f(U) \subseteq g(U)$.

In what follows, we recall some univalence criteria and other usefull lemmas that we need for proving the results from the next chapters.

Lemma 1.1.6. (Ozaki-Nunokawa Lemma, [81]) *If $f \in \mathcal{A}$ satisfies the condition*

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| \leq 1, z \in \mathcal{U} \tag{1.1.2}$$

then f is univalent in $\mathcal{U}$.

Lemma 1.1.7. (Becker Lemma, [81]) *If the function f is regular in the open unit disk $\mathcal{U}$, $f(z) = z + a_2 z^2 + \ldots$ and*

$$\left(1 - |z|^2\right)\left|\frac{z f''(z)}{f'(z)}\right| \leq 1 \tag{1.1.3}$$

for all $z \in \mathcal{U}$, then the function f is univalent in $\mathcal{U}$.

Lemma 1.1.8. (Ahlfors-Becker Lemma, [81]) *Let c be a complex number, $|c| \leq 1, c \neq -1$. If $f(z) = z + a_2 z^2 + \ldots.$ is a regular function in $\mathcal{U}$ and*

$$\left|c\,|z|^2 + \left(1 - |z|^2\right)\frac{z f''(z)}{f'(z)}\right| \leq 1, \tag{1.1.4}$$

for all $z \in \mathcal{U}$, then the function f is univalent in $\mathcal{U}$.

Lemma 1.1.9. (Generalized Schwarz Lemma, [81]) *Let f be a regular function in the disk having the radius R, $\mathcal{U}_R = \{z \in C; |z| < R\}$, with $|f(z)| < M$, M fixed. If f has in $z = 0$ one zero with multiplicity $\geq m$, then*

$$|f(z)| < \frac{M}{R^m} |z|^m, \quad z \in \mathcal{U}_R \tag{1.1.5}$$

the equality (for $z \neq 0$) takes place only for $f(z) = e^{i\theta} \frac{M}{R^m} z^m$,, where θ is a constant.

Among the classes of univalent functions, the most used are those having geometric properties, the class of convex functions, respectively, the class of starlike functions. We recall here the definitions of these classes and also the definitions of some other classes of analytic functions by using their analytic characterization (see [65]).

Definition 1.1.10. *We call the class of starlike functions of order α, $0 \leq \alpha < 1$, the class of functions satisfying the following analytic condition:*

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : \mathrm{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, z \in \mathcal{U} \right\}. \tag{1.1.6}$$

Remark 1.1.11. The class of starlike functions of order α was introduced by Robertson. For $\alpha = 0$, we have the class of starlike functions introduced by Alexander (see [65]).

Definition 1.1.12. *We call the class of convex functions of order α, $0 \leq \alpha < 1$, the class of functions satisfying the following analytic condition:*

$$\mathcal{K}(\alpha) = \left\{ f \in \mathcal{A} : \mathrm{Re}\left\{ \frac{zf''(z)}{f'(z)} + 1 \right\} > \alpha, z \in \mathcal{U} \right\}. \tag{1.1.7}$$

Remark 1.1.13. The class of convex functions of order α was introduced by Robertson. For $\alpha = 0$, we have the class of convex functions introduced by Study (see [65]).

Definition 1.1.14. *We call the class of Caratheodory functions, the class of functions satisfying the following analytic condition*:

$$\mathcal{P} = \{ f \text{ analytic} : f(0) = 1, Re\ f(z) > 0, z \in \mathcal{U} \}. \tag{1.1.8}$$

The following lemma (**Mocanu and Şerb Lemma**) constitutes a criterion for a function to be in a subclass of starlike function, hence it is also a criterion of starlikeness and consequently, a criterion of univalence.

Lemma 1.1.15. (Mocanu and Şerb Lemma, ([66]) *Let* $M_0 = 1.5936...$ *the positive solution of equation*

$$(2-M)e^M = 2. \tag{1.1.9}$$

If $f \in \mathcal{A}$ *and*

$$\left|\frac{f''(z)}{f'(z)}\right| \le M_0, \quad z \in U, \tag{1.1.10}$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1, \quad z \in U. \tag{1.1.11}$$

The bound M_0 *is sharp.*

Lemma 1.1.16. (Nehari Lemma, ([68]) *If the function* g *is regular in* U *and* $|g(z)| < 1$ *in* U*, then for all* $\xi \in U$ *and* $z \in U$*, the following inequalities hold:*

$$\left|\frac{g(\xi) - g(z)}{1 - \overline{g(z)}g(\xi)}\right| \le \frac{|\xi - z|}{|1 - \bar{z}\xi|}, \tag{1.1.12}$$

$$|g'(z)| \le \frac{1 - |g(z)|^2}{1 - |z|^2}. \tag{1.1.13}$$

The equalities hold only in the case $g(z) = \frac{\epsilon(z+u)}{1+\bar{u}z}$*, where* $|\epsilon| = 1$ *and* $|u| < 1$.

Remark 1.1.17. (Nehari Remark, ([68]) For $z = 0$, from the inequality (1.1.12), we have

$$\left|\frac{g(\xi) - g(0)}{1 - \overline{g(0)}g(\xi)}\right| \le |\xi| \tag{1.1.14}$$

and hence

$$|g(\xi)| \le \frac{|\xi| + |g(0)|}{1 + |g(0)|\,|\xi|}. \tag{1.1.15}$$

Considering $g(0) = a$ and $\xi = z$, we have

$$|g(z)| \le \frac{|z| + |a|}{1 + |a|\,|z|}. \tag{1.1.16}$$

for all $z \in U$.

Lemma 1.1.18. (Kudriasov Lemma, [65]) *Let $f \in \mathcal{A}$. If*

$$\left|\frac{f''(z)}{f'(z)}\right| \leq K, \quad z \in U, \tag{1.1.17}$$

for all $z \in U$, where $K \cong 3.05$ (see the Remark 1.1.19), then the function f is univalent in U.

Remark 1.1.19. (Kudriasov Remark, [65]) The constant K is a solution of the ecuation,

$$8\left[x\left(x-2\right)^3\right]^{\frac{1}{2}} - 3\left(4-x\right)^2 = 12. \tag{1.1.18}$$

The Kudriasov result is not sharp, but the maximum value M for which the Kudriasov condition implies univalence is proved to be $M \in [K, \pi]$, since the function $f(z) = e^{\lambda \cdot z}$ is univalent if and only if $|\lambda| \leq \pi$ (see [65]).

Lemma 1.1.20. (Mocanu Lemma, [65]) *Let be $f \in \mathcal{A}$. If*

$$\left|\frac{f''(z)}{f'(z)}\right| \leq M, \quad z \in U, \tag{1.1.19}$$

for all $z \in U$, where $M \cong 2.83$, then the function f is starlike in U.

Remark 1.1.21. (Mocanu Remark, [65]) The constant M is $M = \sqrt{1+y_0^2}$, where y_0 is the smallest positive root of the equation,

$$ysiny + cosy = \frac{1}{e}. \tag{1.1.20}$$

The same criterion of starlikeness (and consequently, criterion of univalence) was obtained by V. Anisiu and P.T. Mocanu in the paper [3], using different methods of proving.

1.2 Integral operators

An important field in geometric function theory is given by the study of integral operators on spaces of analytic functions), the first mathematician who introduced an integral operator on a class of univalent functions being J.W. Alexander, in 1915.
In this section we recall some well known integral operators that were used as a basis to define our new introduced integral operators and also, a few classical univalence criteria

that were the most used in this thesis, in order to prove our results. Other known results will be recalled within the next chapters, only in the sections where they are needed.
The integral operators bellow are mentioned by their analytic formula and a denotation that is used all over the thesis, mentioning also the mathematicians who introduced them. More details on these operators can be found in various books in the literature, as for example in the books of D. Breaz et al, [7] and [81].

- Alexander operator, 1915

$$I_1(f)(z) = \int_0^z \frac{f(t)}{t} dt \tag{1.2.1}$$

- Kim-Merkes operator (also atributed to Causey), 1963, α complex number

$$I_2(f)(z) = \int_0^z \left[\frac{f(t)}{t}\right]^\alpha dt \tag{1.2.2}$$

- Libera operator, 1965

$$I_3(f)(z) = \frac{2}{z}\int_0^z f(t)\, dt \tag{1.2.3}$$

- Bernardi operator, 1969, γ complex number

$$I_4(f)(z) = \frac{1+\gamma}{z^\gamma}\int_0^z f(t)\, t^{\gamma-1} dt \tag{1.2.4}$$

- Pfaltzgraff operator, 1975, α complex number

$$I_5(f)(z) = \int_0^z \left[f'(t)\right]^\alpha dt \tag{1.2.5}$$

- Mocanu-Miller-Reade operator, 1978, $\alpha, \beta, \gamma, \delta$, complex numbers, $\beta \neq 0$, $\alpha + \delta = \beta + \gamma, \mathrm{Re}(\alpha + \delta) > 0$, Φ, φ functions of the form $g(z) = z + \sum\limits_{k=n}^{\infty} a_k z^k$, $\Phi(z)\varphi(z) \neq 0, z \in \mathcal{U}$

$$I_6(f)(z)=\left[\frac{\beta+\gamma}{z^{\gamma}\Phi(z)}\int_0^z f^{\alpha}(t)\,\varphi(t)\,t^{\delta-1}dt\right]^{\frac{1}{\beta}} \tag{1.2.6}$$

Over the years, many mathematicians have studied the univalence of these and other integral operators, among we recall here the following two results:

Lemma 1.2.1. (N.N. Pascu univalence criterion, ([76]) *Fie $\alpha\in\mathbb{C},\mathrm{Re}\alpha>0$, $f\in\mathcal{A}$. If*

$$\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right|\le 1,\forall z\in\mathcal{U} \tag{1.2.7}$$

then for $\forall\beta\in\mathbb{C},\mathrm{Re}\beta\ge\mathrm{Re}\alpha$, we have

$$F_{\beta}(z)=\left[\beta\int_0^z t^{\beta-1}f'(t)\,dt\right]^{1/\beta}\in\mathcal{S}. \tag{1.2.8}$$

Lemma 1.2.2. (Pescar univalence criterion, ([79]) *Let α, c complex numbers, $\mathrm{Re}\alpha>0$, $|c|\le 1, c\ne -1$ and f a regular function $\mathcal{U}$. If*

$$\left|c\,|z|^{2\alpha}+\left(1-|z|^{2\alpha}\right)\frac{zf''(z)}{\alpha f'(z)}\right|\le 1, z\in\mathcal{U} \tag{1.2.9}$$

then

$$F_{\alpha}(z)=\left[\alpha\int_0^z u^{\alpha-1}f'(u)\,du\right]^{\frac{1}{\alpha}}\in\mathcal{S}. \tag{1.2.10}$$

Chapter 2

Contributions

2.1 General integral operators

In the papers [9], [10], together with D. Breaz, we introduced four integral operators which extend some of the classical operators recalled in the previous chapter, by using more than one function in the construction of the operators. For these operators, we obtained univalence criteria which generalize the univalence criteria given by V. Pescar, respectiely V. Pescar and S. Owa. The paper [9] has over 100 citations, being a reference for other new integral operators that in the meantime were defined by us or by other mathematicians. In what follows, we present these integral operators, by mentioning the type of them and also the authors who's integral operator was generalized by ours:

• Kim-Merkes type operator - generalization of Kim-Merkes operator

$$J_1(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt \qquad (2.1.1.)$$

Remark 2.1.1. The operator (1.2.2), introduced by Kim and Merkes can be obtained for $n = 1$.

• Kim-Merkes type operator - generalization of Pascu - Pescar operator

$$J_2(z) = \left[\beta \int_0^z t^{\beta-1} \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt\right]^{\frac{1}{\beta}} \qquad (2.1.2.)$$

Remark 2.1.2. For $n = 1$, we get the integral operator introduced by N.N. Pascu and V. Pescar (see [81]).

• Pfaltzgraff type operator - generalization of Pascu - Pescar, respectively Pescar- Owa operators

$$J_3(z) = \left\{ \beta \int_0^z t^{\beta-1} \cdot [f_1'(t^n)]^{\gamma_1} \cdot \ldots \cdot \left[f_p'(t^n)\right]^{\gamma_p} dt \right\}^{1/\beta} \tag{2.1.3.}$$

Remark 2.1.3. For $p = 1$, $\beta = 1$, we get the integral operator introduced by N.N. Pascu and V. Pescar, respectively, for $p = 1$, we recover the operator introduced by V. Pescar and S. Owa (see [81]).

• Pfaltzgraff type operator - generalization of Pascu, respectively Pescar-Owa operators

$$J_4(z) = \left\{ \beta \int_0^z t^{\beta-1} \cdot [f_1'(t)]^{\gamma_1} \cdot \ldots \cdot [f_n'(t)]^{\gamma_n} dt \right\}^{1/\beta} \tag{2.1.4.}$$

Remark 2.1.4. For $n = 1$, $\gamma_1 = 1$, we get the integral operator introduced by N.N. Pascu, respectively, for $n = 1$, we recover the operator introduced by V. Pescar and S. Owa (see [81]).

We recall here only one result as an example of the univalence criteria that we obtained, the others being accesible through the papers [9], [10], together with the proofs.

Theorem 2.1.5. ([9]) *Let $\alpha_n \in \mathbb{C}$, $f_n \in \mathcal{S}$, $f_n(z) = z + a_2^n z^2 + a_3^n z^3 + \ldots$, $n \in \mathbb{N}^*$. If*

$$\left| \frac{z f_n'(z) - f_n(z)}{z f_n(z)} \right| \leq 1, \ \forall n \in \mathbb{N}^*, \ \forall z \in \mathcal{U}, \tag{2.1.5}$$

$$\frac{|\alpha_1| + |\alpha_2| + \ldots + |\alpha_n|}{|\alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_n|} \leq 1, \tag{2.1.6}$$

$$|\alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_n| \leq \frac{1}{\max_{|z|\leq 1} \left[\left(1 - |z|^2\right) \cdot |z| \cdot \frac{|z|+|c|}{1+|c|\cdot|z|} \right]}, \tag{2.1.7}$$

where

$$|c| = \frac{|\alpha_1 a_2^1 + \ldots + \alpha_n a_2^n|}{|\alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_n|}, \tag{2.1.8}$$

then

$$J_1(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt$$

is univalent.

Proof. In order to prove this result it is sufficient to apply Becker's univalence criterion, Lemma 1.1.7 (see [9]) .

2.2 Starlikeness condition for Bernardi operator

In the paper [11], together with D. Breaz, we studied the starlikeness of the Bernardi operator defined in the formula (1.2.4) and we got the following main result:

Theorem 2.2.1. **([11])** *Let* $\gamma \geq 0, 0 \leq a \leq 1, -1 \leq b \leq 0$ *be real numbers and let be the function*

$$h(z) = \frac{1+az}{1+bz} + \frac{n(a-b)z}{(1+bz)(1+\gamma+(a+b\gamma)z)}. \tag{2.2.1}$$

If

$f \in A_n$, *(*$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots$*),* $z \in U$

and

$$\frac{zf'(z)}{f(z)} \prec h(z) \tag{2.2.2}$$

then

$$I_4 \in S^*\left(\frac{1-a}{1-b}\right).$$

Proof. In order to prove the starlikeness, we need to apply differential subordinatons (see [11]).

In the same paper, we considered the following more general integral operator:

$$F_\Sigma(z) = \frac{1+\sum_{i=1}^k \beta_i}{z^{\sum_{i=1}^k \beta_i}} \int_0^z \left(\prod_{i=1}^k f_i(t)\right) t^{\sum_{i=1}^k (\beta_i - 1)} dt, \beta_i \geq 0, i = \overline{1,k}. \tag{2.2.3}$$

Remark 2.2.2. It can be easily noticed that by chosing suitable values of the parameters that define the integral operator from above, we can obtain as particular cases, both operators (1.2.1) and (1.2.4), the integral operators introduced by Alexander and respectively, by Bernardi.

In the paper [11], we obtained for this general operator, a similar starlikeness condition as that presented in Theorem 2.2.1, thus the result covers starlikeness conditions for both Bernardi and Alexander operators.

2.3 The univalence of Mocanu-Miller-Reade type operator on the Ozaki-Nunokawa class

In the papers [26] and [14], together with D. Breaz and H. M. Srivastava, we proved two univalence criteria of an integral operator defined as a generalization for n functions of an operator given by V. Pescar. Both ours and Pescar operators are particular cases of Mocanu-Miller-Reade operator, (1.2.6), hence we call them here as Mocanu-Miller-Reade type operators. The operator was studied on a class of univalent functions, introduced by Ozaki and Nunokawa, defined by the formula (2.3.2). The general integral operator introduced and studied by us is defined as follows:

- Mocanu-Miller-Reade type operator - generalization of Pescar operator

$$J_5(z) = \left\{ [n(\alpha-1)+1] \int_0^z g_1^{\alpha-1}(t) \cdot \ldots \cdot g_n^{\alpha-1}(t)\, dt \right\}^{\frac{1}{n(\alpha-1)+1}} \tag{2.3.1}$$

Theorem 2.3.1. **([14])** *Let $M \geq 1$, $g_i \in \mathcal{A}$, $\forall\, i = \overline{1,n}$, $n \in \mathbb{N}^*$ satisfying Ozaki-Nunokawa condition,*

$$\left| \frac{z^2 g_i'(z)}{g_i^2(z)} - 1 \right| \leq 1,\ \forall\, z \in \mathcal{U},\ \forall\, i = \overline{1,n} \tag{2.3.2}$$

and $\alpha \in \mathbb{C}$, satisfying conditions

$$|\alpha - 1| \leq \frac{\mathrm{Re}\alpha}{n(2M+1)},$$

$$\mathrm{Re}\{n(\alpha-1)+1\} \geq \mathrm{Re}\alpha. \tag{2.3.3}$$

If

$$|g_i(z)| \leq M, \ \forall \ z \in U, \forall \ i = \overline{1,n}, \tag{2.3.4}$$

then $J_5(z)$ *is univalent.*

Proof. We apply Schwarz Lemma, 1.1.9 and N.N. Pascu univalence criterion for integral operators, Lemma 1.2.1.

Theorem 2.3.2. ([26]) *Let* $M \geq 1$, $g_i \in \mathcal{A}$, $\forall \ i = \overline{1,n}$, $n \in \mathbb{N}^*$ *satisfying Ozaki-Nunokawa condition,*

$$\left| \frac{z^2 g_i'(z)}{g_i^2(z)} - 1 \right| \leq 1, \ \forall \ z \in \mathcal{U}, \ \forall \ i = \overline{1,n} \tag{2.3.5}$$

and $\alpha \in \mathbb{R}$, *satisfying conditions*

$$1 \leq \alpha \leq \frac{(2M+1)\, n}{(2M+1)\, n - 1} \tag{2.3.6}$$

If $c \in \mathbb{C}$, *with*

$$|c| \leq 1 + \left(\frac{1-\alpha}{\alpha} \right) (2M+1)\, n \tag{2.3.7}$$

and

$$|g_i(z)| \leq M, \ \forall \ z \in U, \forall \ i = \overline{1,n}, \tag{2.3.8}$$

then $J_5(z)$ *is univalent.*

Proof. In order to prove the univalence, we apply the univalence criterion of Ahlfors-Becker type, given by Pescar, Lemma 1.2.2.

2.4 Convexity properties for a general integral operator on some classes of univalent functions

The results from this section are based on the paper [15]. Together with D. Breaz, we have studied convexity properties for the following general integral operator of Kim-Merkes type, introduced by us:

- Kim-Merkes type operator - generalization of Kim-Merkes operator

$$J_1(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt \qquad (2.4.1)$$

The following classes of univalent functions were used:

- Stankiewicz-Wisniowska class $SH(\beta)$, $\beta > 0$, ([98])

$$\mathcal{SH}(\beta) = \left\{ f \in \mathcal{S} : \left| \frac{zf'(z)}{f(z)} - 2\beta\left(\sqrt{2}-1\right) \right| < \operatorname{Re}\left\{ \sqrt{2}\frac{zf'(z)}{f(z)} \right\} + 2\beta\left(\sqrt{2}-1\right), z \in \mathcal{U} \right\} \qquad (2.4.2)$$

- SP, Ronning class

$$\mathcal{SP} = \left\{ f \in \mathcal{S} : \operatorname{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \left| \frac{zf'(z)}{f(z)} - 1 \right|, z \in \mathcal{U} \right\} \qquad (2.4.3)$$

- Ronning class $SP(\alpha, \beta)$, $\alpha > 0, \beta \in [0, 1)$

$$\mathcal{SP}(\alpha, \beta) = \left\{ f \in \mathcal{S} : \left| \frac{zf'(z)}{f(z)} - (\alpha + \beta) \right| \leq \operatorname{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} + \alpha - \beta, z \in \mathcal{U} \right\} \qquad (2.4.4)$$

Theorem 2.4.1.([15]) *Let $\alpha_i, i \in \{1, ..., n\}$ be real numbers with the property $\alpha_i > 0$, $i \in \{1, ..., n\}$ and*

$$\sum_{i=1}^{n} \alpha_i \leq \frac{\sqrt{2}}{2\beta\left(\sqrt{2}-1\right) + \sqrt{2}}. \qquad (2.4.5)$$

If $f_i \in SH(\beta)$, $i = \{1, ..., n\}$, $\beta > 0$, then the integral operator J_1 is convex.

Proof. In order to prove the convexity, we need to apply the analytic characterization of a convex function.

Theorem 2.4.2.([15]) *Let $\alpha_i, i \in \{1, ..., n\}$ be real numbers, having the property $\alpha_i > 0$, $i \in \{1, ..., n\}$ and $\sum_{i=1}^{n} \alpha_i < 1$. Daca $f_i \in \mathcal{SP}$, $i = \{1, ..., n\}$,then the integral operator J_1 is convex of order $1 - \sum_{i=1}^{n} \alpha_i$.*

Proof. We apply the analytic characterization of convex functions of a given order, (1.1.7).

Corollary 2.4.3.([15]) *The Alexander operator maps the functions from the class* $\mathcal{SP}$ *in the class of convex functions.*

Theorem 2.4.4.([15])] *Let* $\alpha_i, i \in \{1, ..., n\}$ *be the real numbers having the property* $\alpha_i > 0$, $i \in \{1, ..., n\}$ *and*

$$\sum_{i=1}^{n} \alpha_i < \frac{1}{\alpha - \beta + 1} \tag{2.4.6}$$

If $f_i \in \mathcal{SP}(\alpha, \beta)$, $i = \{1, ..., n\}$, $\alpha > 0, \beta \in [0, 1)$, *then the integral operator* J_1 *is convex of order* $(\beta - \alpha - 1)\sum_{i=1}^{n} \alpha_i + 1$.

2.5 Univalence criterion for Senivasagan-Breaz integral operator

In the paper [19], we proved an univalence criterion and some secondary results for the following Kim-Merkes type operator, introduced by Senivasagan-Breaz in 2007:

$$H_{\gamma_1,\gamma_2,...,\gamma_n,\beta}(z) = \left[\beta \int_0^z u^{\beta-1} \prod_{j=1}^{n} \left(\frac{f_j(u)}{u}\right)^{\frac{1}{\gamma_j}} du\right]^{\frac{1}{\beta}}, \tag{2.5.1}$$

where $f_j \in \mathcal{A}$, β, γ_j complex numbers, $\beta \neq 0$, $\gamma_j \neq 0$, $j = \overline{1, n}$, $n \in \mathbb{N} - \{0\}$.

The operator is studied on the class of univalent functions given by Ozaki and Nunokawa, using the condition (2.5.2). Here we recall only the main result.

Theorem 2.5.1. ([19]) *Let* $M \geq 1$, $f_j \in \mathcal{A}$, *satisfying Ozaki-Nunokawa condition,*

$$\left|\frac{z^2 f_j'(z)}{f_j^2(z)} - 1\right| \leq 1, \ \forall\, z \in \mathcal{U}, \ \forall\, j = \overline{1, n}, \tag{2.5.2}$$

β *real number, with*

$$\beta \geq \sum_{j=1}^{n} (2M + 1) / |\gamma_j| \tag{2.5.3}$$

and c, complex number. If

$$|c| \leq 1 - \frac{1}{\beta}\sum_{j=1}^{n}\frac{2M+1}{|\gamma_j|} \tag{2.5.4}$$

and

$$|f_j(z)| \leq M, \forall\, j = \overline{1,n}, \tag{2.5.5}$$

then the operator, $H_{\gamma_1,\gamma_2,...,\gamma_n,\beta}$ *is in the class* $\mathcal{S}$.

Proof. We apply Schwarz Lemma, 1.1.9 and the univalence criterion of Ahlfors-Becker type, given by Pescar, Lemma 1.2.2.

2.6 Class preserving properties for a Pfaltzgraff type general integral operator

In the paper [29], we introduced and studied together with D. Breaz and S. Owa, a new general integral operator of Pfaltzgraff type, obtaining some class preserving conditions, on various classes as univalent, starlike, convex, convex of a given order and respectively, uniformly convex functions classes.

The following operator was considered:

$$J_6(z) = \int_0^z (f_1'(t))^{\alpha_1} ... (f_n'(t))^{\alpha_n}\, dt, \alpha_i > 0. \tag{2.6.1}$$

Besides the classes recalled in the Section 1.1, we used also the class of uniform convex functions, introduced by Goodman as follows:

$$UCV = \left\{ f \in \mathcal{A} : \mathbf{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \left|\frac{zf''(z)}{f'(z)}\right|, z \in \mathcal{U} \right\}. \tag{2.6.2}$$

Next result gives an univalence criterion for the operator (2.6.1):

Theorem 2.6.1. **([29])** *Let* $\alpha_i \geq 0$, $f_i \in \mathcal{A}$, $i \in \{1, ..., n\}$, *satisfying the Kudriasov condition,*

$$\left|\frac{f_i''(z)}{f_i'(z)}\right| \le K \qquad (z \in U), i \in \{1, ..., n\}, \tag{2.6.3}$$

where $K = 3.05...$ *and* $\sum_{i=1}^{n} \alpha_i \le 1$*, then* $J_6 \in S$.

Proof. In order to prove the univalence we apply Kudriasov Lemma, 1.1.18.

In what follows, a starlikeness theorem is presented:

Theorem 2.6.2. **([29])** *Let* $\alpha_i \ge 0$*,* $f_i \in \mathcal{A}$*,* $i \in \{1, ..., n\}$*, satisfying Mocanu condition,*

$$\left|\frac{f_i''(z)}{f_i'(z)}\right| \le M \qquad (z \in U), i \in \{1, ..., n\}, \tag{2.6.4}$$

where $M = 2.83...$ *and* $\sum_{i=1}^{n} \alpha_i \le 1$*, then* $J_6 \in S^*$.

Proof. To prove the starlikness, Mocanu Lemma, 1.1.20 was used.

The integral operator that we studied preserves also the class of convex functions, as it can be seen in the next result:

Theorem 2.6.3. **([29])** *Let* $\alpha_i \ge 0$*,* $f_i \in \mathcal{K}$*,* $i \in \{1, ..., n\}$*, then* $J_6 \in \mathcal{K}$.

Proof. We apply the analythic characterization of the convexity recalled in the Section 1.1.

In the next theorem, we prove the convexity of a given order for our integral operator:

Theorem 2.6.4. **([29])** *Let* $\alpha_i \ge 0$*,* $f_i \in \mathcal{K}(\beta_i)$*,* $0 \le \beta_i < 1$*,* $i \in \{1, ..., n\}$*. If* $\sum_{i=1}^{n} \alpha_i (\beta_i - 1) + 1 \ge 0$*, then* $J_6 \in \mathcal{K}\left(\sum_{i=1}^{n} \alpha_i (\beta_i - 1) + 1\right)$.

Proof. To prove this result, we use the analythic characterization of the convexity of a given order, (1.1.7).

Finally, we found also conditions for the integral operator to be uniformly convex:

Theorem 2.6.5. **([29])** *Let* $\alpha_i \ge 0$*,* $f_i \in \mathcal{UCV}$*,* $i \in \{1, ..., n\}$*. If* $\sum_{i=1}^{n} \alpha_i \le 1$*, then* $J_6 \in \mathcal{K}\left(1 - \sum_{i=1}^{n} \alpha_i\right)$.

Proof. The analythic characterization of the uniform convexity, (2.6.2), is applied.

2.7 The behaviour of two general integral operators on functions of complex order and real type

In the paper [8], together with D. Breaz and M.K. Aouf we studied the behaviour of the following general integral operators, of type Kim-Merkes and Pfaltzgraff, on some classes of analytic functions of complex order and real type:

- Kim-Merkes type general integral operator Breaz-Breaz

$$J_1(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt \tag{2.7.1.}$$

- Pfaltzgraff type general integral operator Breaz-Owa-Breaz

$$J_6(z) = \int_0^z (f_1'(t))^{\alpha_1} \ldots (f_n'(t))^{\alpha_n}\, dt \tag{2.7.2}$$

B. Frasin introduced the so-called classes of analytic functions of complex order and real type:

- The class of starlike functions of complex order b and real type α, $S_\alpha^*(b)$, $b \in \mathbb{C} - \{0\}$, $0 \le \alpha < 1$

$$S_\alpha^*(b) = \left\{ f \in \mathcal{A} : \mathbf{Re}\left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > \alpha \right\} \tag{2.7.3}$$

- The class of convex functions of complex order b and real type α, $C_\alpha(b)$, $b \in \mathbb{C} - \{0\}$, $0 \le \alpha < 1$

$$C_\alpha(b) = \left\{ f \in \mathcal{A} : \mathbf{Re}\left\{1 + \frac{1}{b}\frac{zf''(z)}{f'(z)}\right\} > \alpha \right\} \tag{2.7.4}$$

Remark 2.7.1 i) For $b = 1$, we have $S_\alpha^*(1) = S^*(\alpha)$, the class of starlike functions of order α, respectively, $C_\alpha(1) = K(\alpha)$, the class of convex functions of order α.
ii) For $\alpha = 0$, $S_0^*(b)$, the class of starlike functions of complex order b was introduced by Nasr and Aouf.
iii) For $\alpha = 0$, $C_0(b)$, the class of convex functions of complex order b was introduced by Wiatrowski.

Using the analythic characterization for the above mentioned classes, we have the following results:

Theorem 2.7.2. ([8]) *Let* $\alpha_i, i \in \{1, ..., n\}$, *be real numbers having the property* $\alpha_i > 0$, $i \in \{1, ..., n\}$ *and* $\sum_{i=1}^{n} \alpha_i \leq 1$. *If* $f_i \in S_0^*(b)$, $i = \{1, ..., n\}$, $b \in \mathbb{C} - \{0\}$, *then the integral operator* $J_1 \in C_\gamma(b)$, *where* $\gamma = 1 - \sum\limits_{i=1}^{n} \alpha_i$.

Theorem 2.7.3. ([8]) *Let* $\alpha_i, i \in \{1, ..., n\}$, *real numbers having the property* $\alpha_i > 0$, $i \in \{1, ..., n\}$ *and* $\sum_{i=1}^{n} \alpha_i \leq 1$. *If* $f_i \in C_0(b)$, $i = \{1, ..., n\}$, $b \in \mathbb{C} - \{0\}$, *then the integral operator*$J_6 \in C_\gamma(b)$, *where* $\gamma = 1 - \sum\limits_{i=1}^{n} \alpha_i$.

2.8 Convexity order and coefficients estimates for two general integral operators, on convex functions related to a hyperbola

In the papers [100] and [32], together with N. Ularu, respectively, D. Breaz and M. Acu, we found the convexity order for two general integral operators and obtained an estimation for the first two coefficients of the operators, on the class of functions $CVH(\beta)$, $\beta > 0$ (convex functions related to a hyperbola), introduced by M. Acu and S. Owa as follows:

$$CVH(\beta) =$$
$$\left\{ f \in \mathcal{A} : \left| \frac{zf''(z)}{f'(z)} - 2\beta\left(\sqrt{2} - 1\right) + 1 \right| < \mathbf{Re}\left\{ \sqrt{2}\frac{zf''(z)}{f'(z)} \right\} + 2\beta\left(\sqrt{2} - 1\right) + \sqrt{2}, z \in \mathcal{U} \right\}. \tag{2.8.1}$$

The following two general integral operarors are considered:

- Pfaltzgraff type general integral operator Breaz-Owa-Breaz

$$J_6(z) = \int_0^z \left(f_1'(t)\right)^{\gamma_1} ... \left(f_n'(t)\right)^{\gamma_n} dt, \gamma_i > o, i = \bar{1,n}. \tag{2.8.2}$$

• Pfaltzgraff/Kim-Merkes type general integral operator Pescar

$$K(z) = \int_0^z \prod_{i=1}^n \left(f_i'(t)\right)^{\gamma_i} \left(\frac{g_i(t)}{t}\right)^{\eta_j} dt, \eta_i, \gamma_i > o, i = \bar{1, n}. \tag{2.8.3}$$

First we get a convexity of a given order criterion for each of the operators mentioned above, by applying the analythic characterization of the class $CVH\left(\beta\right)$:

Theorem 2.8.1.([32]) *If $f_i \in \mathcal{CVH}\left(\beta_i\right)$, $\beta_i > 0$, $\gamma_i > 0$, $i \in \overline{1,n}$, then the integral operator J_6 is in the class $\mathcal{K}\left(\alpha\right)$, with the condition, $0 \leq \alpha < 1$ where*

$$\alpha = 1 - \sum_{i=1}^n \gamma_i - \left(2 - \sqrt{2}\right) \sum_{i=1}^n \gamma_i \beta_i. \tag{2.8.4}$$

Theorem 2.8.2.([100]) *If $f_i \in \mathcal{CVH}\left(\beta_i\right)$, $\beta_i > 0$, $g_i \in \mathcal{S}^*\left(\alpha_i\right)$, $0 \leq \alpha_i < 1$, $\gamma_i, \eta_i > 0$, $i \in \overline{1,n}$, then the integral operator K is in the class $\mathcal{K}\left(\alpha\right)$, with, $0 \leq \alpha < 1$ where*

$$\alpha = 1 - \sum_{i=1}^n \gamma_i - \left(2 - \sqrt{2}\right) \sum_{i=1}^n \gamma_i \beta_i + \sum_{i=1}^n \eta_i \left(\alpha_i - 1\right). \tag{2.8.5}$$

The following result gives some coefficient estimates for the operator J_6, on the above mentioned class:

Theorem 2.8.3. ([32]) *Let be $f_i \in \mathcal{CVH}\left(\beta_i\right)$, $\beta_i > 0$, $i \in \overline{1,n}$,*

$$f_i(z) = z + \sum_{j=2}^{\infty} a_{i,j} z^j, i = \overline{1,n}. \tag{2.8.6}$$

If we consider the operator J_6, with $n = 1$ and the analytic form of this is

$$J_6(z) = z + \sum_{j=2}^{\infty} b_j z^j, \tag{2.8.7}$$

then:

$$|b_2| \leq \frac{1}{2} \sum_{i=1}^n \frac{1 + 4\beta_i}{(1 + 2\beta_i)}, \tag{2.8.8}$$

$$|b_3| \leq \sum_{i=1}^{n} \frac{(1+4\beta_i)(3+16\beta_i+24\beta_i^2)}{12(1+2\beta_i)^3} +$$

$$\frac{1}{3}\sum_{k=1}^{n-1}\left(\frac{1+4\beta_k}{(1+2\beta_k)}\cdot\sum_{i=k+1}^{n}\frac{1+4\beta_i}{(1+2\beta_i)}\right). \tag{2.8.9}$$

Proof. We apply the estimation obtained by M. Acu and S. Owa, for the coefficients of the functions from the class $CVH\left(\beta\right)$.

Remark 2.8.4. The estimation of the coefficient was given also for the operator$K$, in [100], for the case when both sets of functions are in the classes of type $CVH\left(\beta\right)$.

2.9 Univalence of a general integral operator based on a number of functions, related to a complex number

Together with D. Breaz and V. Pescar, we studied in some papers, various general integral operators for which the number of functions that compose the operator depends on a complex number. For these operators we obtained various univalence criteria as for example, those from [39].

We defined the following integral operator:

- Kim-Merkes type general integral operator, Breaz-Pescar-Breaz

$$J_7(z) = \left\{\eta\beta\int_0^z u^{\eta\beta-1}\left(\frac{f_1(u)}{u}\right)^{\frac{1}{\gamma_1}}\cdots\left(\frac{f_{|[Re\eta]|}(u)}{u}\right)^{\frac{1}{\gamma_{|[Re\eta]|}}}du\right\}^{\frac{1}{\eta\beta}},$$

$$\beta,\eta,\gamma_j, complex\ numbers, \gamma_j\neq 0, \beta\neq 0, Re\eta\notin[0,1)\,, j=\overline{1,|[Re\eta]|}. \tag{2.9.1}$$

For this operator, we recall here the following univalence criterion obtained in the paper [39]:

Theorem 2.9.1. ([39]) *Let $\beta,\eta,\alpha,\gamma_j$, complex numbers, $\gamma_j\neq 0, \beta\neq 0, a=Re\alpha>0, Re\eta\notin[0,1)$ and $f_j\in\mathcal{S}$, $j=\overline{1,|[Re\eta]|}$. If*

$$\sum_{j=1}^{|[Re\eta]|}\frac{1}{|\gamma_j|}\leq\frac{a}{2},\quad for\quad 0<a<\frac{1}{2} \tag{2.9.2}$$

or

$$\sum_{j=1}^{|[Re\eta]|} \frac{1}{|\gamma_j|} \leq \frac{1}{4}, \quad for \quad a \geq \frac{1}{2}, \tag{2.9.3}$$

then for every complex number, $\beta, Re\ \eta\beta \geq a$, the integral operator J_7 is in the class $\mathcal{S}$.

Proof. To prove the univalence, we apply Becker type criterion given by N.N. Pascu, Lemma 1.2.1.

2.10 General integral operators, on the subclasses of univalent functions $S(\alpha)$ and $T_{2,\mu}$

Together with V. Pescar, in the paper [83], we studied the behaviour of the general integral operator (2.10.3), introduced by Senivasagan and Breaz, on the subclasses of univalent functions $S(\alpha)$ and $T_{2,\mu}$, studied by Ozaki, Nunokawa, Yang, Liu, Singh and others.

In what follows, we recall the analytic definitions of these classes:

- The class $T_{2,\mu}$, $0 < \mu < 1$

$$\mathcal{T}_{2,\mu} = \left\{ f \in \mathcal{S}, f(z) = z + \sum_{k=3}^{\infty} a_k z^k : \left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < \mu, z \in \mathcal{U} \right\} \tag{2.10.1}$$

- The class $S(\alpha)$, $0 < \alpha \leq 2$

$$\mathcal{S}(\alpha) = \left\{ f \in \mathcal{A} : \left| \left(\frac{z}{f(z)} \right)'' \right| \leq \alpha, f(z) \neq 0, z \in \mathcal{U} \right\} \tag{2.10.2}$$

and also the formula of the general integral operator of Kim-Merkes type, introduced by Senivasagan and Breaz

$$H_{\gamma_1,\gamma_2,\ldots,\gamma_n,\beta}(z) = \left[\beta \int_0^z u^{\beta-1} \prod_{j=1}^{n} \left(\frac{f_j(u)}{u} \right)^{\frac{1}{\gamma_j}} du \right]^{\frac{1}{\beta}}, \tag{2.10.3.}$$

with $f_j \in \mathcal{A}$, β, γ_j complex numbers, $\beta \neq 0$, $\gamma_j \neq 0$, $j = \overline{1,n}$, $n \in \mathbb{N} - \{0\}$.

On the above mentioned classes, we obtained the following univalence criteria:

Theorem 2.10.1. **([83])** *Let γ_j, α be complex numbers, $\gamma_j \neq 0$, $j = \overline{1,n}$,* Re$\alpha > 0$, *M_j real pozitive numbers, $M_j > 1$ and $f_j \in \mathcal{T}_{2,\mu}$, $f_j(z) = z + a_{3j}z^3 + \ldots$, $j = \overline{1,n}$.*

If

$$|f_j(z)| \le M_j,\ \left(z \in \mathcal{U};\ j = \overline{1,n}\right) \tag{2.10.4}$$

and

$$\sum_{j=1}^{n} \frac{(\mu+1)\, M_j + 1}{|\gamma_j|} \le \mathrm{Re}\alpha, \tag{2.10.5}$$

then for any complex number, β, $\mathrm{Re}\beta \ge \mathrm{Re}\alpha$, *the function*

$$H_{\gamma_1,\gamma_2,\dots,\gamma_n,\beta}(z) = \left\{ \beta \int\limits_0^z u^{\beta-1} \left(\frac{f_1(u)}{u}\right)^{\frac{1}{\gamma_1}} \dots \left(\frac{f_n(u)}{u}\right)^{\frac{1}{\gamma_n}} du \right\}^{\frac{1}{\beta}} \tag{2.10.6}$$

is in the class $\mathcal{S}$.

Proof. We apply Schwarz Lemma 1.1.9 and the univalence criterion for integral operators given by N.N. Pascu, Lemma 1.2.1.

Theorem 2.10.2. **([83])** *Let* γ_j *be complex numbers,* $\gamma_j \ne 0$, $j = \overline{1,n}$, α *real pozitive number,* $0 < \alpha \le 2$, M_j *real pozitive numbers,* $M_j > 1$ *and* $f_j \in \mathcal{S}(\alpha)$, $f_j(z) = z + a_{2j}z^2 + a_{3j}z^3 + \dots$, $j = \overline{1,n}$.

If

$$|f_j(z)| \le M_j,\ \ \left(z \in \mathcal{U};\ j = \overline{1,n}\right) \tag{2.10.7}$$

and

$$\alpha^2 \sum_{j=1}^{n} \frac{M_j}{|\gamma_j|} + (\alpha+1)^{\frac{\alpha+1}{\alpha}} \sum_{j=1}^{n} \frac{M_j + 1}{|\gamma_j|} \le \alpha\,(\alpha+1)^{\frac{\alpha+1}{\alpha}}, \tag{2.10.8}$$

then for any complex number β, $\mathrm{Re}\beta \ge \alpha$, *the integral operator* $H_{\gamma_1,\gamma_2,\dots,\gamma_n,\beta}$ *belongs to the class* $\mathcal{S}$.

2.11 Convexity of two general integral operators on the classes of some special analytic functions

In [33], together with D. Breaz and M. Darus, we studied the convexity of two of our general integral operators, on the classes of some special analytic functions.

First, we recall the classes of analytic functions, introduced by M. Darus. It can be noticed that for $\alpha = 0$, $\beta = 1$, the classes are reduced at those introduced by Goodman, respectively, the class of uniform convex functions and the class of uniform starlike functions:

- The class of β-uniform convex functions of order α, $\beta - \mathcal{UCV}(\alpha)$, $-1 \le \alpha \le 1$, $\beta > 0$

$$\beta-\mathcal{UCV}(\alpha)=\left\{f\in\mathcal{A}:\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}-\alpha\right\}\geq\beta\left|\frac{zf''(z)}{f'(z)}\right|,z\in\mathcal{U}\right\} \tag{2.11.1}$$

- The class of β-uniform starlike functions of order α, $\beta-\mathcal{S}_p(\alpha)$, $-1\leq\alpha\leq1$, $\beta>0$,

$$\beta-\mathcal{S}_p(\alpha)=\left\{f\in\mathcal{A}:\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}-\alpha\right\}\geq\beta\left|\frac{zf'(z)}{f(z)}-1\right|,z\in\mathcal{U}\right\} \tag{2.11.2}$$

On these classes the following general integral operators are studied:

- Kim-Merkes type general integral operator Breaz-Breaz

$$J_1(z)=\int_0^z\left(\frac{f_1(t)}{t}\right)^{\gamma_1}\cdot\ldots\cdot\left(\frac{f_n(t)}{t}\right)^{\gamma_n}dt \tag{2.11.3}$$

- Pfaltzgraff type general integral operator Breaz-Owa-Breaz

$$J_6(z)=\int_0^z\left(f_1'(t)\right)^{\gamma_1}\ldots\left(f_n'(t)\right)^{\gamma_n}dt \tag{2.11.4}$$

Applying the analythic characterization for the above mentioned classes, we get the following convexity properties (convexity of a given order):

Theorem 2.11.1.([33]) *If* $f_i\in\beta_i-\mathcal{UCV}(\alpha_i)$, $-1\leq\alpha_i\leq1$, $\beta_i>0$, $\gamma_i>0$, $i\in\{1,...,n\}$ *and* $\sum\limits_{i=1}^n\gamma_i\leq\frac{1}{2}$, *then* $J_6\in\mathcal{K}(\rho)$,*where* $\rho=1+\sum\limits_{i=1}^n\gamma_i(\alpha_i-1)$.

Theorem 2.11.2.([33]) *If* $f_i\in\beta_i-\mathcal{S}_p(\alpha_i)$, $-1\leq\alpha_i\leq1$, $\beta_i>0$, $\gamma_i>0$, $i\in\{1,...,n\}$ *and* $\sum\limits_{i=1}^n\gamma_i\leq\frac{1}{2}$, *then* $J_1\in\mathcal{K}(\rho)$, *where* $\rho=1+\sum\limits_{i=1}^n\gamma_i(\alpha_i-1)$.

2.12 Univalence of a general integral operator based on regular and Caratheodory functions

In the paper [38], together with V. Pescar, we obtained three univalence criteria for a general integral operator built on two sets of functions, respectively regular and Caratheodory functions (Definition 1.1.14), defined as follows:

- Breaz-Pescar general integral operator

$$J_8(z) = \left(\delta \int_0^z u^{\delta-1} \prod_{j=1}^{n} \left(\frac{f_j(u)}{u} \right)^{\alpha_j} (g_j(u))^{\beta_j} \, du \right)^{\frac{1}{\delta}}, \tag{2.12.1}$$

where δ, α_j, β_j are complex numbers, $\delta \neq 0$, $f_j \in \mathcal{A}$, $g_j \in \mathcal{P}$, $j = \overline{1, n}$.

The operator is studied on the following classes of functions:

- The class of functions $\mathcal{A}_M$

$$\mathcal{A}_M = \left\{ f \in \mathcal{A} : \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq M, \ M \geq 1 \right\}, \tag{2.12.2}$$

- The class of functions $\mathcal{P}_L$

$$\mathcal{P}_L = \left\{ f \in \mathcal{P} : \left| \frac{zf'(z)}{f(z)} \right| \leq L, \ L > 0 \right\}. \tag{2.12.3}$$

For various combinations of functions that compose the operator, we gave the following three univalence criteria:

Theorem 2.12.1. **([38])** *Let δ, α_j, β_j be complex numbers, $Re\delta \geq 1$, M_j, L_j pozitive real numbers, $M_j \geq 1$, $j = \overline{1, n}$ and the functions $f_j \in \mathcal{A}_{M_j}$, $g_j \in \mathcal{P}_{L_j}$, $j = \overline{1, n}$.*
If

$$\sum_{j=1}^{n} [|\alpha_j| M_j + |\beta_j| L_j] \leq \frac{3\sqrt{3}}{2}, \tag{2.12.4}$$

then the integral operator J_8, is in the class $\mathcal{S}$.

Proof. We apply Pascu criterion, Lemma 1.2.1 and Schwarz Lemma 1.1.9.

Theorem 2.12.2. **([38])** *Let γ, δ, α_j, β_j be complex numbers, $j = \overline{1, n}$, $Re\ \delta \geq Re\ \gamma > 0$ and the functions $f_j \in \mathcal{A}_\mu$, $g_j \in \mathcal{P}_\mu$, $\mu \geq 1$, with*

$$\mu = \frac{(2Re\gamma + 1)^{1 + \frac{1}{2Re\gamma}}}{2}. \tag{2.12.5}$$

If

$$\sum_{j=1}^{n} [|\alpha_j| + |\beta_j|] \leq 1, \tag{2.12.6}$$

then the integral operator J_8 is in the class $\mathcal{S}$.

Theorem 2.12.3. **([38])** *Let γ, δ, α_j, β_j be complex numbers, $j = \overline{1, n}$, $Re\ \delta \geq Re\ \gamma > 0$ and $f_j \in \mathcal{S}$, $g_j \in \mathcal{P}$. If*

$$2\sum_{j=1}^{n}|\alpha_j|+\sum_{j=1}^{n}|\beta_j|\leq \min\left\{\frac{Re\,\gamma}{2},\frac{1}{2}\right\} \tag{2.12.7}$$

then the integral operator J_8 *is in the class* $\mathcal{S}$.

2.13 Kudriasov type univalence conditions for two general integral operators

Together with V. Pescar, in the paper [85], we obtained sufficient univalence conditions of Kudriasov type for two of our general integral operators, one of Kim-Merkes type and the other, of Pfaltzgraff type. From these conditions some corollaries for various particular operators can be derived, but we present here only the main results.

The following general integral operators are studied:

- Kim-Merkes type Breaz-Breaz operator

$$J_2(z)=\left[\beta\int_0^z u^{\beta-1}\left(\frac{f_1(u)}{u}\right)^{\gamma_1}\cdot\ldots\cdot\left(\frac{f_n(u)}{u}\right)^{\gamma_n}du\right]^{\frac{1}{\beta}} \tag{2.13.1}$$

- Pfaltzgraff type Breaz-Breaz operator

$$J_4(z)=\left\{\beta\int_0^z u^{\beta-1}\cdot\left[f_1'(u)\right]^{\gamma_1}\cdot\ldots\cdot\left[f_n'(u)\right]^{\gamma_n}du\right\}^{1/\beta} \tag{2.13.2}$$

The next theorems state for univalence criteria of Kudriasov type, for the above mentioned operators:

Theorem 2.13.1 ([85]) *Let* α, γ_j, *be complex numbers,* $\mathrm{Re}\alpha>0$, *the functions* $f_j\in\mathcal{A}$, $j=\overline{1,n}$, $n\in\mathbb{N}-\{0\}$ *and* K *pozitive real number,* $K\cong 3.05$. *If*

$$\left|\frac{f_j''(z)}{f_j'(z)}\right|\leq K,\quad z\in U, j=\overline{1,n} \tag{2.13.3}$$

and

$$|\gamma_1|+|\gamma_2|+\ldots+|\gamma_n|\leq min\left\{\frac{\mathrm{Re}\alpha}{4},\frac{1}{4}\right\} \tag{2.13.4}$$

then $f_j\in\mathcal{S}$, $j=\overline{1,n}$ *and for any complex number,* β, $\mathrm{Re}\beta\geq\mathrm{Re}\alpha$, *the integral operator* J_2 *belongs to the class* $\mathcal{S}$.

Proof. We consider the regular function

$$h_n(z) = \int_0^z \left(\frac{f_1(u)}{u}\right)^{\gamma_1} ... \left(\frac{f_n(u)}{u}\right)^{\gamma_n} du. \tag{2.13.5}$$

Aiming to apply univalence criterion given by N.N. Pascu, Lemma 1.2.1, after some calculus, we obtain

$$\frac{zh''_n(z)}{h'_n(z)} = \gamma_1 \left(\frac{zf'_1(z)}{f_1(z)} - 1\right) + ... + \gamma_n \left(\frac{zf'_n(z)}{f_n(z)} - 1\right), z \in U \tag{2.13.6}$$

If we take the modulus, we get

$$\left|\frac{zh''_n(z)}{h'_n(z)}\right| \leq |\gamma_1| \left(\left|\frac{zf'_1(z)}{f_1(z)}\right| + 1\right) + ... + |\gamma_n| \left(\left|\frac{zf'_n(z)}{f_n(z)}\right| + 1\right). \tag{2.13.7}$$

From Kudriasov type hypothesis condition, taking into account Kudriasov Lemma 1.1.18, we have that $f_j \in \mathcal{S}$, $j = \overline{1,n}$, hence,

$$\left|\frac{zf'_j(z)}{f_j(z)}\right| \leq \frac{1+|z|}{1-|z|}, \quad z \in U, j = \overline{1,n}. \tag{2.13.8}$$

From (2.13.7) and (2.13.8), we get

$$\frac{1-|z|^{2\text{Re}\alpha}}{\text{Re}\alpha} \left|\frac{zh''_n(z)}{h'_n(z)}\right| \leq \frac{1-|z|^{2\text{Re}\alpha}}{\text{Re}\alpha} \frac{2}{1-|z|} \left(|\gamma_1| + ... + |\gamma_n|\right). \tag{2.13.9}$$

We consider two cases:

1) $0 < \text{Re}\alpha < 1$. The function $s : (0,1) \to \text{Re}$, $s(x) = 1 - a^{2x}$, $x = \text{Re}\alpha$, $a = |z|$, $(0 < a < 1)$ is an increasing function, hence,

$$1 - |z|^{2\text{Re}\alpha} < 1 - |z|^2, \quad z \in U. \tag{2.13.10}$$

Using this inequality in the estimation (2.13.9), we have

$$\frac{1-|z|^{2\text{Re}\alpha}}{\text{Re}\alpha} \left|\frac{zh''_n(z)}{h'_n(z)}\right| < \frac{4}{\text{Re}\alpha} \left(|\gamma_1| + ... + |\gamma_n|\right), \; z \in U. \tag{2.13.11}$$

Now, we use the hypothesis conditions on the parameters and further, we get,

$$\frac{1-|z|^{2\text{Re}\alpha}}{\text{Re}\alpha} \left|\frac{zh''_n(z)}{h'_n(z)}\right| < 1, \quad z \in U. \tag{2.13.12}$$

2) $\text{Re}\alpha \geq 1$. Since the function q is a decreasing function

$$q : [1, \infty) \to \mathbb{R}, \; q(x) = \frac{1-a^{2x}}{x}, \; x = \text{Re}\alpha, \; a = |z|, \; (0 < a < 1)$$

we get

$$\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha} \leq 1-|z|^2\,,\ z\in U. \tag{2.13.13}$$

Further, using the last formula in the estimation (2.13.9), we obtain

$$\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zh_n''(z)}{h_n'(z)}\right| \leq 4\left(|\gamma_1|+...+|\gamma_n|\right). \tag{2.13.14}$$

Now we use the hypothesis conditions on the coefficients, hence we obtain

$$\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zh_n''(z)}{h_n'(z)}\right| \leq 1,\ \ z\in U. \tag{2.13.15}$$

In both cases, we can apply the univalence criterion given by N.N. Pascu, Lemma 1.2.1 and find that $J_2 \in \mathcal{S}$.

Theorem 2.13.2 ([85]) *Let α, γ_j, be complex numbers, $j=\overline{1,n}$, $\mathrm{Re}\alpha>0$, the functions $f_j\in\mathcal{A}$, $f_j(z)=z+a_{2j}z^2+...$, $j=\overline{1,n}$, $n\in\mathbb{N}-\{0\}$ and K pozitive real number, $K\cong 3.05$. If*

$$\left|\frac{f_j''(z)}{f_j'(z)}\right| \leq K,\ \ z\in U,\ j=\overline{1,n} \tag{2.13.16}$$

and

$$|\gamma_1|+|\gamma_2|+...+|\gamma_n| \leq \frac{(2\mathrm{Re}\alpha+1)^{\frac{2\mathrm{Re}\alpha+1}{2\mathrm{Re}\alpha}}}{2K} \tag{2.13.17}$$

then the functions $f_j\in\mathcal{S}$, $j=\overline{1,n}$ and, for any complex number β, $\mathrm{Re}\beta\geq\mathrm{Re}\alpha$, the integral operator J_4 is in the class $\mathcal{S}$.

Proof. Using both Kudriasov type hypothesis condition and Kudriasov Lemma 1.1.18, it comes that $f_j\in\mathcal{S}$, $j=\overline{1,n}$.

We consider the regular function

$$p_n(z)=\int_0^z \left(f_1'(u)\right)^{\gamma_1}...\left(f_n'(u)\right)^{\gamma_n}du \tag{2.13.18}$$

After some calculation, we have

$$\frac{zp_n''(z)}{p_n'(z)}=\gamma_1\frac{zf_1''(z)}{f_1'(z)}+...+\gamma_n\frac{zf_n''(z)}{f_n'(z)},\ \ z\in U. \tag{2.13.19}$$

Aiming to apply N.N. Pascu criterion, Lemma 1.2.1, we get the evaluation

$$\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zp_n''(z)}{p_n'(z)}\right| \leq \frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}|z|\left[|\gamma_1|\left|\frac{f_1''(z)}{f_1'(z)}\right|+...+|\gamma_n|\left|\frac{f_n''(z)}{f_n'(z)}\right|\right]. \tag{2.13.20}$$

If in this inequality, we apply the hypothesis Kudriasov type condition, we obtain

$$\frac{1-|z|^{2\text{Re}\alpha}}{\text{Re}\alpha}\left|\frac{zp_n''(z)}{p_n'(z)}\right| \leq \left[\frac{1-|z|^{2\text{Re}\alpha}}{\text{Re}\alpha}|z|\right](K|\gamma_1|+...+K|\gamma_n|), z \in U. \tag{2.13.21}$$

Now, we consider the function $G : [0,1] \to \mathbb{R}$, $G(x) = \frac{1-x^{2a}}{a}x$, $x = |z|$, $a = \text{Re}\alpha$. We can prove that

$$\max_{x\in[0,1]} G(x) = \frac{2}{(2a+1)^{\frac{2a+1}{2a}}}. \tag{2.13.22}$$

If we use (2.13.22) and the hypothesis conditions on the parameter, in the last formula, we get

$$\frac{1-|z|^{2\text{Re}\alpha}}{\text{Re}\alpha}\left|\frac{zp_n''(z)}{p_n'(z)}\right| \leq 1, z \in U. \tag{2.13.23}$$

Applying N.N. Pascu criterion, Lemma 1.2.1, it comes that $J_4 \in \mathcal{S}$.

2.14 Coefficient estimates and modified Hadamard product for classes of analytic functions, p-valent, with negative coefficients

In the paper [43], together with R. El-Ashwah and M. Aouf, we gave coefficients estimates and studied the modified Hadamard product for some classes of starlike, respectively convex functions of order α, p-valent, with negative coefficients, defined with a differential operator. After that, in [36], we extended these results for some other classes of analytic functions, p-valent, with negative coefficients. Here we present only the more general results.

We denote by $T_0(p)$, the class of analytic functions, p-valent in the open unit disk, $\mathcal{U}$, having the form:

$$f(z) = a_p \cdot z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, (a_{p+n} \geq 0; p \in N = \{1, 2, ...\}, a_p > 0) \tag{2.14.1}$$

For $a_p = \frac{1}{p!}$, we have: $f(0) = f'(0) = ... = f^{(p-1)}(0) = 0$ and $f^{(p)}(0) = 1$. We denote by $T(p)$, the class $T_0(p)$ with $a_p = 1$.

We introduced the following classes:

• The class $\beta - UST_0(p, q, \alpha)$ of the p-valent functions, with negative coefficients, β - uniformly starlike of order α, with respect to the differentiation of order q, Breaz-El.Ashwah

$$\beta - UST_0(p,q,\alpha) =$$
$$= \left\{ f \in T_0(p) : \mathrm{Re}\left\{ \frac{zf^{(1+q)}(z)}{f^{(q)}(z)} \right\} \geq \beta \left| \left\{ \frac{zf^{(1+q)}(z)}{f^{(q)}(z)} \right\} - 1 \right| + \alpha \right\}, \tag{2.14.2}$$
$$\left(z \in U; 0 \leq \alpha < p-q; p \in N; p > q; q \in N_0 = N \cup \{0\},\ \beta \geq 0\right)$$

• The class $\beta - UCV_0(p,q,\alpha)$ of the p-valent functions, with negative coefficients, β - uniformly convex of order α, with respect to the differentiation of order q, Breaz-El.Ashwah

$$\beta - UCV_0(p,q,\alpha) =$$
$$= \left\{ f \in T_0(p) : \mathrm{Re}\left\{ 1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} \right\} \geq \beta \left| \left\{ \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} \right\} \right| + \alpha \right\}, \tag{2.14.3}$$
$$\left(z \in U; 0 \leq \alpha < p-q; p \in N; p > q; q \in N_0 = N \cup \{0\},\ \beta \geq 0\right)$$

Particular classes of functions with negative coefficients can be derived if we take different values for the parameters involved in the definitions:

i) $\beta = 0,\ a_p = 1$: $\beta - UST_0(p,q,\alpha) = S(p,q,\alpha)$ and $\beta - UCV_0(p,q,\alpha) = C(p,q,\alpha)$, Chen, Irmak and Srivastava; for $\beta = 0,\ a_p = 1$ si $q = 0$, the classes were studied by Owa, Salagean, Hossen, Aouf and Sekine.
ii) $p = 1,\ q = 0$: the classes of functions β-uniformly starlike, respectively convex of order α, Bharati, Frasin; for $p = 1,\ q = 0,\ \alpha = 0$: classes of functions β - uniformly convex, respectively starlike, Kanas and Wisniowska.

Next two theorems give information about the coefficients of the functions considered:

Theorem 2.14.1. **([36])** *Every function $f \in \beta - UST_0(p,q,\alpha)$ satisfies the inequality*

$$\sum_{n=1}^{\infty} [(n+p-q-\alpha) + \beta(n+p-q-1)] \cdot \delta(n+p,q) \cdot a_{n+p}$$
$$\leq [(p-q-\alpha) + \beta(p-q-1)] \cdot \delta(p,q) \cdot a_p, \tag{2.14.4}$$

where

$$\delta(p,q) = \frac{p!}{(p-q)!} = \begin{cases} p(p-1)...(p-q+1) & (q \neq 0) \\ 1 & (q = 0). \end{cases} \tag{2.14.5}$$

Proof. It can be proved that $f \in S(p,q,\gamma)$, $\gamma = \frac{\alpha+\beta}{1+\beta}$ and then, the characterization of the coefficients from this class, given by Chen, Irmak and Srivastava, is used.

Theorem 2.14.2. **([36])** *Every functions $f \in \beta - UCV_0(p, q, \alpha)$ satisfies the inequality*

$$\sum_{n=1}^{\infty} \left(\frac{n+p-q}{p-q}\right) [(n+p-q-\alpha) + \beta(n+p-q-1)] \cdot \delta(n+p, q) \cdot a_{n+p}$$
$$\leq [(p-q-\alpha) + \beta(p-q-1)] \cdot \delta(p, q) \cdot a_p. \tag{2.14.6}$$

Proof. We show that $f \in C(p, q, \gamma)$, $\gamma = \frac{\alpha+\beta}{1+\beta}$.

Now, let's consider the following class:

- The class $T_0(k, p, q, \alpha)$, $k \geq 0$

$$T_0(k, p, q, \alpha) = \{f \in T_0(p) :$$
$$\sum_{n=1}^{\infty} \left(\frac{n+p-q}{p-q}\right)^k [(n+p-q-\alpha) + \beta(n+p-q-1)] \, \delta(n+p, q) a_{p+n}$$
$$\leq [(p-q-\alpha) + \beta(p-q-1)] \, \delta(p, q) a_p \} . \tag{2.14.7}$$

We can see that the class $T_0(k, p, q, \alpha)$, $k \geq 0$ is not empty, containing at least, the function:

$$f(z) = a_p z^p -$$
$$\sum_{n=1}^{\infty} \frac{[(p-q-\alpha) + \beta(p-q-1)] \, \delta(p, q) a_p \cdot \lambda_{p+n} z^{p+n}}{\left(\frac{n+p-q}{p-q}\right)^k [(n+p-q-\alpha) + \beta(n+p-q-1)] \, \delta(n+p, q)}, \tag{2.14.8}$$

with $a_p > 0$, $\lambda_{p+n} > 0$ si $\sum\limits_{n=1}^{\infty} \lambda_{p+n} \leq 1$.

The following inclusions hold:

$$\beta - T_0(k, p, q, \alpha) \subset \beta - T_0(c, p, q, \alpha) \text{ for } k > c \geq 0.$$
$$\beta - UST_0(p, q, \alpha) \subset \beta - T_0(0, p, q, \alpha)$$
$$\beta - UCV_0(p, q, \alpha) \subset \beta - T_0(1, p, q, \alpha). \tag{2.14.9}$$

Further we will consider the modified Hadamard product, on the class $T_0(p)$, defined as:

$$(f_i * g_j)(z) = a_{p,i} b_{p,j} z^p - \sum_{n=1}^{\infty} a_{p+n,i} b_{p+n,j} z^{p+n} \qquad (i, j = 1, 2, 3, ...) . \tag{2.14.10}$$

where

$$f_i(z) = a_{p,i} z^p - \sum_{n=1}^{\infty} a_{p+n,i} z^{p+n} \qquad (a_{p,i} > 0; a_{p+n,i} \geq 0),$$

$$g_j(z) = b_{p,j} z^p - \sum_{n=1}^{\infty} b_{p+n,j} z^{p+n} \qquad (b_{p,j} > 0; b_{p+n,j} \geq 0). \tag{2.14.11}$$

Next theorem describes the behaviour of the modified Hadamard product on our classes:

Theorem 2.14.3 ([36]) *Let be the functions $f_i(z)$ from the classes $\beta - UST_0(p, q, \alpha_i)(i = 1, 2, 3, ..., m)$ and the functions $g_j(z)$ from the classes $\beta - UCV_0(p, q, \gamma_j)(j = 1, 2, 3, ..., d)$. Then the modified Hadamard product $f_1 * f_2 * f_3 * ... * f_m * g_1 * g_2 * g_3 * ... * g_d(z)$ belongs to the class $\beta - T_0(m + 2d - 1, p, q, \rho)$, with*

$$\rho = \max\{\alpha_1, \alpha_2, \alpha_3, ..., \alpha_m, \gamma_1, \gamma_2, \gamma_3, ..., \gamma_d\} . \tag{2.14.12}$$

Proof. We prove the theorem only for the case $m = d = 1$ and $\alpha_i = \gamma_j = \alpha$, namely, we show that if $f \in \beta - UST_0(p, q, \alpha)$, $g \in \beta - UCV_0(p, q, \alpha)$ then $f * g \in \beta - T_0(2, p, q, \alpha)$.

From $f \in \beta - UST_0(p, q, \alpha)$, using the Theorem of coefficient estimates, 2.14.1, we obtain

$$a_{n+p} \leq \left[\frac{[(p - q - \alpha) + \beta(p - q - 1)] \cdot \delta(p, q)}{[(n + p - q - \alpha) + \beta(n + p - q - 1)] \cdot \delta(n + p, q)} \right] \cdot a_p. \tag{2.14.13}$$

We denote by $H(\alpha)$, the function

$$H(\alpha) = \frac{[(p - q - \alpha) + \beta(p - q - 1)]}{[(n + p - q - \alpha) + \beta(n + p - q - 1)]} \tag{2.14.14}$$

and by $G(\beta)$, the function

$$G(\beta) = \frac{[(p - q) + \beta(p - q - 1)]}{[(n + p - q) + \beta(n + p - q - 1)]}. \tag{2.14.15}$$

Since the functions $H(\alpha)$, $G(\beta)$ are decreasing functions

$$\frac{\delta(p, q)}{\delta(n + p, q)} \leq 1, \tag{2.14.16}$$

further, we get,

$$a_{n+p} \leq \frac{p - q}{n + p - q} \cdot a_p. \tag{2.14.17}$$

On the other hand, since $g \in \beta - UCV_0(p,q,\alpha)$, we have, according to the Theorem of coefficient estimates, 2.14.2,

$$\sum_{n=1}^{\infty} \left(\frac{n+p-q}{p-q}\right) [(n+p-q-\alpha) + \beta(n+p-q-1)] \, \delta(n+p,q) b_{p+n}$$

$$\leq [(p-q-\alpha) + \beta(p-q-1)] \, \delta(p,q) b_p. \tag{2.14.18}$$

Combining the last two inequalities, we obtain that the Hadamard product is in the class $\beta - T_0(2,p,q,\alpha)$, because

$$\sum_{n=1}^{\infty} \left(\frac{n+p-q}{p-q}\right)^2 [(n+p-q-\alpha) + \beta(n+p-q-1)] \cdot$$

$$\cdot \delta(n+p,q) a_{p+n} b_{p+n}$$

$$\leq [(p-q-\alpha) + \beta(p-q-1)] \, \delta(p,q) a_p b_p. \tag{2.14.19}$$

These three theorems extend known results. For example, for $p = 1$ and $q = 0$ we get the results given by B. Frasin.

2.15 Mocanu and Şerb type univalence criterion for some general integral operators

Here, we obtain new conditions of univalence for two general integral operators, T_n and B_n, by applying the improvement of Becker univalence criterion, obtained by Pascu in the paper [76]. Also, a lemma given by Mocanu and Şerb in the paper [66], will be used to get some parts of the results. These univalence conditions were published as a joint work with V. Pescar in the paper [86].

In the last decade, some general integral operators, defined as a family of integral operators, using more than one analytic function in their definition, have been studied with respect to their univalence (see for example, the works [9], [10] and [81], and many other recent paper as [38], [57], [97]).

In this section, the univalence study is focused on the following general integral operators:

$$T_n(z) = \left\{ \beta \int_0^z u^{\beta-1} \left(f_1'(u)\right)^{\gamma_1} \ldots \left(f_n'(u)\right)^{\gamma_n} du \right\}^{\frac{1}{\beta}}, \tag{2.15.1}$$

$$B_n(z) = \left\{ \beta \int_0^z u^{\beta-1} \left(\frac{f_1(u)}{u} \right)^{\mu_1} \dots \left(\frac{f_n(u)}{u} \right)^{\mu_n} \left(g_1'(u) \right)^{\eta_1} \dots \left(g_n'(u) \right)^{\eta_n} du \right\}^{\frac{1}{\beta}}, \tag{2.15.2}$$

β, γ_j, μ_j, η_j complex numbers, $\beta \neq 0$, $f_j, g_j \in \mathcal{A}$, $j = \overline{1,n}$.

Remark 2.15.1. ([86])

(i) The integral operator T_n, introduced by Breaz and Breaz in the paper [10] is a general integral operator of Pfaltzgraff type which extends also the operator introduced by Pescar and Owa([87]), derived from (2.15.1) , for $n = 1$. This operator has been studied with respect to its univalence, in many papers (see for example [81] and [85]).

(ii) Let's consider also the integral operator

$$H_n(z) = \left\{ \beta \int_0^z u^{\beta-1} \left(\frac{f_1(u)}{u} \right)^{\gamma_1} \dots \left(\frac{f_n(u)}{u} \right)^{\gamma_n} du \right\}^{\frac{1}{\beta}}. \tag{2.15.3}$$

The integral operator H_n, introduced by Breaz and Breaz in the paper [9] is a general integral operator of Kim-Merkes type, which extends also the operator introduced in [78], by Pascu and Pescar, derived from (2.15.3) , for $n = 1$.

Thus, the integral operator B_n, introduced here by the formula (2.15.2) , can be considered as an extension of both H_n (for $\eta_j = 0, j = \overline{1,n}$) and T_n (for $\mu_j = 0, j = \overline{1,n}$).

Moreover, if in the definition of B_n, we take $g = f$, we obtain the general integral operator given by Frasin ([45]), from which, if we take $n = 1$, we can derive further the operator given by Ovesea ([70]). Also, some different versions of this operator, B_n, were studied in other papers as for example [38] and [57].

In what follows, we present conditions of univalence for these two general integral operators, T_n and B_n, and related results.

Theorem 2.15.2. ([86]) *Let α, γ_j be complex numbers, $j = \overline{1,n}$, $\mathrm{Re}\alpha > 0$ and the functions $f_j \in \mathcal{A}$, $f_j(z) = z + a_{2j}z^2 + \dots$, $j = \overline{1,n}$, $n \in \mathbb{N} - \{0\}$, M a positive real number.*

If

$$\left| \frac{f_j''(z)}{f_j'(z)} \right| \leq M, \quad z \in U, \ j = \overline{1,n} \tag{2.15.4}$$

and

$$|\gamma_1| + |\gamma_2| + ... + |\gamma_n| \leq \frac{(2\text{Re}\alpha + 1)^{\frac{2\text{Re}\alpha+1}{2\text{Re}\alpha}}}{2M} \tag{2.15.5}$$

then for every complex number β, $\text{Re}\beta \geq \text{Re}\alpha$, *the integral operator* T_n *belongs to the class* $\mathcal{S}$.

Proof. We consider the function

$$t_n(z) = \int_0^z \left(f_1'(u)\right)^{\gamma_1} ... \left(f_n'(u)\right)^{\gamma_n} du \tag{2.15.6}$$

which is regular in U and $t_n(0) = t_n'(0) - 1 = 0$.

After some calculus we have the following evaluation for the expression involved in the hypothesis of N.N.Pascu univalence criterion, Lemma 1.2.1, [76],

$$\frac{1-|z|^{2\text{Re}\alpha}}{\text{Re}\alpha}\left|\frac{zt_n''(z)}{t_n'(z)}\right| \leq \frac{1-|z|^{2\text{Re}\alpha}}{\text{Re}\alpha}|z|\left[|\gamma_1|\left|\frac{f_1''(z)}{f_1'(z)}\right| + ... + |\gamma_n|\left|\frac{f_n''(z)}{f_n'(z)}\right|\right], \tag{2.15.7}$$

for all $z \in U$.

On the other hand it can be proved that

$$\max_{|z|\in[0,1]} \frac{1-|z|^{2\text{Re}\alpha}}{\text{Re}\alpha}|z| = \frac{2}{(2\text{Re}\alpha+1)^{\frac{2\text{Re}\alpha+1}{2\text{Re}\alpha}}}. \tag{2.15.8}$$

Hence, if we apply hypothesis conditions (2.15.4), (2.15.5) and also (2.15.8) in the formula (2.15.7) , the condition of N.N.Pascu univalence criterion, Lemma 1.2.1 ([76]) is satisfied, consequently $T_n \in \mathcal{S}$.

Remark 2.15.3. **([86])**If in Theorem 2.15.2, we take different values for the positive constant M, we can obtain also some information about the functions f_j, $j = \overline{1,n}$, not only about the integral operator. Thus:

(i) For $M = K \cong 3.05$ (Kudriasov constant), we have that the functions f_j, $j = \overline{1,n}$ are univalent (see Kudriasov Lemma 1.1.18). Thus, Theorem 2.15.2 extends the result obtained by us, using Kudriasov constant in the paper [85].

(ii) For $M \cong 2.83$ (Mocanu constant), f_j, $j = \overline{1,n}$ are starlike and consequently univalent (see Mocanu Lemma 1.1.20).

(iii) For $M = M_0 \cong 1.5936$ (Mocanu and Şerb constant), the functions f_j, $j = \overline{1,n}$ belongs to some special class of starlike functions, consequently they are univalent (see Mocanu and Şerb Lemma 1.1.15).

Corollary 2.15.4. **([86])***Let* α, γ_j *be complex numbers,* $j = \overline{1,n}$, $0 \leq \text{Re}\alpha \leq 1$ *and the* *functions* $f_j \in \mathcal{A}$, $f_j(z) = z + a_{2j}z^2 + ...$, $j = \overline{1,n}$, $n \in \mathbb{N} - \{0\}$, M *the positive real number. If*

$$\left|\frac{f_j''(z)}{f_j'(z)}\right| \leq M, \quad z \in U, \ j = \overline{1,n} \tag{2.15.9}$$

and

$$|\gamma_1| + |\gamma_2| + ... + |\gamma_n| \leq \frac{(2\text{Re}\alpha + 1)^{\frac{2\text{Re}\alpha+1}{2\text{Re}\alpha}}}{2M} \tag{2.15.10}$$

then the integral operator K_n *defined by*

$$K_n(z) = \int_0^z \left(f_1'(u)\right)^{\gamma_1} ... \left(f_n'(u)\right)^{\gamma_n} du \tag{2.15.11}$$

is in the class $\mathcal{S}$.

Proof. We take $\beta = 1$.

Remark 2.15.5. (**[86]**)The general integral operator of Pfaltzgraff type K_n was introduced by Breaz et al. in the paper [29].

Theorem 2.15.6. (**[86]**)*Let* α, μ_j, η_j, *be complex numbers,* $\text{Re}\alpha > 0$, $f_j, g_j \in \mathcal{A}$, $j = \overline{1,n}$, $n \in \mathbb{N} - \{0\}$, M *a positive real number and* $M_0 = 1.5936...$ *the positive solution of equation*

$$(2 - M)e^M = 2. \tag{2.15.12}$$

If

$$\left|\frac{f_j''(z)}{f_j'(z)}\right| \leq M_0, \quad z \in U, j = \overline{1,n}, \tag{2.15.13}$$

$$\left|\frac{g_j''(z)}{g_j'(z)}\right| \leq M, \quad z \in U, j = \overline{1,n}, \tag{2.15.14}$$

and

$$\sum_{j=1}^{n} |\mu_j| + M \cdot \sum_{j=1}^{n} |\eta_j| \leq \text{Re}\alpha, \tag{2.15.15}$$

then $f_j \in \mathcal{S}$, $j = \overline{1,n}$ *and for every complex number* β, $\text{Re}\beta \geq \text{Re}\alpha$, *we have* $B_n \in \mathcal{S}$.

Proof. We apply N.N.Pascu univalence criterion, Lemma 1.2.1 for the regular function

$$b_n(z) = \int_0^z \prod_{j=1}^{n} \left(\frac{f_j(u)}{u}\right)^{\mu_j} \prod_{j=1}^{n} \left(g_j'(u)\right)^{\eta_j} du. \tag{2.15.16}$$

After some derivative calculus, we get

$$\frac{1-|z|^{2\text{Re}\alpha}}{\text{Re}\alpha}\left|\frac{zb_n''(z)}{b_n'(z)}\right| \le \frac{1-|z|^{2\text{Re}\alpha}}{\text{Re}\alpha}\sum_{j=1}^{n}\left[|\mu_j|\left|\frac{zf_j'(z)}{f_j(z)}-1\right| + |\eta_j|\,|z|\left|\frac{g_j''(z)}{g_j'(z)}\right|\right], \tag{2.15.17}$$

for all $z \in U$.

Applying all hypothesis conditions and further, Mocanu and Şerb Lemma 1.1.15, for f_j, $j = \overline{1, n}$, we have

$$\frac{1-|z|^{2\text{Re}\alpha}}{\text{Re}\alpha}\left|\frac{zb_n''(z)}{b_n'(z)}\right| \le 1, \tag{2.15.18}$$

which according to N.N.Pascu univalence criterion, Lemma 1.2.1 implies that $B_n \in \mathcal{S}$.

Remark 2.15.7. ([86])

(i) For $\eta_j = 0, j = \overline{1, n}$, we get the same univalence criterion for the integral operator H_n, recalled in the Remark 2.15.1 (ii), as it was obtained in [80].

(ii) For $\mu_j = 0, j = \overline{1, n}$, we get a new univalence criterion for the integral operator T_n, based on the condition $\sum_{j=1}^{n} |\eta_j| \le \frac{\text{Re}\alpha}{M}$ and the Remark 2.15.1 could be also reiterated.

2.16 A subclass of multivalent functions involving higher-order derivatives

In this section we present a new class of analytic and p-valent functions involving higher-order derivatives. For this p-valent function class, we derive several interesting properties including coefficient inequalities, distortion theorems, extreme points, and the radii of close-to-convexity, starlikeness and convexity. Several applications involving an integral operator are also considered. Finally, we obtain some results for the modified Hadamard product of the functions belonging to the p-valent function class which is introduced here. The results are published as a joint work with H.M. Srivastava and R. El-Ashwah in the paper [96].

Let $\mathcal{A}(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \qquad (p \in \mathbb{N} = \{1, 2, 3, \cdots\}), \tag{2.16.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

A function $f(z) \in \mathcal{A}(p)$ is said to be in the class $\mathcal{UST}(p, \alpha, \beta)$ of p-valent β-uniformly starlike functions of order α in $\mathbb{U}$ if and only if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}-\alpha\right) \geqq \beta\left|\frac{zf'(z)}{f(z)}-p\right| \qquad (z \in \mathbb{U};\ -p \leqq \alpha < p;\ \beta \geqq 0). \tag{2.16.2}$$

On the other hand, a function $f(z) \in \mathcal{A}(p)$ is said to be in the class $\mathcal{UCV}(p,\alpha,\beta)$ of p-valent β-uniformly convex functions of order α in $\mathbb{U}$ if and only if

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}-\alpha\right) \geqq \beta\left|1+\frac{zf''(z)}{f'(z)}-p\right| \ (z \in \mathbb{U};\ -p \leqq \alpha < p;\ \beta \geqq 0). \tag{2.16.3}$$

The above-defined function classes $\mathcal{UST}(p,\alpha,\beta)$ and $\mathcal{UCV}(p,\alpha,\beta)$ were introduced recently by Khairnar and More [52]. Various analogous classes of analytic and univalent or multivalent functions were studied in many papers (see, for example, [2], [36] and [51]).

We notice from the inequalities (2.16.2) and (2.16.3) that

$$f(z) \in \mathcal{UCV}(p,\alpha,\beta) \iff \frac{zf'(z)}{p} \in \mathcal{UST}(p,\alpha,\beta). \tag{2.16.4}$$

Now, for each $f(z) \in \mathcal{A}(p)$, it is easily seen upon differentiating both sides of (2.16.1) q times with respect to z that

$$f^{(q)}(z) = \delta(p,q)z^{p-q} + \sum_{k=p+1}^{\infty} \delta(k,q)a_k z^{k-q} \qquad (q \in \mathbb{N}_0 := \mathbb{N}\cup\{0\};\ p > q), \tag{2.16.5}$$

where, and in what follows, $\delta(p,q)$ denotes the q-permutations of p objects $(p \geqq q \geqq 0)$, that is,

$$\delta(p,q) := \frac{p!}{(p-q)!} = \begin{cases} p(p-1)\cdots(p-q+1) & (q \neq 0) \\ 1 & (q = 0), \end{cases}$$

which may also be identified with the notation $\{p\}_q$ for the *descending factorial.*

Let

$$-\delta(p-q,m) \leqq \alpha < \delta(p-q,m), \quad \beta \geqq 0 \qquad \text{and} \qquad p > q+m \qquad (p \in \mathbb{N};\ q,m \in \mathbb{N}_0).$$

We then denote by $\mathcal{US}_m(p,q;\alpha,\beta)$ the subclass of the p-valent function class $\mathcal{A}(p)$ consisting of functions $f(z)$ of the form (2.16.1), which also satisfy the following analytic criterion:

$$\mathrm{Re}\left(\frac{z^m f^{(q+m)}(z)}{f^{(q)}(z)}-\alpha\right) \geqq \beta\left|\frac{z^m f^{(q+m)}(z)}{f^{(q)}(z)}-\delta(p-q,m)\right| \qquad (z \in \mathbb{U}). \tag{2.16.6}$$

We also denote by $\mathcal{T}(p)$ the subclass of $\mathcal{A}(p)$ consisting of functions of the following form:

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k \qquad (a_k \geqq 0;\ p \in \mathbb{N}). \tag{2.16.7}$$

Further, we define the class

$$\mathcal{UST}_m(p, q; \alpha, \beta) = \mathcal{US}_m(p, q; \alpha, \beta) \cap \mathcal{T}(p). \tag{2.16.8}$$

For suitable choices of $p, q,\ m\ and\ \beta$, we obtain the following known subclasses:
(i) It is easily verified that $\big($see Liu and Liu [56] (with $\gamma = 1$ and $n = 1$)$\big)$

$$\mathcal{UST}_m\,(p, q, \alpha, 0) = \mathcal{A}^*_{1,p}\,(m, q, \alpha, 1)$$

$$\big(0 \leqq \alpha < \delta(p - q, m);\ p \in \mathbb{N};\ m, q \in \mathbb{N}_0;\ p > q + m\big);$$

(ii) We observe that (see Khairnar and More [52])

$$\mathcal{UST}_1(p, 0; \alpha, \beta) = \mathcal{UST}(p, \alpha, \beta) \qquad (-p \leqq \alpha < p;\ \beta \geqq 0;\ p \in \mathbb{N})$$

and

$$\mathcal{UST}_1(p, 1; \alpha, \beta) = \mathcal{UCV}(p, \gamma, \beta) \qquad (-p \leqq \gamma = \alpha + 1 < p;\ \beta \geqq 0;\ p \in \mathbb{N});$$

(iii) It is easy to see that $\big($see Aouf [4] (with $\beta = 1$ and $n = 1$)$\big)$

$$\mathcal{UST}_1\,(p, q, \alpha, 0) = \mathcal{S}_1\,(p, q, \alpha, 1) \qquad (0 \leqq \alpha < p - q;\ p \in \mathbb{N};\ q \in \mathbb{N}_0;\ p > q + 1)$$

and

$$\mathcal{UST}_1\,(p, q, \alpha, 0) = \mathcal{C}_1\,(p, t, \gamma, 1) \qquad (0 \leqq \alpha < p - q;\ p, q \in \mathbb{N};\ p > q + 1;\ t = q - 1;\ \gamma = \alpha + 1);$$

(iv) We notice that $\big($see Chen *et al.* [40] (with $n = 1$)$\big)$

$$\mathcal{UST}_1\,(p, q, \alpha, 0) = \mathcal{S}_1\,(p, q, \alpha) \qquad (0 \leqq \alpha < p - q;\ p \in \mathbb{N};\ q \in \mathbb{N}_0;\ p > q + 1)$$

and

$$\mathcal{UST}_1\,(p, q, \alpha, 0) = \mathcal{C}_1\,(p, t, \gamma) \qquad (0 \leqq \alpha < p - q;\ p, q \in \mathbb{N};\ p > q + 1;\ t = q - 1;\ \gamma = \alpha + 1).$$

In what follows we obtain several properties of the class $\mathcal{UST}_m(p, q; \alpha, \beta)$. Various other papers were dedicated to the study of such aspects as we have considered in this section. For example, different classes of functions with negative coefficients, defined by using some derivative operators, were studied with respect to their Hadamard product in the paper [1] in the case of multivalence or coefficient estimates, distorsion bounds and Hadamard product, and in the paper [50] in the case of univalence. Another class of analytic and multivalent functions was studied in the paper [67], where the class was proved as being closed under the convolution and some integral operators (see also the recent works [41], [94] and [95]).

The following results state for coefficient estimates:

Unless otherwise mentioned, we assume throughout this paper that

$$-\delta(p-q,m) \leqq \alpha < \delta(p-q,m), \quad \beta \geqq 0, \quad q,m \in \mathbb{N}_0, \quad p \in \mathbb{N} \quad \text{and} \quad p > q+m.$$

Our first result (Theorem 2.16.1 below) provides the coefficient inequalities for functions in the class $\mathcal{US}_m(p,q;\alpha,\beta)$.

Theorem 2.16.1. **([96])** *A function $f(z)$ of the form (2.16.1) is in the class $\mathcal{US}_m(p,q;\alpha,\beta)$ if*

$$\sum_{k=p+1}^{\infty} \left[(1+\beta)[\delta(k-q,m)-\delta(p-q,m)] + [\delta(p-q,m)-\alpha]\right]\delta(k,q)a_k \leqq [\delta(p-q,m)-\alpha]\,\delta(p,q). \tag{2.16.9}$$

Proof. It is easy to show that

$$\beta\left|\frac{z^m f^{(q+m)}(z)}{f^{(q)}(z)} - \delta(p-q,m)\right| - \Re\left(\frac{z^m f^{(q+m)}(z)}{f^{(q)}(z)} - \delta(p-q,m)\right) \leqq [\delta(p-q,m)-\alpha],$$

which implies the result (2.16.9) asserted by Theorem 2.16.1.

Theorem 2.16.2 **([96])** *A necessary and sufficient condition for $f(z)$ of the form (2.16.7) to be in the class $\mathcal{UST}_m(p,q;\alpha,\beta)$ is that*

$$\sum_{k=p+1}^{\infty} \left[(1+\beta)[\delta(k-q,m)-\delta(p-q,m)] + [\delta(p-q,m)-\alpha]\right]\delta(k,q)a_k \leqq [\delta(p-q,m)-\alpha]\,\delta(p,q). \tag{2.16.10}$$

Proof. In view of Theorem 2.16.1, we need only to prove the necessity.
If $f(z) \in \mathcal{UST}_m(p,q;\alpha,\beta)$ and z is a real number, then

$$\frac{z^m f^{(q+m)}(z)}{f^{(q)}(z)} - \alpha \geqq \beta\left|\frac{z^m f^{(q+m)}(z)}{f^{(q)}(z)} - \delta(p-q,m)\right|.$$

By making some calculations and letting $z \to 1-$ along the real axis, we have the desired inequality (2.16.10)

Remark 2.16.3. **([96])** (i) Taking $\beta = 0$, Theorem 2.16.2 extends the result for the coefficient estimates related to the class $\mathcal{A}^*_{1,p}(m, q, \alpha, 1)$, which is due to Liu and Liu [56] (with $\gamma = 1$ and $n = 1$);
(ii) Taking $q = 0$ and $p = m = 1$, Theorem 2.16.2 extends the result for the coefficient estimates related to the class $\mathcal{ST}_0(\alpha, \beta)$, which is due to Frasin [44] (with $a_1 = 1$);
(iii) Taking $p = m = 1$, $q = t+1, t = 0$ and $\alpha = \gamma - 1$, Theorem 2.16.2 extends the result for the coefficient estimates related to the class $\mathcal{UCT}_0(\gamma, \beta)$, which is due to Frasin [44] (with $a_1 = 1$);
(iv) Taking $\beta = 0$ and $m = 1$, Theorem 2.16.2 extends the result or the coefficient estimates related to the class $\mathcal{S}_1(p, q, \alpha, 1)$, which is due to Aouf [4] (with $\beta = 1$ and $n = 1$);
(v) Taking $\beta = 0$, $m = 1$, $q = t + 1$ and $\alpha = \gamma - 1$, Theorem 2.16.2 extends the result for the coefficient estimates related to the class $\mathcal{C}_1(p, t, \gamma, 1)$, which is due to Aouf [4] (with $\beta = 1$ and $n = 1$);
(vi) Taking $\beta = 0$ and $m = 1$, Theorem 2.16.2 extends the result for the coefficient estimates related to the class $\mathcal{S}(p, q, \alpha)$, which is due to Chen *et al.* [40] (with $n = 1$);
(vii) Taking $\beta = 0$, $m = 1$, $q = t + 1$ and $\alpha = \gamma - 1$, Theorem 2.16.2 extends the result for the coefficient estimates related to the class $\mathcal{C}(p, t, \gamma)$, which is due to Chen *et al.* [40] (with $n = 1$).

Corollary 2.16.4. **([96])** *Let the function $f(z)$ defined by (2.16.7) be in the class $\mathcal{UST}_m(p, q; \alpha, \beta)$. Then*

$$a_k \leqq \frac{[\delta(p-q,m)-\alpha]\,\delta(p,q)}{\big[(1+\beta)[\delta(k-q,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\big]\delta(k,q)} \qquad (k \geqq p+1). \quad (2.16.11)$$

The result is sharp for the functions $f_k(z)$ given by

$$f_k(z) = z^p - \frac{[\delta(p-q,m)-\alpha]\,\delta(p,q)}{\big[(1+\beta)[\delta(k-q,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\big]\delta(k,q)} z^k \qquad (k \geqq p+1). \quad (2.16.12)$$

The following two results give distorsion theorems for the functions from our class:

Theorem 2.16.5. **([96])** *Let the function $f(z)$ defined by (2.16.7) be in the class $\mathcal{UST}_m(p, q; \alpha, \beta)$. Then, for $|z| = r < 1$,*

$$|f(z)| \geqq r^p - \frac{[\delta(p-q,m)-\alpha]\delta(p,q)}{\big[(1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\big]\delta(p+1,q)} r^{p+1} \quad (2.16.13)$$

and

$$|f(z)| \leqq r^p + \frac{[\delta(p-q,m)-\alpha]\,\delta(p,q)}{\big[(1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\big]\delta(p+1,q)}\, r^{p+1}, \tag{2.16.14}$$

The equalities in (2.16.13) and (2.16.14) are attained for the function $f(z)$ *given by*

$$f(z) = z^p - \frac{[\delta(p-q,m)-\alpha]\,\delta(p,q)}{\big[(1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\big]\delta(p+1,q)}\, z^{p+1} \tag{2.16.15}$$

at $z=r$ *and* $z=re^{i(2s+1)\pi}$ $(s\in\mathbb{Z})$.

Proof. For $k \geqq p+1$, we have

$$\big[(1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\big]\delta(p+1,q)$$
$$\leqq \big[(1+\beta)[\delta(k-q,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\big]\delta(k,q).$$

Now, using the hypothesis of Theorem 2.16.2, we get

$$\sum_{k=p+1}^{\infty} a_k \leqq \frac{[\delta(p-q,m)-\alpha]\,\delta(p,q)}{\big[(1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\big]\delta(p+1,q)}. \tag{2.16.16}$$

Lastly, by using the form (2.16.7) of the function, the proof of Theorem 2.16.5 is completed.

Theorem 2.16.6. **([96])** *Let the function* $f(z)$ *defined by (2.16.7) be in the class* $\mathcal{UST}_m(p,q;\alpha,\beta)$. *Then, for* $|z|=r<1$,

$$\big|f'(z)\big| \geqq pr^{p-1} - \frac{(p+1)\,[\delta(p-q,m)-\alpha]\,\delta(p,q)}{\big[(1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\big]\delta(p+1,q)}\, r^{p} \tag{2.16.17}$$

and

$$\big|f'(z)\big| \leqq pr^{p-1} + \frac{(p+1)\,[\delta(p-q,m)-\alpha]\,\delta(p,q)}{\big[(1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\big]\delta(p+1,q)}\, r^{p}. \tag{2.16.18}$$

The result is sharp for the function $f(z)$ *given by (2.16.15).*

Proof. Using similar techniques as in our demonstration of Theorem 2.16.5, we get

$$\sum_{k=p+1}^{\infty} k a_k \leqq \frac{(p+1)\,[\delta(p-q,m)-\alpha]\,\delta(p,q)}{\big[(1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\big]\delta(p+1,q)},$$

which leads us to the completion of the proof of Theorem 2.16.6.

Remark 2.16.7. (**[96]**) Taking $\beta = 0$, in the above theorems, we obtain results similar to those obtained by Liu and Liu [56] (with $\gamma = 1$ and $n = 1$).

By applying Theorem 2.16.2, we can prove that our class is closed under convex linear combinations as a corollary of the next result.

Theorem 2.16.8. (**[96]**) *Let* $\mu_\nu \geqq 0$ *for* $\nu = 1, 2, \cdots, l$ *and* $\sum_{\nu=1}^{l} \mu_\nu \leqq 1$. *If the functions* $f_\nu(z)$ *defined by*

$$f_\nu(z) = z^p - \sum_{k=p+1}^{\infty} a_{k,\nu} z^k \qquad (a_{k,\nu} \geqq 0;\ \nu = 1, 2, \cdots, l), \tag{2.16.19}$$

are in the class $\mathcal{UST}_m(p, q; \alpha, \beta)$ *for every* $\nu = 1, 2, \cdots, l$, *then the function* $f(z)$ *given by*

$$f(z) = z^p - \sum_{k=p+1}^{\infty} \left(\sum_{\nu=1}^{l} \mu_\nu a_{k,\nu} \right) z^k,$$

is also in the class $\mathcal{UST}_m(p, q; \alpha, \beta)$.

Proof. In order to proof this result, the assertion of Theorem 2.16.2 is used.

Theorem 2.16.9. (**[96]**) *Let* $f_p(z) = z^p$ *and*

$$f_k(z) = z^p - \frac{[\delta(p-q,m) - \alpha]\,\delta(p,q)}{\big[(1+\beta)[\delta(k-q,m) - \delta(p-q,m)] + [\delta(p-q,m) - \alpha]\big]\delta(k,q)}\, z^k \qquad (k \geqq p+1). \tag{2.16.20}$$

Then $f(z)$ *is in the class* $\mathcal{UST}_m(p, q; \alpha, \beta)$ *if and only if it can be expressed in the following form*:

$$f(z) = \sum_{k=p}^{\infty} \mu_k f_k(z), \tag{2.16.21}$$

where

$$\mu_k \geqq 0, \quad k \geqq p \qquad \text{and} \qquad \sum_{k=p}^{\infty} \mu_k = 1.$$

Proof. The part related to sufficiency is easily proved by using again the assertion of Theorem 2.16.2. For the necessity condition, we can see that the function $f(z)$ can be expressed in the form (2.16.21) if we set

$$\mu_k = \frac{\big[(1+\beta)[\delta(k-q,m) - \delta(p-q,m)] + [\delta(p-q,m) - \alpha]\big]\delta(k,q) a_k}{[\delta(p-q,m) - \alpha]\,\delta(p,q)} \qquad (k \geqq p+1)$$

and

$$\mu_p = 1 - \sum_{k=p+1}^{\infty} \mu_k,$$

such that $\mu_p \geqq 0$. This is already assured by Corollary 2.16.4.

Corollary 2.16.10. (**[96]**) *The extreme points of the class $\mathcal{UST}_m(p, q; \alpha, \beta)$ are the functions $f_p(z) = z^p$ and*

$$f_k(z) = z^p - \frac{[\delta(p-q,m) - \alpha]\,\delta(p,q)}{\big[(1+\beta)[\delta(k-q,m) - \delta(p-q,m)] + [\delta(p-q,m) - \alpha]\big]\delta(k,q)}\, z^k \qquad (k \geqq p+1).$$

In what follows we will see results related to radii of close-to-convexity, starlikeness and convexity:

Theorem 2.16.11. (**[96]**) *Let the function $f(z)$ defined by (2.16.7) be in the class $\mathcal{UST}_m(p, q; \alpha, \beta)$. Then $f(z)$ is a p-valent close-to-convex function of order ξ $(0 \leqq \xi < p)$ for $|z| \leqq r_1(p, q; \alpha, \beta; \xi)$, where*

$$r_1 = \inf_{k \geqq p+1} \left\{ \frac{\big[(1+\beta)[\delta(k-q,m) - \delta(p-q,m)] + [\delta(p-q,m) - \alpha]\big]\delta(k,q)}{[\delta(p-q,m) - \alpha]\,\delta(p,q)} \left(\frac{p-\xi}{k}\right) \right\}^{\frac{1}{k-p}}. \tag{2.16.22}$$

The result is sharp and the extremal functions are given by (2.16.12).

Proof. By applying Corollary 2.16.4 and the form (2.16.7), we see that, for $|z| \leqq r_1$, we have

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leqq p - \xi \;\; for \; |z| \leqq r_1(p, q; \alpha, \beta; \xi)\ , \tag{2.16.23}$$

which completes the proof of Theorem 2.16.11.

Theorem 2.16.12. (**[96]**) *Let the function $f(z)$ defined by (2.16.7) be in the class $\mathcal{UST}_m(p, q; \alpha, \beta)$.Then $f(z)$ is a p-valent starlike function of order ξ $(0 \leqq \xi < p)$ for $|z| \leqq r_2(p, q, \alpha, \beta, \xi)$, where*

$$r_2 = \inf_{k \geqq p+1} \left\{ \frac{\big[(1+\beta)[\delta(k-q,m) - \delta(p-q,m)] + [\delta(p-q,m) - \alpha]\big]\delta(k,q)}{[\delta(p-q,m) - \alpha]\,\delta(p,q)} \left(\frac{p-\xi}{k-\xi}\right) \right\}^{\frac{1}{k-p}}. \tag{2.16.24}$$

The result is sharp and the extremal functions are given by (2.16.12).

Proof. Using the same steps as in the proof of Theorem 2.16.11, it is seen that

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leqq p - \xi \ for\,|z| \leqq r_2(p,q,\alpha,\beta,\xi). \tag{2.16.25}$$

Corollary 2.16.13. ([96]) *Let the function $f(z)$ defined by (2.16.7) be in the class $\mathcal{UST}_m(p,q;\alpha,\beta)$. Then $f(z)$ is a p-valent convex function of order ξ $(0 \leqq \xi < p)$ for $|z| \leqq r_3(p,q,\alpha,\beta,\xi)$, where*

$$r_3 = \inf_{k \geqq p+1}\left\{\frac{[(1+\beta)[\delta(k-q,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]]\,\delta(k,q)}{[\delta(p-q,m)-\alpha]\,\delta(p,q)}\left(\frac{p(p-\xi)}{k(k-\xi)}\right)\right\}^{\frac{1}{k-p}}. \tag{2.16.26}$$

The result is sharp and the extremal function is given by (2.16.12).

In view of Theorem 2.16.2, we see that the function:

$$z^p - \sum_{k=p+1}^{\infty} d_k z^k$$

is in the class $\mathcal{UST}_m(p,q;\alpha,\beta)$ as long as $0 \leqq d_k \leqq a_k$ for all $k \geqq p+1$, where a_k is the coefficient corresponding to a function which is in the class $\mathcal{UST}_m(p,q;\alpha,\beta)$. We are thus led to the next theorem.

Theorem 2.16.14.([96]) *Let the function $f(z)$ defined by (2.16.7) be in the class $\mathcal{UST}_m(p,q;\alpha,\beta)$. Also let c be a real number such that $c > -p$. Then the function $F(z)$ defined by*

$$F(z) = \frac{c+p}{z^c}\int_0^z t^{c-1} f(t)dt \qquad (c > -p) \tag{2.16.27}$$

also belongs to the class $\mathcal{UST}_m(p,q;\alpha,\beta)$.

Proof. From the representation (2.16.27) of $F(z)$, it follows that

$$F(z) = z^p - \sum_{k=p+1}^{\infty} d_k z^k,$$

where

$$d_k = \left(\frac{c+p}{k+c}\right) a_k \leqq a_k \qquad (k \geqq p+1).$$

Putting $c = 1-p$ in Theorem 2.16.14, we get the following corollary.

Corollary 2.16.15. ([96]) *Let the function $f(z)$ defined by (2.16.7) be in the class*

$\mathcal{UST}_m(p,q;\alpha,\beta)$. *Also let* $F(z)$ *be defined by*

$$F(z)=\frac{1}{z^{1-p}}\int_0^z \frac{f(t)}{t^p}dt. \tag{2.16.28}$$

Then $F(z)\in\mathcal{UST}_m(p,q;\alpha,\beta)$.

Remark 2.16.16. ([96]) The converse of Theorem 2.16.14 is not true. This observation leads to the following result involving the radius of p-valence.

Theorem 2.16.17. ([96]) *Let the function*

$$F(z)=z^p-\sum_{k=p+1}^{\infty}a_kz^k \qquad (a_k\geqq 0)$$

be in the class $\mathcal{UST}_m(p,q;\alpha,\beta)$. *Also let* c *be a real number such that* $c>-p$. *Then the function* $f(z)$ *given by (2.16.27) is* p*-valent in* $|z|<R_p^*$, *where*

$$R_p^*=\inf_{k\geqq p+1}\left\{\frac{\left[(1+\beta)[\delta(k-q,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\right]\delta(k,q)}{[\delta(p-q,m)-\alpha]\,\delta(p,q)}\left(\frac{p(c+p)}{k(c+k)}\right)\right\}^{\frac{1}{k-p}}. \tag{2.16.29}$$

The result is sharp.

Proof. From the definition (2.16.27), we have

$$f(z)=\frac{z^{1-c}[z^c\ F(z)]'}{c+p}=z^p-\sum_{k=p+1}^{\infty}\frac{k+c}{c+p}a_kz^k \qquad (c>-p).$$

In order to obtain the required result, it suffices to show that

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|\leqq p \quad \text{for} \quad |z|<R_p^*\ ,$$

where R_p^* is given by (2.16.29). Making use of Theorem 2.16.2, we get that the required inequality is true if

$$|z|\leqq\left(\frac{\left[(1+\beta)[\delta(k-q,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\right]\delta(k,q)}{[\delta(p-q,m)-\alpha]\,\delta(p,q)}\left(\frac{p(c+p)}{k(c+k)}\right)\right)^{\frac{1}{k-p}} \tag{2.16.30}$$

$$(k\geqq p+1).$$

The result is sharp for the function $f(z)$ given by

$$f(z)=z^p-$$

$$\frac{(c+k)\,[\delta(p-q,m)-\alpha]\,\delta(p,q)}{(c+p)\big[(1+\beta)[\delta(k-q,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\big]\delta(k,q)}\,z^k \qquad (k \geqq p+1). \tag{2.16.31}$$

The next two results are related to the modified Hadamard product:

Let the functions $f_\nu(z)$ $(\nu = 1, 2)$ be defined by (2.16.19). The *modified* Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z^p - \sum_{k=p+1}^{\infty} a_{k,1}a_{k,2}z^k. \tag{2.16.32}$$

Theorem 2.16.18. ([96]) *Let the functions $f_\nu(z)$ $(\nu = 1, 2)$, defined by (2.16.19) be in the class $\mathcal{UST}_m(p, q; \alpha, \beta)$. Then $(f_1 * f_2)(z) \in \mathcal{UST}_m(p, q; \eta, \beta)$, where*

$$\eta = \delta(p-q,m) - \frac{[\delta(p-q,m)-\alpha]^2(1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)]\delta(p,q)}{\big[(1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\big]^2\delta(p+1,q)-[\delta(p-q,m)-\alpha]^2\,\delta(p,q)}. \tag{2.16.33}$$

The result is sharp when

$$f_1(z) = f_2(z) = f(z),$$

where the function $f(z)$ is given by

$$f(z) = z^p - \frac{[\delta(p-q,m)-\alpha]\,\delta(p,q)}{\big[(1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\big]\delta(p+1,q)}\,z^{p+1}. \tag{2.16.34}$$

Proof. Employing the technique used earlier by Schild and Silverman [92], we need to find the largest η such that

$$\sum_{k=p+1}^{\infty} \frac{\big[(1+\beta)[\delta(k-q,m)-\delta(p-q,m)]+[\delta(p-q,m)-\eta]\big]\delta(k,q)}{[\delta(p-q,m)-\eta]\,\delta(p,q)}a_{k,1}a_{k,2} \leqq 1. \tag{2.16.35}$$

Using the inequalities for the coefficients of the functions in the class $\mathcal{UST}_m(p, q; \eta, \beta)$, and by applying the Cauchy-Schwarz inequality, it is sufficient to show that

$$\eta \leqq \delta(p-q,m) - \frac{[\delta(p-q,m)-\alpha]^2(1+\beta)[\delta(k-q,m)-\delta(p-q,m)]\delta(p,q)}{\big[(1+\beta)[\delta(k-q,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\big]^2\delta(k,q)-[\delta(p-q,m)-\alpha]^2\,\delta(p,q)}. \tag{2.16.36}$$

Now, defining the function $G(k)$ by

$$G(k) = \delta(p-q,m) - \frac{[\delta(p-q,m)-\alpha]^2(1+\beta)[\delta(k-q,m)-\delta(p-q,m)]\delta(p,q)}{\Big[(1+\beta)[\delta(k-q,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\Big]^2\delta(k,q)-[\delta(p-q,m)-\alpha]^2\,\delta(p,q)}, \tag{2.16.37}$$

we see that $G(k)$ is an increasing function of k, $k \geqq p+1$, which obviously completes the proof.

Using similar arguments to those from the proof of Theorem 2.16.18, we obtain the following result.

Theorem 2.16.19. **([96])** *Let the function $f_1(z)$ defined by (2.16.19) be in the class $\mathcal{UST}_m(p,q;\alpha,\beta)$. Suppose also that the function $f_2(z)$ defined by (2.16.19) be in the class $\mathcal{UST}_m(p,q;\varphi,\beta)$. Then*

$$(f_1 * f_2)(z) \in \mathcal{UST}_m(p,q;\zeta,\beta),$$

where

$$\zeta = \delta(p-q,m) - \frac{[\delta(p-q,m)-\alpha][\delta(p-q,m)-\varphi](1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)]\delta(p,q)}{[(1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]][(1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)]+[\delta(p-q,m)-\varphi]]\delta(p+1,q)-\Omega} \tag{2.16.38}$$

with

$$\Omega = [\delta(p-q,m)-\alpha][\delta(p-q,m)-\varphi]\delta(p,q).$$

The result is sharp for the functions $f_\nu(z)$ $(\nu = 1,2)$ given by

$$f_1(z) = z^p - \frac{[\delta(p-q,m)-\alpha]\,\delta(p,q)}{\big[(1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\big]\delta(p+1,q)}\,z^{p+1} \tag{2.16.39}$$

and

$$f_2(z) = z^p - \frac{[\delta(p-q,m)-\varphi]\,\delta(p,q)}{\big[(1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)]+[\delta(p-q,m)-\varphi]\big]\delta(p+1,q)}\,z^{p+1}. \tag{2.16.40}$$

Theorem 2.16.20. **([96])** *Let the functions $f_\nu(z)$ $(\nu = 1,2)$ defined by (2.16.19) be in the class $\mathcal{UST}_m(p,q;\alpha,\beta)$. Then the function $h(z)$ given by*

$$h(z) = z^p - \sum_{k=p+1}^{\infty}(a_{k,1}^2 + a_{k,2}^2)z^k \tag{2.16.41}$$

belongs to the class $\mathcal{UST}_m(p,q;\alpha,\phi)$, *where*

$$\phi = \delta(p-q,m) - \frac{2[\delta(p-q,m)-\alpha]^2(1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)]\delta(p,q)}{\big[(1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\big]^2\delta(p+1,q)-2[\delta(p-q,m)-\alpha]^2\delta(p,q)}. \tag{2.16.42}$$

The result is sharp for

$$f_1(z) = f_2(z) = f(z),$$

where the function $f(z)$ *is given by (2.16.34).*

Proof. If we combine the assertions of Theorem 2.16.2 for both of the functions $f_1(z)$ and $f_2(z)$, we get

$$\sum_{k=p+1}^{\infty} \frac{1}{2}\left(\frac{\big[(1+\beta)[\delta(k-q,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\big]\delta(k,q)}{[\delta(p-q,m)-\alpha]\delta(p,q)}\right)^2 (a_{k,1}^2 + a_{k,2}^2) \leqq 1. \tag{2.16.43}$$

Therefore, we need to find the largest $\phi = \phi(p,q,\alpha,\beta)$ such that

$$\frac{\big[(1+\beta)[\delta(k-q,m)-\delta(p-q,m)]+[\delta(p-q,m)-\phi]\big]\delta(k,q)}{[\delta(p-q,m)-\phi]\delta(p,q)} \leqq \frac{1}{2}\left(\frac{\big[(1+\beta)[\delta(k-q,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\big]\delta(k,q)}{[\delta(p-q,m)-\alpha]\delta(p,q)}\right)^2. \tag{2.16.44}$$

Since $D(k)$ given by

$$D(k) = \delta(p-q,m) - \frac{2[\delta(p-q,m)-\alpha]^2(1+\beta)[\delta(k-q,m)-\delta(p-q,m)]\delta(p,q)}{\big[(1+\beta)[\delta(k-q,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\big]^2\delta(k,q)-2[\delta(p-q,m)-\alpha]^2\delta(p,q)}$$

is an increasing function of k $(k \geqq p+1)$, we obtain $\phi \leqq D(p+1)$. The proof of Theorem 2.16.20 is thus completed.

2.17 Fractional calculus of analytic functions concerned with Möbius transformations

Applying the Möbius transformations, we consider in this section, some properties of fractional calculus of $f(z) \in \mathcal{A}$. Also some interesting examples for fractional calculus are given. The results are published with D. Breaz and S. Owa in [34].

In this section we work with the class $\mathcal{A}$ of functions $f(z)$:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{2.17.1}$$

analytic in the open unit disk, $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

and with the characterization

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U}) \tag{2.17.2}$$

with real $\alpha\,(0 \leqq \alpha < 1)$, for $f(z)$ starlike of order α in $\mathbb{U}$. We denote by $\mathcal{S}^*(\alpha)$ the class of all starlike functions $f(z)$ of order α in $\mathbb{U}$ and $\mathcal{S}^*(0) \equiv \mathcal{S}^*$. Also we will use the characterization

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathbb{U}) \tag{2.17.3}$$

with some real $\alpha\,(0 \leqq \alpha < 0)$, for $f(z)$ convex of order α in $\mathbb{U}$. We denote by $\mathcal{K}(\alpha)$ the class of all such functions $f(z)$ and $\mathcal{K}(0) \equiv \mathcal{K}$. In view of definitions for the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, we know that

(i) $f(z) \in \mathcal{K}(\alpha)$ if and only if $zf'(z) \in \mathcal{S}^*(\alpha)$

and

(ii) $f(z) \in \mathcal{S}^*(\alpha)$ if and only if $\displaystyle\int_0^z \frac{f(t)}{t}dt \in \mathcal{K}(\alpha)$.

Further, MacGregor [58] and Wilken and Feng [101] have the sharp inclusion relation that $\mathcal{K}(\alpha) \subset \mathcal{S}^*(\beta)$ for each $\alpha(0 \leqq \alpha < 1)$ with

$$\beta = \begin{cases} \dfrac{1-2\alpha}{2^{2(1-\alpha)}(1-2^{2\alpha-1})} & (\alpha \neq \frac{1}{2}) \\[2ex] \dfrac{1}{2\log 2} \quad (\alpha = \frac{1}{2}). \end{cases} \tag{2.17.4}$$

For $\alpha = 0$, Marx [59] and Strohhäcker [99] showed that $\mathcal{K} \subset \mathcal{S}^*(\frac{1}{2})$. Also, by Robertson [89], we know that the extremal function $f(z)$ for the class $\mathcal{S}^*(\alpha)$ is

$$f(z) = \frac{z}{(1-z)^{2(1-\alpha)}} = z + \sum_{k=2}^{\infty} \frac{\prod_{j=2}^{k}(j-2\alpha)}{(k-1)!} z^k \tag{2.17.5}$$

and the extremal function $f(z)$ for the class $\mathcal{K}(\alpha)$ is

$$f(z) = \begin{cases} \dfrac{1-(1-z)^{2\alpha-1}}{2\alpha-1} = z + \displaystyle\sum_{k=2}^{\infty} \frac{\prod_{j=2}^{k}(j-2\alpha)}{k!} z^k \quad (\alpha \neq \frac{1}{2}) \\[2ex] -\log(1-z) = z + \displaystyle\sum_{k=2}^{\infty} \frac{1}{k} z^k \quad (\alpha = \frac{1}{2}). \end{cases} \tag{2.17.6}$$

For $f(z) \in \mathcal{A}$, we apply the following Möbius transformation

$$w(\zeta) = \frac{z+\zeta}{1+\bar{z}\zeta} \qquad (\zeta \in \mathbb{U}) \tag{2.17.7}$$

for a fixed $z \in \mathbb{U}$. This Möbius transformation $w(\zeta)$ maps $\mathbb{U}$ onto itself and $\zeta = 0$ to $w(0) = z$.

From among the various definitions for fractional calculus (that is, fractional derivatives and fractional integrals) given in the literature, we have to recall here the following definitions for fractional calculus which are used by Owa [71], [72] and by Owa and Srivastava [74].

Definition 2.17.1. *The fractional integral of order λ is defined, for $f(z) \in \mathcal{A}$, by*

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta, \tag{2.17.8}$$

where $\lambda > 0$ and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z - \zeta > 0$.

Definition 2.17.2. *The fractional derivative of order λ is defined, for $f(z) \in \mathcal{A}$, by*

$$D_z^{\lambda} f(z) = \frac{d}{dz}\left(D_z^{\lambda-1} f(z)\right) \tag{2.17.9}$$

$$= \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta,$$

where $0 \leqq \lambda < 1$ and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed as in Definition 2.17.1 above.

Definition 2.17.3. *Under the hypotheses of Definition 2.17.2, the fractional derivative of order $n + \lambda$ is defined by*

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n}\left(D_z^{\lambda} f(z)\right), \tag{2.17.10}$$

where $0 \leqq \lambda < 1$ and $n \in \mathbb{N}_0 = 0, 1, 2, \cdots$.

Remark 2.17.4. ([34]) In view of definitions for the fractional calculus of $f(z) \in \mathcal{A}$, we see that

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(2+\lambda)} z^{1+\lambda} + \frac{2!}{\Gamma(3+\lambda)} a_2 z^{2+\lambda} + \cdots + \frac{k!}{\Gamma(k+1+\lambda)} a_k z^{k+\lambda} + \cdots \tag{2.17.11}$$

$$= \frac{1}{\Gamma(2+\lambda)} z^{1+\lambda} + \sum_{k=2}^{\infty} \frac{k!}{\Gamma(k+1+\lambda)} a_k z^{k+\lambda} \quad (\lambda > 0),$$

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(2-\lambda)} z^{1-\lambda} + \frac{2!}{\Gamma(3-\lambda)} a_2 z^{2-\lambda} + \cdots + \frac{k!}{\Gamma(k+1-\lambda)} a_k z^{k-\lambda} + \cdots \tag{2.17.12}$$

$$= \frac{1}{\Gamma(2-\lambda)} z^{1-\lambda} + \sum_{k=2}^{\infty} \frac{k!}{\Gamma(k+1-\lambda)} a_k z^{k-\lambda} \quad (0 \leqq \lambda < 1),$$

and

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n}\left(\frac{1}{\Gamma(2-\lambda)} z^{1-\lambda} + \sum_{k=2}^{\infty} \frac{k!}{\Gamma(k+1-\lambda)} a_k z^{k-\lambda}\right) \tag{2.17.13}$$

$$= \frac{1}{\Gamma(2-n-\lambda)} z^{1-n-\lambda} + \sum_{k=2}^{\infty} \frac{k!}{\Gamma(k+1-n-\lambda)} a_k z^{k-n-\lambda}$$

for $0 \leqq \lambda < 1$ and $n \in \mathbb{N}_0$.
Therefore, we can write that

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n}\left(D_z^{\lambda} f(z)\right) = D_z^{\lambda}\left(\frac{d^n}{dz^n} f(z)\right) \tag{2.17.14}$$

and

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(2-\lambda)} z^{1-\lambda} + \sum_{k=2}^{\infty} \frac{k!}{\Gamma(k+1-\lambda)} a_k z^{k-\lambda} \tag{2.17.15}$$

for any real number λ.

Using the fractional calculus (2.17.15), we define

$$F(z) = \Gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z) = z + \sum_{k=2}^{\infty} \frac{k!\Gamma(2-\lambda)}{\Gamma(k+1-\lambda)} a_k z^k \quad (\lambda \in \mathbb{R}, \lambda \neq 2). \tag{2.17.16}$$

If we take $\lambda = -1$ in (2.17.16), then

$$F(z) = \Gamma(3) z^{-1} D_z^{-1} f(z) = \frac{2}{z}\int_0^z f(t)dt = z + \sum_{k=2}^{\infty} \frac{2}{k+1} a_k \cdot z^k$$

implies the Libera integral operator defined by Libera [54]. Therefore, $F(z)$ given by (2.17.16) is the generalization operator of Libera integral operator.

Let us give two examples for the fractional operator $F(z)$ defined in (2.17.16).

Example 2.17.5. ([34]) Let us define $f(z)$ by

$$f(z) = z + \frac{2-\lambda}{6} z^2 \in \mathcal{A} \qquad (-1 \leqq \lambda < 2). \tag{2.17.17}$$

Then, we have that

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) = \mathrm{Re}\left(2 - \frac{1}{1+Mz}\right) \tag{2.17.18}$$

$$= 2 - \frac{1+M\cos\theta}{1+M^2+2M\cos\theta} \qquad (z = e^{i\theta}),$$

where $M = \dfrac{2-\lambda}{6} > 0$. If we define

$$h(t) = \frac{1+Mt}{1+M^2+2Mt} \qquad (t = \cos\theta), \tag{2.17.19}$$

then

$$h'(t) = \frac{M(M+1)(M-1)}{(1+M^2+2Mt)^2} < 0 \qquad (0 < M \leqq \frac{1}{2}). \tag{2.17.20}$$

This shows us that

$$h(t) \leqq h(-1) = \frac{1}{1-M}, \tag{2.17.21}$$

that is, that

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 2 - \frac{1}{1-M} = \frac{2+2\lambda}{4+\lambda} > 0 \qquad (z \in \mathbb{U}). \tag{2.17.22}$$

Therefore, $f(z) \in \mathcal{S}^*\left(\dfrac{2+2\lambda}{4+\lambda}\right)$.

For $f(z)$ given by (2.17.17), $F(z)$ becomes

$$F(z) = \Gamma(2-\lambda)z^\lambda D_z^\lambda f(z) = z + \frac{1}{3}z^2 \qquad (-1 \leqq \lambda < 2). \tag{2.17.23}$$

Then, we see that $F(z) \in \mathcal{S}^*\left(\dfrac{1}{2}\right)$.

Next, let us consider the function $g(\zeta)$ given by

$$g(\zeta) = \frac{(F \circ w)(\zeta) - F(z)}{(1-|z|^2)F'(z)} \qquad (\zeta \in \mathbb{U}) \tag{2.17.24}$$

for a fixed $z \in \mathbb{U}$, where $w(\zeta)$ is given by (2.17.7). Then, it is easy to see that $g(\zeta) \in \mathcal{A}$. Taking $z = \dfrac{1}{2}$ in (2.17.24), we have that

$$g(\zeta) = \frac{\zeta(11\zeta+16)}{4(\zeta+2)^2} \qquad (\zeta \in \mathbb{U}) \tag{2.17.25}$$

and

$$\mathrm{Re}\left(\frac{\zeta g'(\zeta)}{g(\zeta)}\right) = \mathrm{Re}\left(1 - \frac{\zeta(11\zeta+10)}{(\zeta+2)(11\zeta+16)}\right) \tag{2.17.26}$$

$$= 1 - \frac{704\cos^2\theta + 848\cos\theta + 149}{1408\cos^2\theta + 3268\cos\theta + 1885} \quad (\theta = e^{i\theta}).$$

Letting

$$H(t) = \frac{704t^2+848t+149}{1408t^2+3268t+1885} \qquad (t = \cos\theta), \tag{2.17.27}$$

we obtain that

$$H'(t) = \frac{12(9224t^2+186208t+92629)}{(1408t^2+3268t+1885)^2}. \tag{2.17.28}$$

This shows that $H'(-1) < 0, H'(0) > 0$, and $H'(1) > 0$. Therefore, there exists some t_0 such that $H'(t_0) = 0$ for $-1 < t_0 < 0$. It follows that

$$\mathrm{Max}_{-1 \leqq t \leqq 1} H(t) = \mathrm{Max}\{H(-1), H(1)\} = H(1) = \frac{7}{27}. \tag{2.17.29}$$

Thus, we say that

$$\mathrm{Re}\left(\frac{\zeta g'(\zeta)}{g(\zeta)}\right) > 1 - \frac{7}{27} = \frac{20}{27} \qquad (\zeta \in \mathbb{U}). \tag{2.17.30}$$

Consequently, we say that $F(z) \in \mathcal{S}^*\left(\frac{1}{2}\right), g(\zeta) \in \mathcal{S}^*\left(\frac{20}{27}\right)$ for $f(z) \in \mathcal{S}^*\left(\frac{2+2\lambda}{4+\lambda}\right)$ given by (2.17.17).
If $\lambda = -\frac{1}{2}$, then

$$f(z) = z + \frac{5}{12}z^2 \in \mathcal{S}^*\left(\frac{2}{7}\right).$$

The open unit disk $\mathbb{U}$ is mapped on a starlike domain of order $\frac{2}{7}$.

If $\lambda = \frac{1}{3}$, then

$$f(z) = z + \frac{5}{18}z^2 \in \mathcal{S}^*\left(\frac{8}{13}\right).$$

Thus, $f(z)$ maps $\mathbb{U}$ on to a starlike domain of order $\frac{8}{13}$.

Example 2.17.5 means that there is some function $f(z) \in \mathcal{S}^*(\alpha)$ such that $F(z) \in \mathcal{S}^*(\beta)$ and $g(\zeta) \in \mathcal{S}^*(\gamma)$.

Next, we consider

Example 2.17.6. ([34]) Let a function $f(z)$ be given by

$$f(z) = z + \frac{2-\lambda}{12}z^2 \in \mathcal{A} \qquad (-1 \leqq \lambda < 2). \tag{2.17.31}$$

Then, we have that

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) = \mathrm{Re}\left(2 - \frac{1}{1+2Mz}\right) \tag{2.17.32}$$
$$= 2 - \frac{1+2M\cos\theta}{1+4M^2+4M\cos\theta} \qquad (z = e^{i\theta}),$$

where $M = \frac{2-\lambda}{12} > 0$. Defining $h(t)$ by

$$h(t) = \frac{1+2Mt}{1+4M^2+4Mt} \qquad (t = \cos\theta), \tag{2.17.33}$$

we have that

$$h'(t) = \frac{2M(2M+1)(2M-1)}{(1+4M^2+4Mt)^2} < 0 \qquad (0 < M \leqq \frac{1}{4}) \tag{2.17.34}$$

which shows us that

$$h(t) \leqq h(-1) = \frac{1}{1-2M}. \tag{2.17.35}$$

Thus, we obtain that

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > 2 - \frac{1}{1-2M} = \frac{2+2\lambda}{4+\lambda} > 0 \qquad (z \in \mathbb{U}). \tag{2.17.36}$$

This gives us that $f(z) \in \mathcal{K}\left(\frac{2+2\lambda}{4+\lambda}\right)$.

For $f(z)$ given by (2.17.31), $F(z)$ becomes

$$F(z) = \Gamma(2-\lambda)z^{\lambda}D_z^{\lambda}f(z) = z + \frac{1}{6}z^2 \qquad (-1 \leqq \lambda < 2). \tag{2.17.37}$$

Then, it is easy to see that $F(z) \in \mathcal{K}\left(\frac{1}{2}\right)$.

For this $F(z)$, we consider $g(\zeta)$ defined by (2.17.24). If we take $z = \frac{1}{2}$ for $g(\zeta)$, we have that

$$g(\zeta) = \frac{\zeta(17\zeta+28)}{7(\zeta+2)^2} \qquad (\zeta \in \mathbb{U}) \tag{2.17.38}$$

and

$$\mathrm{Re}\left(1+\frac{\zeta g''(\zeta)}{g'(\zeta)}\right) = \mathrm{Re}\left(1 - \frac{\zeta(10\zeta+11)}{(5\zeta+7)(\zeta+2)}\right) \tag{2.17.39}$$

$$= 1 - \frac{280\cos^2\theta + 379\cos\theta + 97}{280\cos^2\theta + 646\cos\theta + 370} \qquad (\zeta = e^{i\theta}).$$

If we write that

$$H(t) = \frac{280t^2+379t+97}{280t^2+646t+370} \qquad (t = \cos\theta), \tag{2.17.40}$$

then

$$H'(t) = \frac{24(3115t^2+6370t+3232)}{(280t^2+646t+370)^2}. \tag{2.17.41}$$

Since $H'(-1) < 0, H'(0) > 0$, and $H'(1) > 0$, there exists some t_0 such that $H'(t_0) = 0$ for $-1 < t_0 < 0$. This gives us that

$$\mathrm{Max}_{-1 \leqq t \leqq 1} H(t) = \mathrm{Max}\{H(-1), H(1)\} = H(1) = \frac{7}{12}. \tag{2.17.42}$$

It follows that

$$\mathrm{Re}\left(1+\frac{\zeta g''(\zeta)}{g'(\zeta)}\right) > 1 - \frac{7}{12} = \frac{5}{12} \qquad (\zeta \in \mathbb{U}). \tag{2.17.43}$$

Therefore, we say that $F(z) \in \mathcal{K}\left(\frac{1}{2}\right), g(\zeta) \in \mathcal{K}\left(\frac{5}{12}\right)$ for $f(z) \in \mathcal{K}\left(\frac{2+2\lambda}{4+\lambda}\right)$.

If $\lambda = -\frac{2}{3}$, then

$$f(z) = z + \frac{2}{9}z^2 \in \mathcal{K}\left(\frac{1}{5}\right)$$

maps $\mathbb{U}$ on to a convex domain of order $\frac{1}{5}$.

If $\lambda = \frac{3}{2}$, then

$$f(z) = z + \frac{1}{24}z^2 \in \mathcal{K}\left(\frac{10}{11}\right).$$

This function $f(z)$ maps $\mathbb{U}$ on to a convex domain of order $\frac{10}{11}$.

Example 2.17.6 says that there exists some function $f(z) \in \mathcal{K}(\alpha)$ such that $F(z) \in \mathcal{K}(\beta)$ and $g(\zeta) \in \mathcal{K}(\gamma)$.

In view of the previous examples, we introduce

Definition 2.17.7. ([34]) *Let $f(z) \in \mathcal{A}, F(z) = \Gamma(2-\lambda)z^{\lambda}D_z^{\lambda}f(z)$ with $-1 \leqq \lambda < 2$ and let $g(\zeta)$ be defined by (2.17.24) for a fixed $z \in \mathbb{U}$. Then we say that*
(i) $f(z) \in \mathcal{S}_0$ if $g(\zeta)$ is univalent in $\mathbb{U}$,
(ii) $f(z) \in \mathcal{S}_0^*(\alpha)$ if $g(\zeta) \in \mathcal{S}^*(\alpha)$
and
(iii) $f(z) \in \mathcal{K}_0(\alpha)$ if $g(\zeta) \in \mathcal{K}(\alpha)$.

Also, we write that $\mathcal{S}_0^*(0) \equiv \mathcal{S}_0^*$ and $\mathcal{K}_0(0) \equiv \mathcal{K}_0$ when $\alpha = 0$.

In order to discuss our classes $\mathcal{S}_0, \mathcal{S}_0^*(\alpha)$ and $\mathcal{K}_0(\alpha)$, we need the following lemma due to Robertson [89] (also see Duren [42]).

Lemma 2.17.8 *If $f(z) \in \mathcal{S}^*(\alpha)$, then*

$$|a_k| \leqq \frac{\prod_{j=2}^{k}(j-2\alpha)}{(k-1)!} \qquad (k = 2, 3, 4, \cdots) \tag{2.17.44}$$

with the equality in (2.17.44) with $f(z)$ given by (2.17.5). If $f(z) \in \mathcal{K}(\alpha)$, then

$$|a_k| \leqq \frac{\prod_{j=2}^{k}(j-2\alpha)}{k!} \qquad (k = 2, 3, 4, \cdots) \tag{2.17.45}$$

with the equality in (2.17.45) with $f(z)$ given by (2.17.6).

We also need the following lemma.

Lemma 2.17.9. ([34]) *If $g(\zeta)$ is defined by*

$$g(\zeta) = \frac{(f \circ w)(\zeta) - f(z)}{(1 - |z|^2) f'(z)} \qquad (\zeta \in \mathbb{U}) \tag{2.17.46}$$

for a fixed $z \in \mathbb{U}$ for $f(z) \in \mathcal{A}$, then

$$\frac{\dfrac{d^n}{d\zeta^n}(f \circ w)(\zeta)}{f'(z)} \tag{2.17.47}$$

$$= \frac{n!(n-1)!(1+\overline{z}\zeta)^{2n}}{(1-|z|^2)^{n-1}} \left(\sum_{j=0}^{n-1} \frac{g^{(n-j)}(\zeta)\overline{z}^j}{(n-j)!(n-j-1)!j!(1+\overline{z}\zeta)^j} \right)$$

for $n = 1, 2, 3, \ldots$, where $w(\zeta)$ is given by (2.17.7).

Proof We use the mathematical induction to prove (2.40). For $n = 1$, the right-hand side of (2.17.47) becomes that

$$(1 + \overline{z}\zeta)^2 g'(\zeta) = \frac{\dfrac{d}{d\zeta}(f \circ w)(\zeta)}{f'(z)}, \tag{2.17.48}$$

which is given the left-hand side of (2.17.47) for $n = 1$. Therefore, (2.17.47) holds true for $n = 1$. Assume that the relation (2.17.47) is true for a fixed positive integer n. Then, some calculations lead us to

$$\frac{\dfrac{d^{n+1}}{d\zeta^{n+1}}(f \circ w)(\zeta)(1 - |z|^2)}{(1 + \overline{z}\zeta)^2 f'(z)} \tag{2.17.49}$$

$$= \frac{n!(n-1)!(1+\overline{z}\zeta)^{2n}}{(1-|z|^2)^{n-1}} \left\{ \sum_{j=0}^{n-1} \frac{1}{(n-j)!(n-j-1)!j!} \left(g^{(n+1-j)}(\zeta) + \frac{(2n-j)g^{(n-j)}(\zeta)\overline{z}}{1+\overline{z}\zeta} \right) \frac{\overline{z}^j}{(1+\overline{z}\zeta)^j} \right\}$$

$$= \frac{(n+1)!n!(1+\overline{z}\zeta)^{2n}}{(1-|z|^2)^{n-1}} \left(\sum_{j=0}^{n} \frac{g^{(n+1-j)}(\zeta)\overline{z}^j}{(n+1-j)!(n-j)!j!(1+\overline{z}\zeta)^j} \right).$$

This means that the relation (2.17.47) holds true for $n+1$. Thus, by applying the mathematical induction, we complete the proof of the lemma.

Taking $\zeta = 0$ in Lemma 2.17.9, we have

Corollary 2.17.10. ([34]) *If $g(\zeta)$ is defined by (2.17.46) for $f(z) \in \mathcal{A}$, then we have*

$$\left| \frac{f^{(n)}(z)}{f'(z)} \right| \leqq \frac{n!(n-1)!}{(1-|z|^2)^{n-1}} \left(\sum_{j=0}^{n-1} \frac{|g^{(n-j)}(0)||z|^j}{(n-j)!(n-j-1)!j!} \right) \tag{2.17.50}$$

for $z \in \mathbb{U}$. Furthermore, we have

$$\left|\frac{f''(z)}{f'(z)}\right| \leqq \frac{|g''(0)| + 2|g'(0)||z|}{1-|z|^2} \quad (z \in \mathbb{U}). \tag{2.17.51}$$

Applying Corollary 2.17.10, we have

Theorem 2.17.11. ([34]) *Let $F(z)$ be defined by (2.17.16) for $f(z) \in \mathcal{A}$ with $-1 \leqq \lambda < 2$.*
(i) If $f(z) \in \mathcal{S}_0$, then

$$\left|\frac{F^{(n)}(z)}{F'(z)}\right| \leqq \frac{n!(n+|z|)}{(1-|z|)^{n-1}(1+|z|)} \qquad (n = 1, 2, 3, \cdots) \tag{2.17.52}$$

with the equality for $g(\zeta)$ given by

$$g(\zeta) = \frac{\zeta}{(1+e^{i\theta}\zeta)^2} \qquad (\theta \in \mathbb{R}). \tag{2.17.53}$$

(ii) If $f(z) \in \mathcal{S}_0^*(\alpha)$, then

$$\left|\frac{F^{(n)}(z)}{F'(z)}\right| \leqq \frac{n!(n-1)!}{(1-|z|^2)^{n-1}} \left(\sum_{j=0}^{n-1} \frac{\prod_{k=2}^{n-j}(k-2\alpha)}{j!((n-j-1)!)^2}|z|^j\right) \tag{2.17.54}$$

with the equality for $g(\zeta)$ given by

$$g(\zeta) = \frac{\zeta}{(1+e^{i\theta}\zeta)^{2(1-\alpha)}} \qquad (\theta \in \mathbb{R}). \tag{2.17.55}$$

(iii) If $f(z) \in \mathcal{K}_0(\alpha)$, then

$$\left|\frac{F^{(n)}(z)}{F'(z)}\right| \leqq \frac{n!(n-1)!}{(1-|z|^2)^{n-1}} \left(\sum_{j=0}^{n-1} \frac{\prod_{k=2}^{n-j}(k-2\alpha)}{j!(n-j)!(n-j-1)!}|z|^j\right) \quad (n = 1, 2, 3, \ldots) \tag{2.17.56}$$

with the equality for $g(\zeta)$ given by

$$g(\zeta) = \begin{cases} \dfrac{1-(1-\zeta)^{2\alpha-1}}{2\alpha-1} & \left(\alpha \neq \dfrac{1}{2}\right), \\ -\log(1-\zeta) & \left(\alpha = \frac{1}{2}\right). \end{cases} \tag{2.17.57}$$

Proof Note that

$$g(\zeta) = \frac{(F \circ w)(\zeta) - F(z)}{(1-|z|^2)F'(z)} \qquad (\zeta \in \mathbb{U})$$

for $F(z) = \Gamma(2-\lambda)z^\lambda D_z^\lambda f(z)$. Therefore, Corollary 2.17.10 gives us that

$$\left|\frac{F^{(n)}(z)}{F'(z)}\right| \leqq \frac{n!(n-1)!}{(1-|z|^2)^{n-1}} \left(\sum_{j=0}^{n-1} \frac{|g^{(n-j)}(0)||z|^j}{(n-j)!(n-j-1)!j!}\right). \tag{2.17.58}$$

According to Lemma 2.17.8, we have

$$|g^{(n-j)}(0)| \leqq (n-j)!(n-j), \tag{2.17.59}$$

if $f(z) \in \mathcal{S}_0$, then we calculate that

$$\left|\frac{F^{(n)}(z)}{F'(z)}\right| \leqq \frac{n!(n-1)!}{(1-|z|^2)^{n-1}} \left(\sum_{j=0}^{n-1} \frac{n-j}{(n-j-1)!j!}|z|^j\right) \tag{2.17.60}$$

$$= \frac{n!(n+|z|)}{(1-|z|)^{n-1}(1+|z|)},$$

because

$$\sum_{j=0}^{n-1} \frac{n-j}{(n-j-1)!j!}|z|^j = \frac{(n+|z|)(1+|z|)^{n-2}}{(n-1)!}. \tag{2.17.61}$$

If $f(z) \in \mathcal{S}_0^*(\alpha)$, then

$$|g^{(n-j)}(0)| \leqq (n-j)\prod_{k=2}^{n-j}(k-2\alpha) \quad (g'(0)=1) \tag{2.17.62}$$

by means of (2.17.44). This implies the inequality (2.17.54) for $f(z) \in \mathcal{S}_0^*(\alpha)$. Furthermore, if $f(z) \in \mathcal{K}_0(\alpha)$, then $g(\zeta)$ satisfies

$$|g^{(n-j)}(0)| \leqq \prod_{k=2}^{n-j}(k-2\alpha) \quad (g'(0)=1), \tag{2.17.63}$$

which implies the inequality (2.17.56). Consequently, we complete the proof of the theorem.

Since $g'(0)=1$, letting $n=2$ in Theorem 2.17.11, we have

Corollary 2.17.12. ([34]) *Let $f(z) \in \mathcal{A}$ with $-1 \leqq \lambda < 2$.*
(i) If $f(z) \in \mathcal{S}_0$, then

$$\left|\frac{\lambda(\lambda-1)D_z^\lambda f(z) + 2\lambda z D_z^{\lambda+1} f(z) + z^2 D_z^\lambda f(z)}{z(\lambda D_z^\lambda f(z) + zD_z^{\lambda+1}f(z)}\right| \leqq \frac{2(2+|z|)}{1-|z|^2} \tag{2.17.64}$$

for $z \in \mathbb{U}$.

(ii) *If* $f(z) \in \mathcal{S}_0^*(\alpha)$, *then*

$$\left|\frac{\lambda(\lambda-1)D_z^{\lambda}f(z)+2\lambda z D_z^{\lambda+1}f(z)+z^2D_z^{\lambda}f(z)}{z(\lambda D_z^{\lambda}f(z)+zD_z^{\lambda+1}f(z)}\right| \leqq \frac{2(2(1-\alpha)+|z|)}{1-|z|^2} \tag{2.17.65}$$

for $z \in \mathbb{U}$.

(iii) *If* $f(z) \in \mathcal{K}_0(\alpha)$, *then*

$$\left|\frac{\lambda(\lambda-1)D_z^{\lambda}f(z)+2\lambda z D_z^{\lambda+1}f(z)+z^2D_z^{\lambda}f(z)}{z(\lambda D_z^{\lambda}f(z)+zD_z^{\lambda+1}f(z)}\right| \leqq \frac{2(1-\alpha+|z|)}{1-|z|^2} \tag{2.17.66}$$

for $z \in \mathbb{U}$.

Taking $\lambda = 0$ in Corollary 2.17.12, we have

Corollary 2.17.13. ([34]) *If* $f(z) \in \mathcal{S}_0$, *then*

$$\left|\frac{f(z)}{f'(z)}\right| \leqq \frac{2(2+|z|)}{1-|z|^2} \qquad (z \in \mathbb{U}), \tag{2.17.67}$$

if $f(z) \in \mathcal{S}_0^*(\alpha)$, *then*

$$\left|\frac{f(z)}{f'(z)}\right| \leqq \frac{2(2(1-\alpha)+|z|)}{1-|z|^2} \qquad (z \in \mathbb{U}), \tag{2.17.68}$$

and if $f(z) \in \mathcal{K}_0(\alpha)$, *then*

$$\left|\frac{f(z)}{f'(z)}\right| \leqq \frac{2(1-\alpha+|z|)}{1-|z|^2} \qquad (z \in \mathbb{U}), \tag{2.17.69}$$

Now, having in view to discuss the univalence of fractional calculus $F(z)$ given by (2.17.16), we need the following lemma due to Miller and Mocanu [61] (or due to Jack [49]).

Lemma 2.17.14. ([34]) *Let the function* $w(z)$ *be analytic in* $\mathbb{U}$ *with* $w(0) = 0$. *If there exists a point* $z_0 \in \mathbb{U}$ *such that*

$$\mathrm{Max}_{|z| \leqq |z_0|}|w(z)| = |w(z_0)|, \tag{2.17.70}$$

then

$$\frac{z_0 w'(z_0)}{w(z_0)} = k \tag{2.17.71}$$

and

$$\mathrm{Re}\left(1+\frac{z_0 w''(z_0)}{w'(z_0)}\right) \geqq k, \tag{2.17.72}$$

where $k \geqq 1$.

Now, we derive

Theorem 2.17.15. ([34]) *If* $F(z)$ *defined by (2.17.16) for* $f(z) \in \mathcal{A}$ *satisfies*

$$\left|1+\frac{1}{2}\frac{zF''(z)}{F'(z)}-\frac{zF'(z)}{F(z)}\right|<\frac{2-\alpha}{4\alpha}\quad(z\in\mathbb{U}) \tag{2.17.73}$$

for some real α *which satisfies* $2(\sqrt{2}-1) \leqq \alpha < 1$, *then*

$$\frac{z^2F'(z)}{F(z)^2}\prec\frac{1+(1-\alpha)z}{1-z}\quad(z\in\mathbb{U}). \tag{2.17.74}$$

Proof We define the function $w(z)$ by

$$\frac{z^2F'(z)}{F(z)^2}=\frac{1+(1-\alpha)w(z)}{1-w(z)}\quad(z\in\mathbb{U}) \tag{2.17.75}$$

with $2(\sqrt{2}-1) \leqq \alpha < 1$. Then, we see that $w(z)$ is analytic in $\mathbb{U}$ with $w(0) = 0$, and that

$$1+\frac{1}{2}\frac{zF''(z)}{F'(z)}-\frac{zF'(z)}{F(z)}=\frac{zw'(z)}{2}\left(\frac{1-\alpha}{1+(1-\alpha)w(z)}+\frac{1}{1-w(z)}\right). \tag{2.17.76}$$

If there exists a point $z_0 \in \mathbb{U}$ such that

$$\mathrm{Max}_{|z|\leqq|z_0|}|w(z)|=|w(z_0)|=1,$$

then Lemma 2.17.14 gives us that

$$z_0w'(z_0)=kw(z_0)\quad(k\geqq 1) \tag{2.17.77}$$

and $w(z_0) = e^{i\theta}$. This implies that

$$\left|1+\frac{1}{2}\frac{z_0F''(z_0)}{F'(z_0)}-\frac{z_0F'(z_0)}{F(z_0)}\right|=\frac{k}{2}\left|\frac{1-\alpha}{1+(1-\alpha)e^{i\theta}}+\frac{1}{1-e^{i\theta}}\right| \tag{2.17.78}$$

$$\geqq\frac{2-\alpha}{2}\left|\frac{1}{(1-e^{i\theta})(1+(1-\alpha)e^{i\theta})}\right|$$

$$=\frac{2-\alpha}{2}\frac{1}{\sqrt{2(1-\cos\theta)(1+(1-\alpha)^2+2(1-\alpha)\cos\theta)}}.$$

Letting

$$g(t)=(1-t)(1+(1-\alpha)^2+2(1-\alpha)t)\quad(t=\cos\theta), \tag{2.17.79}$$

we have that

$$g'(t) = -(\alpha^2 + 4(1-\alpha)t) \leqq -(\alpha^2 + 4\alpha - 4) \leqq 0 \tag{2.17.80}$$

for $2(\sqrt{2}-1) \leqq \alpha < 1$. Thus

$$g(t) \leqq g(-1) = 2\alpha^2. \tag{2.17.81}$$

Therefore, $F(z)$ satisfies

$$\left|1 + \frac{1}{2}\frac{z_0 F''(z_0)}{F'(z_0)} - \frac{z_0 F'(z_0)}{F(z_0)}\right| \geqq \frac{2-\alpha}{4\alpha} \tag{2.17.82}$$

for $2(\sqrt{2}-1) \leqq \alpha < 1$. This contradicts our condition (2.17.73) for $F(z)$. Thus, $w(z)$ satisfies $|w(z)| < 1$ for all $z \in \mathbb{U}$. With this reason above, we conclude that

$$\frac{z^2 F'(z)}{F(z)^2} \prec \frac{1+(1-\alpha)z}{1-z} \quad (z \in \mathbb{U}.$$

Next, we show

Theorem 2.17.16. ([34]) *If $F(z)$ defined by (2.17.16) for $f(z) \in \mathcal{A}$ satisfies*

$$\left|1 + \frac{1}{2}\frac{zF''(z)}{F'(z)} - \frac{zF'(z)}{F(z)}\right| < \frac{\alpha}{2(1+\alpha)} \quad (z \in \mathbb{U}) \tag{2.17.83}$$

for some real $\alpha > 0$, then

$$\left|\frac{z^2 F'(z)}{F(z)^2} - 1\right| < \alpha \quad (z \in \mathbb{U}). \tag{2.17.84}$$

Proof Let us define the function $w(z)$ by

$$\frac{z^2 F'(z)}{F(z)^2} - 1 = \alpha w(z) \quad (z \in \mathbb{U}). \tag{2.17.85}$$

Then

$$\left|1 + \frac{1}{2}\frac{zF''(z)}{F'(z)} - \frac{zF'(z)}{F(z)}\right| = \frac{\alpha}{2}\left|\frac{zw'(z)}{1+\alpha w(z)}\right| < \frac{\alpha}{2(1+\alpha)} \quad (z \in \mathbb{U}). \tag{2.17.86}$$

Suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\mathrm{Max}_{|z| \leqq |z_0|}|w(z)| = |w(z_0)| = 1, \tag{2.17.87}$$

then we can write that

$$z_0 w'(z_0) = k w(z_0) \quad (k \geqq 1)$$

and $w(z_0) = e^{i\theta}$. Therefore, we have that

$$\left|1 + \frac{1}{2}\frac{z_0 F''(z_0)}{F'(z_0)} - \frac{z_0 F'(z_0)}{F(z_0)}\right| = \frac{k\alpha}{2}\left|\frac{1}{1+\alpha e^{i\theta}}\right| \geqq \frac{\alpha}{2(1+\alpha)}, \tag{2.17.88}$$

which contradicts our condition (2.17.83). Thus, we say that there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$. Consequently, letting $|w(z)| < 1$ for all $z \in \mathbb{U}$, we prove the theorem.

Taking $\alpha = 1$ in Theorem 2.17.16, we have

Corollary 2.17.17. ([34]) *If $F(z)$ defined by (2.17.16) for $f(z) \in \mathcal{A}$ satisfies*

$$\left|1 + \frac{1}{2}\frac{zF''(z)}{F'(z)} - \frac{zF'(z)}{F(z)}\right| < \frac{1}{4} \quad (z \in \mathbb{U}), \tag{2.17.89}$$

then

$$\left|\frac{z^2F'(z)}{F(z)^2} - 1\right| < 1 \quad (z \in \mathbb{U}). \tag{2.17.90}$$

Remark 2.17.18. ([34]) In view of the result for the univalence of analytic functions due to Ozaki and Nunokawa [75], we see that $F(z)$ satisfying the inequality (2.17.90) is univalent in $\mathbb{U}$.

Example 2.17.19. ([34]) Let us consider the function $f(z)$ given by

$$f(z) = D_z^{-\lambda}\left(\frac{z^{1-\lambda}}{\Gamma(2-\lambda)}e^{\frac{z}{2}}\right) \quad (-1 \leqq \lambda < 2). \tag{2.17.91}$$

Then we have that

$$F(z) = \Gamma(2-\lambda)z^{\lambda}D_z^{\lambda}f(z) = ze^{\frac{z}{2}}, \tag{2.17.92}$$

$$\frac{zF'(z)}{F(z)} = 1 + \frac{1}{2}z, \tag{2.17.93}$$

and

$$\frac{zF''(z)}{F'(z)} = \frac{1}{2}z + \frac{z}{2+z}. \tag{2.17.94}$$

Therefore, $F(z)$ satisfies

$$\left|1 + \frac{1}{2}\frac{zF''(z)}{F'(z)} - \frac{zF'(z)}{F(z)}\right| = \frac{1}{4}\left|\frac{z^2}{2+z}\right| < \frac{1}{4} \quad (z \in \mathbb{U}). \tag{2.17.95}$$

For such a function $F(z)$, we see that

$$\left|\frac{z^2F'(z)}{F(z)^2} - 1\right| = \left|e^{-\frac{z}{2}}\left(1 + \frac{1}{2}z\right) - 1\right| \leqq c \quad (z \in \mathbb{U}). \tag{2.17.96}$$

By using the computer, we know that $c < 0.18 < 1$. Indeed, the function $F(z)$ satisfying (2.17.93) implies that

$$\mathrm{Re}\left(\frac{zF'(z)}{F(z)}\right) > \frac{1}{2} \quad (z \in \mathbb{U}). \tag{2.17.97}$$

This shows us that $F(z) \in \mathcal{S}^*\left(\frac{1}{2}\right)$.

2.18 Classes of analytic functions, based on subordinations

Applying the extremal function for the subclass $\mathcal{S}^*(\alpha)$ of $\mathcal{A}$, new classes $\mathcal{P}^*(\alpha)$ and $\mathcal{Q}^*(\alpha)$ are considered using certain subordinations. The object of this section is to present some interesting properties for $f(z)$ belonging to the classes $\mathcal{P}^*(\alpha)$ and $\mathcal{Q}^*(\alpha)$. The results were obtained together with S. Owa and published in [37].

For the classes defined in the Definitions 1.1.2, 1.1.10 and 1.1.12, we know that $\mathcal{K}(\alpha) \subset \mathcal{S}^*(\alpha) \subset \mathcal{S}^* \subset \mathcal{S} \subset \mathcal{A}$ and that $f(z) \in \mathcal{S}^*(\alpha)$ if and only if $\int_0^z \frac{f(t)}{t}\, dt \in \mathcal{K}(\alpha)$. The function $f(z)$ given by

$$f(z) = \frac{z}{(1-z)^{2(1-\alpha)}} = z + \sum_{n=2}^{\infty} \frac{\prod_{j=2}^{n}(j-2\alpha)}{(n-1)!} z^n \tag{2.18.1}$$

is the extremal function for the class $\mathcal{S}^*(\alpha)$, and the function $f(z)$ given by

$$f(z) = \begin{cases} \dfrac{1-(1-z)^{2\alpha-1}}{2\alpha-1} = z + \sum_{n=2}^{\infty} \dfrac{\prod_{j=2}^{n}(j-2\alpha)}{n!} z^n & \left(\alpha \neq \dfrac{1}{2}\right) \\[2ex] -\log(1-z) = z + \sum_{n=2}^{\infty} \dfrac{1}{n} z^n & \left(\alpha = \dfrac{1}{2}\right) \end{cases} \tag{2.18.2}$$

is the extremal function for the class $\mathcal{K}(\alpha)$ (see [42] or [62]).
Considering the principal value for $\sqrt{z}$, we consider a function $f(z)$ given by

$$f(z) = \frac{z}{(1-\sqrt{z})^{2(1-\alpha)}} = z + \sum_{n=2}^{\infty} \frac{\prod_{j=2}^{n}(j-2\alpha)}{(n-1)!} z^{\frac{n+1}{2}}. \tag{2.18.3}$$

Then, $f(z)$ satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) = \operatorname{Re}\left(\alpha + \frac{1-\alpha}{1-\sqrt{z}}\right) > \frac{1+\alpha}{2} \quad (z \in \mathbb{U}). \tag{2.18.4}$$

Therefore, $f(z)$ given by (2.18.3) is starlike of order $\frac{1+\alpha}{2}$ in $\mathbb{U}$.

In order to introduce our classes we need the subordination definition. Differential subordinations were used in many papers of univalent function theory as for example, [30], [55], [90], [91] and [93].

We know that, for analytic functions, $f(z)$ is subordinated to $g(z)$ if there exists an analytic function $w(z)$ in $\mathbb{U}$ satisfying $w(0) = 0$, $|w(z)| < 1$ $(z \in \mathbb{U})$, $f(z) = g(w(z))$, writing:

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}). \tag{2.18.5}$$

Now, with the function $f(z)$ given by (2.18.3), we introduce a new class of $f(z)$ as follows. Let $\mathcal{A}^*$ be the class of functions $f(z)$ given by

$$f(z) = z + \sum_{n=2}^{\infty} a_{\frac{n+1}{2}} z^{\frac{n+1}{2}} \qquad (z \in \mathbb{U}) \tag{2.18.6}$$

which are analytic in $\mathbb{U}$, where we consider the principal value for $\sqrt{z}$. If $f(z) \in \mathcal{A}^*$ satisfies the following subordination

$$f(z) \prec g(z) = \frac{z}{(1-\sqrt{z})^{2(1-\alpha)}} \qquad (z \in \mathbb{U}) \tag{2.18.7}$$

for some real $\alpha (0 \leqq \alpha < 1)$, then we say that $f(z) \in \mathcal{P}^*(\alpha)$. Also, if $f(z) \in \mathcal{A}^*$ satisfies $zf'(z) \in \mathcal{P}^*(\alpha)$, then we say that $f(z) \in \mathcal{Q}^*(\alpha)$.

Further, we would like to study some properties of functions $f(z) \in \mathcal{A}^*$ concerned with the classes $\mathcal{P}^*(\alpha)$ and $\mathcal{Q}^*(\alpha)$.

Theorem 2.18.1. **([37])** *If $f(z) \in \mathcal{A}^*$ satisfies*

$$\sum_{n=2}^{\infty} (n-\alpha) \left| a_{\frac{n+1}{2}} \right| \leqq 1-\alpha \tag{2.18.8}$$

for some real α $(0 \leqq \alpha < 1)$, then $f(z) \in \mathcal{P}^*(\alpha)$. The result is sharp for $f(z)$ defined by

$$f(z) = z + \sum_{n=2}^{\infty} \frac{(1-\alpha)\varepsilon}{n(n-1)(n-\alpha)} z^{\frac{n+1}{2}} \tag{2.18.9}$$

with $|\varepsilon| = 1$.

Proof. It is easy to know that if $f(z) \in \mathcal{A}^*$ satisfies

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < \frac{1-\alpha}{2} \qquad (z \in \mathbb{U}) \tag{2.18.10}$$

for some real α $(0 \leqq \alpha < 1)$, then

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \frac{1+\alpha}{2} \qquad (z \in \mathbb{U}), \tag{2.18.11}$$

that is, that $f(z) \in \mathcal{P}^*(\alpha)$. In order to get (2.18.10), we notice that we have

$$\left|\frac{zf'(z)-f(z)}{f(z)}\right| = \left|\frac{\sum_{n=2}^{\infty}\frac{n-1}{2}a_{\frac{n+1}{2}}z^{\frac{n+1}{2}}}{1+\sum_{n=2}^{\infty}a_{\frac{n+1}{2}}z^{\frac{n+1}{2}}}\right| < \frac{1-\alpha}{2} \qquad (z \in \mathbb{U}) \tag{2.18.12}$$

if $f(z)$ satisfies

$$\sum_{n=2}^{\infty}\frac{n-1}{2}\left|a_{\frac{n+1}{2}}\right| \leqq \frac{1-\alpha}{2}\left(1-\sum_{n=2}^{\infty}\left|a_{\frac{n+1}{2}}\right|\right), \tag{2.18.13}$$

which is equivalent to

$$\sum_{n=2}^{\infty}(n-\alpha)\left|a_{\frac{n+1}{2}}\right| \leqq 1-\alpha \tag{2.18.14}$$

for some real α $(0 \leqq \alpha < 1)$, then $f(z) \in \mathcal{P}^*((\alpha)$.

Further, if we consider a function $f(z)$ given by (2.18.9), then

$$a_{\frac{n+1}{2}} = \frac{(1-\alpha)\varepsilon}{n(n-1)(n-\alpha)} \qquad (|\varepsilon| = 1). \tag{2.18.15}$$

This shows us that

$$\sum_{n=2}^{\infty}(n-\alpha)\left|a_{\frac{n+1}{2}}\right| = \sum_{n=2}^{\infty}\frac{1-\alpha}{n(n-1)} \tag{2.18.16}$$

$$= (1-\alpha)\sum_{n=2}^{\infty}\left(\frac{1}{n-1}-\frac{1}{n}\right) = 1-\alpha.$$

Taking $\alpha = 0$ in Theorem 2.18.1, we have

Corollary 2.18.2. **([37])***If* $f(z) \in \mathcal{A}^*$ *satisfies*

$$\sum_{n=2}^{\infty} n\left|a_{\frac{n+1}{2}}\right| \leqq 1, \tag{2.18.17}$$

then $f(z) \in \mathcal{P}^*(0)$*. The result is sharp for*

$$f(z) = z + \sum_{n=2}^{\infty}\frac{\varepsilon}{n^2(n-1)}z^{\frac{n+1}{2}} \qquad (|\varepsilon| = 1). \tag{2.18.18}$$

Noting that $f(z) \in \mathcal{Q}^*(\alpha)$ if and only if $zf'(z) \in \mathcal{P}^*(\alpha)$, we have

Theorem 2.18.3. *If $f(z) \in \mathcal{A}^*$ satisfies*

$$\sum_{n=2}^{\infty}(n+1)(n-\alpha)\left|a_{\frac{n+1}{2}}\right| \leqq 2(1-\alpha) \tag{2.18.19}$$

for some real α $(0 \leqq \alpha < 1)$, then $f(z) \in \mathcal{Q}^*(\alpha)$. The result is sharp for $f(z)$ given by

$$f(z) = z + \sum_{n=2}^{\infty} \frac{2(1-\alpha)\varepsilon}{n(n^2-1)(n-\alpha)} z^{\frac{n+1}{2}} \tag{2.18.20}$$

with $|\varepsilon| = 1$.

Letting $\alpha = 0$ in Theorem 2.18.3, we have

Corollary 2.18.4. ([37]) *If $f(z) \in \mathcal{A}^*$ satisfies*

$$\sum_{n=2}^{\infty} n(n+1)\left|a_{\frac{n+1}{2}}\right| \leqq 2, \tag{2.18.21}$$

then $f(z) \in \mathcal{Q}^*(0)$. The result is sharp for $f(z)$ given by

$$f(z) = z + \sum_{n=2}^{\infty} \frac{2\varepsilon}{n^2(n^2-1)} z^{\frac{n+1}{2}} \qquad (|\varepsilon| = 1). \tag{2.18.22}$$

To discuss next properties for $f(z) \in \mathcal{Q}^*(\alpha)$, we have to recall here the following lemma which is called as Carathéodory theorem (see [48], [60], [73]).

Lemma 2.18.5. *Let a function $p(z)$ given by*

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \tag{2.18.23}$$

be analytic in $\mathbb{U}$ and $\mathrm{Re}\,p(z) > 0$ $(z \in \mathbb{U})$. Then

$$|c_n| \leqq 2 \qquad (n = 1, 2, 3, \cdots). \tag{2.18.24}$$

The equality holds true for

$$p(z) = \frac{1+z}{1-z} = 1 + 2\sum_{n=1}^{\infty} z^n. \tag{2.18.25}$$

Applying Lemma 2.18.5, we derive

Theorem 2.18.6. **([37])***If $f(z) \in \mathcal{P}^*(\alpha)$, then*

$$\left|a_{\frac{n+1}{2}}\right| \leqq \frac{1}{(n-1)!}\prod_{j=2}^{n}(j-2\alpha) \tag{2.18.26}$$

for $n = 2, 3, 4, \cdots$. The equality holds true for

$$f(z) = \frac{z}{(1-\sqrt{z})^{2(1-\alpha)}}. \tag{2.18.27}$$

Proof. For $f(z) \in \mathcal{P}^*(\alpha)$, we define a function $p(z)$ by

$$p(z) = \frac{1}{1-\alpha}\left(2\frac{zf'(z)}{f(z)} - (1+\alpha)\right) \qquad (z \in \mathbb{U}) \tag{2.18.28}$$

with

$$p(z) = 1 + \sum_{n=1}^{\infty} p_{\frac{n}{2}} z^{\frac{n}{2}} \qquad (z \in \mathbb{U}), \tag{2.18.29}$$

where we consider the principal value for $\sqrt{z}$.

It follows that

$$2zf'(z) = \{(1-\alpha)p(z) + (1+\alpha)\} f(z). \tag{2.18.30}$$

This gives us that

$$\sum_{n=2}^{\infty}(n+1)a_{\frac{n+1}{2}} z^{\frac{n+1}{2}} = \left(2a_{\frac{3}{2}} + (1-\alpha)p_{\frac{1}{2}}\right) z^{\frac{3}{2}} + \left(2a_2 + (1-\alpha)p_{\frac{1}{2}}a_{\frac{3}{2}} + (1-\alpha)p_1\right) z^2 \tag{2.18.31}$$

$$+ \left(2a_{\frac{5}{2}} + (1-\alpha)p_{\frac{1}{2}}a_2 + (1-\alpha)p_1 a_{\frac{3}{2}} + (1-\alpha)p_{\frac{3}{2}}\right) z^{\frac{5}{2}} + \cdots$$

$$+ \left(2a_{\frac{n+1}{2}} + (1-\alpha)p_{\frac{1}{2}}a_{\frac{n}{2}} + (1-\alpha)p_1 a_{\frac{n-1}{2}} + \cdots + (1-\alpha)p_{\frac{n-2}{2}}a_{\frac{3}{2}} + (1-\alpha)p_{\frac{n-1}{2}}\right) z^{\frac{n+1}{2}} + \cdots .$$

Therefore, we obtain that

$$(n-1)a_{\frac{n+1}{2}} = (1-\alpha)\left(p_{\frac{n-1}{2}} + p_{\frac{n-2}{2}}a_{\frac{3}{2}} + p_{\frac{n-3}{2}}a_2 + \cdots + p_1 a_{\frac{n-1}{2}} + p_{\frac{1}{2}}a_{\frac{n}{2}}\right), n \geq 2, \tag{2.18.32}$$

where $a_1 = 1$. From the definition for $p(z)$, we see that $p(z)$ is analytic in $\mathbb{U}$ and $p(0) = 1$. Furthermore, noting that $\mathrm{Re}p(z) > 0$ $(z \in \mathbb{U})$, Lemma 2.18.5 gives us that

$$\left|p_{\frac{n}{2}}\right| \leqq 2 \qquad (n = 1, 2, 3, \cdots). \tag{2.18.33}$$

Taking $n = 2$ in (2.18.32), we see that

$$\left|a_{\frac{3}{2}}\right| \leqq (1-\alpha)\left|p_{\frac{1}{2}}\right| \leqq 2 - 2\alpha. \tag{2.18.34}$$

If we take $n = 3$ in (2.18.32), then we have that

$$|a_2| \leqq \frac{1-\alpha}{2}\left(\left|p_{\frac{1}{2}}\right|\left|a_{\frac{3}{2}}\right| + |p_1|\right) \leqq (1-\alpha)(3-2\alpha) = \frac{1}{2}(2-2\alpha)(3-2\alpha). \tag{2.18.35}$$

Further, letting $n = 4$ in (2.18.33), we obtain that

$$\left|a_{\frac{5}{2}}\right| \leqq \frac{1-\alpha}{3}\left(\left|p_{\frac{1}{2}}\right||a_2| + |p_1|\left|a_{\frac{3}{2}}\right| + \left|p_{\frac{3}{2}}\right|\right) \tag{2.18.36}$$

$$\leqq \frac{2}{3}(1-\alpha)(2-\alpha)(3-2\alpha) = \frac{1}{6}(2-2\alpha)(3-2\alpha)(4-2\alpha).$$

In view of the above, we assume that

$$\left|a_{\frac{n+1}{2}}\right| \leqq \frac{1}{(n-1)!}\prod_{j=2}^{n}(j-2\alpha) \tag{2.18.37}$$

for $j = 2, 3, 4, \cdots, n$. Then, we see that

$$\left|a_{\frac{n+2}{2}}\right| \leq \frac{2(1-\alpha)}{n}\left(1 + \left|a_{\frac{3}{2}}\right| + |a_2| + \left|a_{\frac{5}{2}}\right| + \cdots + \left|a_{\frac{n}{2}}\right| + \left|a_{\frac{n+1}{2}}\right|\right) \tag{2.18.38}$$

$$\leqq \frac{1}{n!}\prod_{j=2}^{n+1}(j-2\alpha).$$

Thus, applying the mathematical induction, we complete the proof of the theorem.

Theorem 2.18.7. **([37])** *If $f(z) \in \mathcal{Q}^*(\alpha)$, then*

$$\left|a_{\frac{n+1}{2}}\right| \leqq \frac{1}{n!}\prod_{j=2}^{n}(j-2\alpha) \tag{2.18.39}$$

for $n = 2, 3, 4, \cdots$. The equality holds true for $f(z)$ satisfying

$$f'(z) = \frac{1}{(1-\sqrt{z})^{2(1-\alpha)}}. \tag{2.18.40}$$

Further, we consider some distortion inequalities for $f(z)$ in $\mathcal{P}^*(\alpha)$ and $\mathcal{Q}^*(\alpha)$.

Theorem 2.18.8. **([37])** *If $f(z) \in \mathcal{P}^*(\alpha)$, then*

$$\frac{|z|}{(1+\sqrt{|z|})^{2(1-\alpha)}} \leqq |f(z)| \leqq \frac{|z|}{(1-\sqrt{|z|})^{2(1-\alpha)}} \qquad (z \in \mathbb{U}). \tag{2.18.41}$$

The equalities in (2.18.41) are attended for

$$f(z) = \frac{z}{(1-\sqrt{z})^{2(1-\alpha)}}. \tag{2.18.42}$$

Proof. We note that there exists a function $w(z)$ which is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$. This function $w(z)$ also satisfies

$$f(z) = \frac{w(z)}{(1 - \sqrt{w(z)})^{2(1-\alpha)}} \qquad (z \in \mathbb{U}). \tag{2.18.43}$$

If we write that $w(z) = |w(z)|e^{i\theta}$, then $f(z)$ gives us that

$$|f(z)| = \frac{|w(z)|}{\left(1 - \sqrt{|w(z)|}e^{i\frac{\theta}{2}}\right)^{2(1-\alpha)}} \tag{2.18.44}$$

$$= \frac{|w(z)|}{\left\{\left(1 - \sqrt{|w(z)|}cos\frac{\theta}{2}\right)^2 + |w(z)|sin^2\frac{\theta}{2}\right\}^{1-\alpha}}$$

$$= \frac{|w(z)|}{\left(1 + |w(z)| - 2\sqrt{|w(z)|}cos\frac{\theta}{2}\right)^{1-\alpha}}.$$

Applying the Schwarz lemma for $w(z)$, we say that $|w(z)| \leqq |z|$ $(z \in \mathbb{U})$. Therefore, we obtain that

$$\frac{|z|}{(1 + \sqrt{|z|})^{2(1-\alpha)}} \leqq |f(z)| \leqq \frac{|z|}{(1 - \sqrt{|z|})^{2(1-\alpha)}} \tag{2.18.45}$$

for $z \in \mathbb{U}$. This completes the proof of the theorem.

Letting $\alpha = 0$ in Theorem 2.18.8, we have

Corollary 2.18.9. **([37])** *If $f(z) \in \mathcal{P}^*(0)$, then*

$$\frac{|z|}{(1 + \sqrt{|z|})^2} \leqq |f(z)| \leqq \frac{|z|}{(1 - \sqrt{|z|})^2} \qquad (z \in \mathbb{U}). \tag{2.18.46}$$

The equalities in (2.18.46) are attended for

$$f(z) = \frac{z}{(1 - \sqrt{z})^2}. \tag{2.18.47}$$

Further, letting $|z| \to 1$ in Theorem 2.18.8, we see

Corollary 2.18.10. **([37])** *If $f(z) \in \mathcal{P}^*(\alpha)$, then*

$$|f(z)| \geqq \left(\frac{1}{4}\right)^{1-\alpha}. \tag{2.18.48}$$

The equality in (2.18.48) is attended for $f(z)$ given by (2.18.42) with $z = e^{i2\pi}$.

Noting that $f(z) \in \mathcal{Q}^*(\alpha)$ if and only if $zf'(z) \in \mathcal{P}^*(\alpha)$, we also have

Theorem 2.18.11. **([37])** *If $f(z) \in \mathcal{Q}^*(\alpha)$, then*

$$\frac{1}{(1+\sqrt{|z|})^{2(1-\alpha)}} \leqq |f'(z)| \leqq \frac{1}{(1-\sqrt{|z|})^{2(1-\alpha)}} \qquad (z \in \mathbb{U}). \tag{2.18.49}$$

The equalities in (2.18.49) are attended for

$$f(z) = \int_0^z \frac{1}{(1-\sqrt{t})^{2(1-\alpha)}} dt. \tag{2.18.50}$$

Corollary 2.18.12. **([37])** *If $f(z) \in \mathcal{Q}^*(0)$, then*

$$\frac{1}{(1+\sqrt{|z|})^{2}} \leqq |f'(z)| \leqq \frac{1}{(1-\sqrt{|z|})^{2}} \qquad (z \in \mathbb{U}). \tag{2.18.51}$$

The equalities in (2.18.51) are attended for

$$f(z) = \int_0^z \frac{1}{(1-\sqrt{t})^{2}} dt. \tag{2.18.52}$$

Corollary 2.18.13. **([37])** *If $f(z) \in \mathcal{Q}^*(\alpha)$, then*

$$|f'(z)| \geqq \left(\frac{1}{4}\right)^{1-\alpha}. \tag{2.18.53}$$

The equality in (2.18.53) is attended for $f(z)$ given by (2.18.50) with $z = e^{i2\pi}$.

Chapter 3

Further research

3.1 Research directions

In this section, we briefly outline some of the research directions that may characterize our future work, more details being given in the next two sections where we present two of our current research projects.

Motivated by the recent results in the field of geometric function theory and willing to extend our previous work, we have in view three general research directions, namely:

- study of new geometric properties for the operators considered in this thesis with respect to their univalence (research direction A),

- construction of new integral operators that cover the already known operators as particular cases (research direction B),

- construction of the classes of analytic functions having interesting geometric properties (research directions C).

In what follows, all three of them are detailed by considering possible problems to focuse on and by giving some examples of particular works together with some hints on the approaching methods.

• **Research direction A**. We aim to extend the results that we have already obtained for the integral operators J_1 - J_8, most of them on univalence (see Chapter 2, where various univalence conditions were obtained), by investigating other properties of the operators, as convexity and starlikeness for example. In order to approach the study of these operators with respect to other properties, we will consider some particular classes of analytic functions. Within this research direction we have in mind to investigate the following:

Problem 1. One of the issue will be to find what conditions are neccessary to preserve the starlikeness. We will have two possible methods to approach this problem, one based on the analytic characterization of the starlikeness and the other, on differential subordinations.

Example 3.1.1. Let's consider the operators:

$$J_2(z) = \left[\beta \int_0^z u^{\beta-1} \left(\frac{f_1(u)}{u}\right)^{\gamma_1} \cdot \ldots \cdot \left(\frac{f_n(u)}{u}\right)^{\gamma_n} du\right]^{\frac{1}{\beta}} \tag{3.1.1}$$

$$J_4(z) = \left\{\beta \int_0^z u^{\beta-1} \cdot \left[f_1'(u)\right]^{\gamma_1} \cdot \ldots \cdot \left[f_n'(u)\right]^{\gamma_n} du\right\}^{1/\beta}. \tag{3.1.2}$$

We can investigate some conditions for starlikeness, besides the univalence conditions wich we have already obtained for these operators (see for example, Section 2.15). More precisely, we can use the following Mocanu starlikeness condition, in the same manner as we used the Kudriasov univalence condition (see Section 2.13), and see what other suplimentary conditions are necessary for starlikeness:

$$\left|\frac{f''(z)}{f'(z)}\right| \leq M, \quad z \in U, M \cong 2,83. \tag{3.1.3}$$

Problem 2. Another problem that we are interested to study is to find the convexity order for integral operators.

Example 3.1.2. Let's consider the operator:

$$J_1(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt \tag{3.1.4}$$

on the classes of uniformly analytic functions,

$$\beta - UST_0(p,q,\alpha) =$$
$$= \left\{f \in T_0(p) : \mathrm{Re}\left\{\frac{zf^{(1+q)}(z)}{f^{(q)}(z)}\right\} \geq \beta \left|\left\{\frac{zf^{(1+q)}(z)}{f^{(q)}(z)}\right\} - 1\right| + \alpha\right\}, \tag{3.1.5}$$
$$\left(z \in U; 0 \leq \alpha < p-q; p \in N; p > q; q \in N_0 = N \cup \{0\},\ \beta \geq 0\right),$$

$$\beta - UCV_0(p,q,\alpha) =$$

$$= \left\{ f \in T_0(p) : \mathrm{Re}\left\{ 1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} \right\} \geq \beta \left| \left\{ \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} \right\} \right| + \alpha \right\}, \tag{3.1.6}$$

$$\left(z \in U; 0 \leq \alpha < p - q; p \in N; p > q; q \in N_0 = N \cup \{0\},\ \beta \geq 0\right).$$

The approach will be based on a differential operator which will be applied on the functions that compose the operator, such that the new operator will be well defined on the class of functions of the form:

$$f(z) = a_p \cdot z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, (a_{p+n} \geq 0; p \in N = \{1, 2, ...\}, a_p > 0). \tag{3.1.7}$$

Further, we will use the analytic characterization of the classes and of the convexity of a given order.

Problem 3. Preserving of other geometric properties by our integral operators will be another research goal.

Example 3.1.3. We consider the operator (see Section 2.12, where univalence was already studied),

$$J_8(z) = \left(\delta \int_0^z u^{\delta-1} \prod_{j=1}^{n} \left(\frac{f_j(u)}{u} \right)^{\alpha_j} (g_j(u))^{\beta_j} \, du \right)^{\frac{1}{\delta}}, \tag{3.1.8}$$

where δ, α_j, β_j are complex numbers, $\delta \neq 0$, $f_j \in \mathcal{A}$, $g_j \in \mathcal{P}$, $j = \overline{1, n}$.

We will study the operator on various classes of analytic functions as for example, the class $SH(\beta)$, $\beta > 0$, introduced by Stankiewicz-Wisniowska, [98] (see Section 2.4, where other operator was studied on this class), the working tool being the analytic characterization for the related class:

$$\mathcal{SH}(\beta) = \left\{ f \in \mathcal{S} : \left| \frac{zf'(z)}{f(z)} - 2\beta\left(\sqrt{2} - 1\right) \right| < \mathrm{Re}\left\{ \sqrt{2}\frac{zf'(z)}{f(z)} \right\} + 2\beta\left(\sqrt{2} - 1\right), z \in \mathcal{U} \right\}. \tag{3.1.9}$$

• **Research direction B**. Construction of new operators that can cover the already known operators as particular cases:

Regarding this research direction, in order to give consistency to our results, we have to consider the next three aspects for each new introduced integral operator:

- checking the existence of the operator (if it is well defined),

- finding other motivation of the operators, besides their generality, taking into account possible geometric properties and some particular interesting examples,

- investigating geometric properties of the operators.

Example 3.1.4. We currently work on the new integral operator $N_{\alpha,\beta}(f,g)$, with $\alpha = (\alpha_1, \alpha_2, ...\alpha_n)$ si $(f,g) = (f_1, f_2, ...f_n, g_1, g_2, ...g_n)$, α_i, real pozitive numbers, β complex numbers, $Re\beta > 0$, $f_i, g_i \in \mathcal{A}$, $i = \overline{1,n}$, $(f_i * g_i)(z) = z + \sum_{k=2}^{\infty} a_{k,i} b_{k,i} z^k$, Hadamard product ($a_{k,i}, b_{k,i}$, coefficients of the functions f_i, g_i) :

$$N_{\alpha,\beta}(f,g)(z) =$$

$$\left[\int_0^z \beta t^{\beta-1} exp\left(\int_0^t \prod_{i=1}^n \left(\frac{(f_i * g_i)(u)}{u}\right)^{\alpha_i} du\right) dt\right]^{\frac{1}{\beta}} \tag{3.1.10}$$

For $n = 1, \alpha_1 = 1, g_1 = \frac{z}{1-z}$, we find the operator introduced by Attiya ([6]):

$$F_\beta(f)(z) = \left[\int_0^z \beta t^{\beta-1} exp\left(\int_0^t \left(\frac{f(u)}{u}\right) du\right) dt\right]^{\frac{1}{\beta}} \tag{3.1.11}$$

For

$$\prod_{i=1}^n \left(\frac{(f_i * g_i)(z)}{z}\right)^{\alpha_i} = \sum_{i=1}^n \alpha_i \left(\frac{(f_i * g_i)'(z)}{(f_i * g_i)(z)} - \frac{1}{z}\right), \tag{3.1.12}$$

we get the operator given by Frasin ([46]),

$$I_{\alpha,\beta}(f,g)(z) = \left[\int_0^z \beta t^{\beta-1} \prod_{i=1}^n \left(\frac{(f_i * g_i)(t)}{t}\right)^{\alpha_i} dt\right]^{\frac{1}{\beta}}. \tag{3.1.13}$$

• **Research problem C**. Construction of new classes of analytic functions:

For each of the new introduced classes, we have in view to study at least the following lines:

- finding examples of functions that proves the nontriviality,

- study of Hadamard product on those classes (or some modified version of Hadamard product),

- characterization of the classes by finding coefficients estimates,

- finding class preserving properties for some integral operators.

Example 3.1.5. (see Section 3.3) We currently work on some new classes defined by the conditions ($\alpha > 1$):

$$0 < \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < \alpha \qquad (z \in \mathbb{U}) \tag{3.1.14}$$

respectively,

$$0 < \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \alpha \qquad (z \in \mathbb{U}). \tag{3.1.15}$$

Other lines of work that we aim to follow are related to:

- extending of other type of univalence criteria of a function to integral operator, in the same manner as we worked with Pascu criterion (see Chapter 2),

- study of some integro-differential operators,

- the analysis of already obtained results through the extremal function issue,

- finding some applications for the theoretical results (see the detailed result from the Section 3.2)

- using of specialized software to outline the geometric properties of some integral operators mapping.

3.2 Univalence of the solution of the inverse boundary problem

In this project, together with V. Pescar, we aim to obtain some application of integral operators. First, we study the univalence of a particular integral operator. Then the univalence criterion is extended for a more general integral operator and also finally, derived for a particular integral operator which can be viewed as a solution of the inverse boundary problem. This project is part of a more general one, started together with D. Breaz and V. Pescar, based on the goal of finding of some applications for the univalence of the integral operators in the field of fluid mechanics.

The following integral operators are studied:

$$F_\beta(z) = \left\{ \beta \int_0^z u^{\beta-1} f'(u) du \right\}^{\frac{1}{\beta}}, \tag{3.2.1}$$

$$T_n(z) = \left\{ \beta \int_0^z u^{\beta-1} \left(f_1'(u)\right)^{\gamma_1} \ldots \left(f_n'(u)\right)^{\gamma_n} du \right\}^{\frac{1}{\beta}}, \tag{3.2.2}$$

β, γ_j complex numbers, $\beta \neq 0$, $j = \overline{1,n}$, $f_j \in \mathcal{A}$, $j = \overline{1,n}$, n positive integer number.

Among the applications of the geometric function theory there is one related to fluid mechanics where we deal with the inverse boundary problem (see [47]). The solution of such problem can be defined as an integral operator having the form,

$$p(z) = \int_0^z e^{h(u)} du, \tag{3.2.3}$$

where h is a regular known function in $\mathcal{U}$. In fluid mechanics it is known that the solution of the inverse boundary problem has to be univalent.

Remark 3.2.1. i) The integral operator F_β was introduced by N.N.Pascu in the paper [77].
ii) Besides the classical integral operators, in the last decade, some general integral operators, defined as a family of integral operators, using more than one analytic function in their definition, have been studied (see for example, the works [38], [57], [85], [81], [97]). The general integral operator T_n is an example of such operator, considered as an operator of Pfaltzgraff type (see [88]). This operator was introduced by D.Breaz and N.Breaz in the paper [10] and has been studied with respect to its univalence, in many other papers (see for example [80]-[81]).
iii) If we consider $n = 1$ and $\gamma_1 = 1$ in (3.2.2), we can see that T_n is viewed as an extension of the operator F_β.
iv) The solution of the inverse boundary problem can be viewed as particular case of the operator F_β. For example, we can chose $\beta = 1$ and set the function f such that $f'(u) = e^{h(u)}$. This connection between the solution of the inverse boundary problem and the integral operator F_β will be exploited in this paper.

The aim here is to obtain new sufficient conditions for the univalence of the integral operator F_β, based on which we can obtain further, univalence of the more general integral operator T_n, but also the univalence of the solution of the inverse boundary problem, the integral operator p.

With the following theorem we achieve univalence criterion for the integral operators F_β and T_n :

Theorem 3.2.2. *Let α be a complex number,* $\mathrm{Re}\alpha > 0$ *and the function f, $f \in \mathcal{A}$, $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. If*

$$\left|\frac{f''(z)}{f'(z)}\right| < M, \tag{3.2.4}$$

for all $z \in U$, where the positive constant M satisfies the inequality,

$$M \leq \frac{1}{max_{|z|<1}\left[\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}|z|\frac{|z|+\frac{2|a_2|}{M}}{1+\frac{2|a_2||z|}{M}}\right]}, \tag{3.2.5}$$

then for all β complex number, with $\mathrm{Re}\beta \geq \mathrm{Re}\alpha$*, the integral operator F_β defined by (3.2.1) is in the class $\mathcal{S}$.*

Proof. Let's consider the function

$$g(z) = \frac{1}{M}\frac{f''(z)}{f'(z)}. \tag{3.2.6}$$

From (3.2.4) and (3.2.6), we have $|g(z)| < 1$, for all $z \in U$. The function g is regular in U and $g(0) = \frac{2a_2}{M}$. From Nehari Remark, we obtain

$$\left|\frac{1}{M}\frac{f''(z)}{f'(z)}\right| \leq \frac{|z|+\frac{2|a_2|}{M}}{1+\frac{2|a_2||z|}{M}}, \tag{3.2.7}$$

for all $z \in U$ and hence, we have

$$\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|z\frac{f''(z)}{f'(z)}\right| \leq M\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}|z|\frac{|z|+\frac{2|a_2|}{M}}{1+\frac{2|a_2||z|}{M}}, \tag{3.2.8}$$

for all $z \in U$.

Let be the function $H : [0,1) \to \mathbb{R}$, $H(x) = \frac{1-x^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}x\frac{x+\frac{2|a_2|}{M}}{1+\frac{2|a_2|x}{M}}$. Since $H(\frac{1}{2}) > 0$, it comes that $max_{x\in[0,1)}H(x) > 0$.

Hence, further, from (3.2.8), we get

$$\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|z\frac{f''(z)}{f'(z)}\right| \leq M \cdot max_{|z|<1}\left[\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}|z|\frac{|z|+\frac{2|a_2|}{M}}{1+\frac{2|a_2||z|}{M}}\right]. \tag{3.2.9}$$

Now, using (3.2.5) and (3.2.9), we have

$$\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \tag{3.2.10}$$

for all $z \in U$.

Finally, from (3.2.10) and N.N. Pascu univalence criterion, it results that $F_\beta \in \mathcal{S}$.

Remark 3.2.3. There is a connection between the coefficient a_2 from the analytic form of the function f and the coefficient A_2 from the analytic form of the operator F_β. The function $F_\beta(z)$ is regular in U and we have

$$F_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u) du\right]^{\frac{1}{\beta}} = z + A_2 z^2 + ... \tag{3.2.11}$$

From (3.2.11) we get $F_\beta^{\beta-1}(z) F'_\beta(z) = z^{\beta-1} f'(z)$ and hence

$$f'(z) = \left(\frac{F_\beta(z)}{z}\right)^{\beta-1} F'_\beta(z), \tag{3.2.12}$$

for all $z \in U$.

From (3.2.12), for β fixed, Re$\alpha > 0$, Re$\beta \geq$ Reα, we obtain the relation,

$$a_2 = \frac{\beta+1}{2} A_2. \tag{3.2.13}$$

From the Theorem 3.2.2, we can derive different univalence conditions, taking some particular cases of the parameters involved.

Corollary 3.2.4. *Let be the function f, $f \in \mathcal{A}$, $f(z) = z + \sum_{k=3}^{\infty} a_k z^k$ and α a complex number,* Re$\alpha > 0$. *If*

$$\left|\frac{f''(z)}{f'(z)}\right| < (\text{Re}\alpha + 1)^{\frac{\text{Re}\alpha+1}{\text{Re}\alpha}}, \tag{3.2.14}$$

for all $z \in U$, then for all β complex number, with Re$\beta \geq$ Reα, *the integral operator F_β defined by (3.2.1) is in the class $\mathcal{S}$.*

Proof. We apply Theorem 3.2.2, taking a positive constant M, having the form,

$$M = (\text{Re}\alpha + 1)^{\frac{\text{Re}\alpha+1}{\text{Re}\alpha}}. \tag{3.2.15}$$

Indeed, since

$$max_{|z|<1}\left[\frac{1-|z|^{2\text{Re}\alpha}}{\text{Re}\alpha}|z|^2\right] = \frac{1}{(\text{Re}\alpha+1)^{\frac{\text{Re}\alpha+1}{\text{Re}\alpha}}}, \tag{3.2.16}$$

the conditions (3.2.4) and (3.2.5) from the Theorem 3.2.2 are satisfied (we are in the case $a_2 = 0$).

Corollary 3.2.5. *Let be the function f, $f \in \mathcal{A}$, $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and α a complex number,* $0 <$ Re$\alpha \leq 1$. *If*

$$\left|\frac{f''(z)}{f'(z)}\right| < M, \tag{3.2.17}$$

for all $z \in U$, where the positive constant M satisfies the inequality

$$M \leq \frac{1}{max_{|z|<1}\left[\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}|z|\frac{|z|+\frac{2|a_2|}{M}}{1+\frac{2|a_2||z|}{M}}\right]}, \tag{3.2.18}$$

then the function f is univalent.

Proof. We apply Theorem 3.2.2 for $\beta = 1$.

Corollary 3.2.6. *Let α be a complex number, $0 < \mathrm{Re}\alpha \leq 1$ and the function f, $f \in \mathcal{A}$, having the form, $f(z) = z + \sum\limits_{k=3}^{\infty} a_k z^k$. If*

$$\left|\frac{f''(z)}{f'(z)}\right| < (\mathrm{Re}\alpha + 1)^{\frac{\mathrm{Re}\alpha+1}{\mathrm{Re}\alpha}}, \tag{3.2.19}$$

for all $z \in U$, then the function f is univalent.

Proof. This result is a combination of the last two corollaries and derives from the Theorem 3.2.2, for the particular case when $a_2 = 0$ (M taking the form (3.2.15)) and $\beta = 1$.

Theorem 3.2.7. *Let α, γ_j $(j = \overline{1,n})$ be complex numbers, $\mathrm{Re}\alpha > 0$ and the functions f_j, $f_j \in \mathcal{A}$, $f_j(z) = z + \sum\limits_{k=2}^{\infty} a_k^j z^k$, $j = \overline{1,n}$, n positive integer number. If*

$$|\gamma_1| + |\gamma_2| + ... + |\gamma_n| \leq 1, \tag{3.2.20}$$

$$\left|\frac{f_j''(z)}{f_j'(z)}\right| < M_j, \tag{3.2.21}$$

for all $z \in U$, $j = \overline{1,n}$, where the positive constants M_j satisfy the inequality,

$$M_j \leq \frac{1}{max_{|z|<1}\left[\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}|z|\frac{|z|+\frac{2|\phi|}{max_{j=\overline{1,n}}M_j}}{1+\frac{2|\phi||z|}{max_{j=\overline{1,n}}M_j}}\right]}, j = \overline{1,n}, \tag{3.2.22}$$

with $\phi = \gamma_1 a_2^1 + \gamma_2 a_2^2 + ... + \gamma_n a_2^n$, then for all β complex number, with $\mathrm{Re}\beta \geq \mathrm{Re}\alpha$, the integral operator T_n defined by (3.2.1) is in the class $\mathcal{S}$.

Proof. We consider the regular function

$$f(z) = \int_0^z (f_1'(u))^{\gamma_1} ... (f_n'(u))^{\gamma_n} du. \tag{3.2.23}$$

It can be easily checked that $f(z) = z + a_2 z^2 + ...$ with

$$a_2 = \phi = \gamma_1 a_2^1 + \gamma_2 a_2^2 + ... + \gamma_n a_2^n \tag{3.2.24}$$

and

$$\frac{f''(z)}{f'(z)} = \gamma_1 \frac{f_1''(z)}{f_1'(z)} + \gamma_2 \frac{f_2''(z)}{f_2'(z)} + ...\gamma_n \frac{f_n''(z)}{f_n'(z)}. \tag{3.2.25}$$

Now we apply the Theorem 3.2.2 for the function f from (3.2.23), taking $M = max_{j=\overline{1,n}} M_j$ which is a positive constant. From (3.2.20), (3.2.21) and (3.2.25) we get that f satisfies the condition (3.2.4). Moreover, M satisfies the condition (3.2.5) if we take into account the formula (3.2.24). Hence, applying the Theorem 3.2.2 and using (3.2.23), the univalence result for the integral operator T_n holds.

Corollary 3.2.8. *Let be the function f_j,$f_j \in \mathcal{A}$, $f_j(z) = z + \sum\limits_{k=3}^{\infty} a_k^j z^k$, $j = \overline{1,n}$ and α, γ_j ($j = \overline{1,n}$) complex numbers,* $\mathrm{Re}\alpha > 0$. *If*

$$|\gamma_1| + |\gamma_2| + ... + |\gamma_n| \leq 1, \tag{3.2.26}$$

$$\left| \frac{f_j''(z)}{f_j'(z)} \right| < (\mathrm{Re}\alpha + 1)^{\frac{\mathrm{Re}\alpha+1}{\mathrm{Re}\alpha}}, \tag{3.2.27}$$

for all $z \in U$, $j = \overline{1,n}$, then for all β complex number, with $\mathrm{Re}\beta \geq \mathrm{Re}\alpha$, *the integral operator T_n defined by (3.2.1) is in the class $\mathcal{S}$.*

Proof. In the Theorem 3.2.7, we take $a_2^j = 0$, $j = \overline{1,n}$ and follow the steps from the proof of the Corollary 3.2.4.

In what follows we work to obtain the univalence of the solution of the inverse boundary problem.

Corollary 3.2.9. *Let α be a complex number,* $0 < \mathrm{Re}\alpha \leq 1$, *the function $h \in \mathcal{A}$ and the function*

$$p(z) = \int_0^z e^{h(u)} du = z + A_2 z^2 + ...$$

If

$$\left|h'(z)\right| < M \tag{3.2.28}$$

for all $z \in U$, where the positive constant M satisfies the inequality

$$M \leq \frac{1}{max_{|z|<1}\left[\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha} |z| \frac{|z| + \frac{2|A_2|}{M}}{1 + \frac{2|A_2||z|}{M}}\right]}, \tag{3.2.29}$$

then $p \in \mathcal{S}$.

Proof. We apply Theorem 3.2.2, for $\beta = 1$. Moreover, in Theorem 3.2.2, we chose the function f such that $f'(z) = e^{h(z)}$ which is a regular function. For the analytic form of this function we use the notation, $f(z) = z + a_2 z^2 + \dots$. Hence, we obtain that $F_1(z) = p(z) = f(z)$ and further $a_2 = A_2$. Moreover, it comes that

$$\frac{f''(z)}{f'(z)} = h'(z)\,. \tag{3.2.30}$$

Taking into account (3.2.30) and (3.2.28) we obtain that function f satisfies the condition (3.2.4) from the Theorem 3.2.2. On the other hand, based on (3.2.29) and on the equality $a_2 = A_2$, the condition (3.2.5) from the Theorem 3.2.2 is satisfied too, hence $p(z) = F_1(z)$ is univalent and the result is proved.

Corollary 3.2.10. *Let* α *be a complex number,* $0 < \mathrm{Re}\alpha \leq 1$*, the function* $h \in \mathcal{A}$ *and the function*

$$p(z) = \int_0^z e^{h(u)} du = z + A_3 z^3 + \dots .$$

If

$$\left|h'(z)\right| < (\mathrm{Re}\alpha + 1)^{\frac{\mathrm{Re}\alpha+1}{\mathrm{Re}\alpha}} \tag{3.2.31}$$

for all $z \in U$*, then* $p \in \mathcal{S}$.

Proof. Taking in Theorem 3.2.2, the function f as in the previous corollary and knowing that $a_2 = A_2 = 0$ the result holds.

The last two corollaries give univalence conditions for the solution of the inverse boundary problem.

3.3 The study of new classes of analytic functions, using differential subordinations

Together with S. Owa, J. Nishiwaki and D. Breaz we introduced two subclasses of analytic functions, respectively $\mathcal{S}_n^*(\alpha)$ and $\mathcal{K}_n(\alpha)$. In this research project we aim to study some interesting properties of the functions from these classes, using differential subordinations.

Here we will work in the framework of the class $\mathcal{A}_n$, defined by the functions

$$f(z) = z + \sum_{k=n}^{\infty} a_k z^k \qquad (n \ = 2,3,4,\cdots), \tag{3.3.1}$$

analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. As we know, if we consider a function $f(z) \in \mathcal{A}_n$ which satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (z \in \mathbb{U}), \tag{3.3.2}$$

then $f(z)$ is starlike with respect to the origin in $\mathbb{U}$. We denote by $\mathcal{S}_n^*$, the subclass of $\mathcal{A}_n$ consisting of starlike functions in $\mathbb{U}$. Also, we know that $f(z)$ is convex in $\mathbb{U}$, $f(z) \in \mathcal{A}_n$, if satisfies

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \qquad (z \in \mathbb{U}) \tag{3.3.3}$$

and this is equivalent to $zf'(z) \in \mathcal{S}_n^*$. We denote by $\mathcal{K}_n$ the subclass of $\mathcal{A}_n$ consisting of all convex functions in $\mathbb{U}$.
Let us consider a function $f(z)$ given by

$$f(z) = z + \frac{1}{n} z^n \qquad (n = 2,3,4,\cdots). \tag{3.3.4}$$

This function $f(z)$ satisfies that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) = n - \operatorname{Re}\left(\frac{n(n-1)}{z^{n-1}+n}\right) = n - \frac{n(n-1)(n + r^{n-1}\cos(n-1)\theta)}{n^2 + r^{2(n-1)} + 2nr^{n-1}\cos(n-1)\theta} \tag{3.3.5}$$

for $z = re^{i\theta} \in \mathbb{U}$. Therefore, we see that

$$0 < \frac{n(1-r^{n-1})}{n-r^{n-1}} < \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < \frac{n(1+r^{n-1})}{n+r^{n-1}} < \frac{2n}{n+1} \qquad (z \in \mathbb{U}). \tag{3.3.6}$$

Further, if we consider a function $f(z)$ given by

$$f(z) = z + \frac{1}{n^2} z^n \qquad (n = 2,3,4,\cdots), \tag{3.3.7}$$

then we have

$$0 < \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{2n}{n+1} \qquad (z \in \mathbb{U}). \tag{3.3.8}$$

Therefore, $f(z)$ given by (3.3.4) is in the class $\mathcal{S}_n^*$ and $f(z)$ given by (3.3.7) is in the class $\mathcal{K}_n$.
In view of the above, we say that $f(z) \in \mathcal{S}_n^*(\alpha)$ if $f(z) \in \mathcal{A}_n$ satisfies

$$0 < \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < \alpha \qquad (z \in \mathbb{U}) \tag{3.3.9}$$

for some real $\alpha > 1$, and that $f(z) \in \mathcal{K}_n(\alpha)$ if $f(z) \in \mathcal{A}_n$ satisfies

$$0 < \text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \alpha \qquad (z \in \mathbb{U}) \tag{3.3.10}$$

for some real $\alpha > 1$. From the above definitions we say that $f(z) \in \mathcal{S}_n^*\left(\frac{2n}{n+1}\right)$ for $f(z)$ given by (3.3.4) and that $f(z) \in \mathcal{K}_n\left(\frac{2n}{n+1}\right)$ for $f(z)$ given by (3.3.7). Also, we know that $\mathcal{S}_n^*(\alpha) \subset \mathcal{S}_n^*(\beta)$ and $\mathcal{K}_n(\alpha) \subset \mathcal{K}_n(\beta)$ for $1 < \alpha < \beta$. It follows that if $f(z)$ is given by (3.3.4), then $f(z) \in \mathcal{S}_n^*(2)$ for any $n = 2, 3, 4, \cdots$, and that if $f(z)$ is given by (3.3.7), then $f(z) \in \mathcal{K}_n(2)$ for any $n = 2, 3, 4, \cdots$.

As we know, if $p(z)$ is subordinated to $q(z)$, $p(z)$, $q(z)$ analytic in $\mathbb{U}$, written as $p(z) \prec q(z)$, then there exists a function $w(z)$ analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$, and such that $p(z) = q(w(z))$. It is well known that if $q(z)$ is univalent in $\mathbb{U}$, then the subordination $p(z) \prec q(z)$ is equivalent to $p(0) = q(0)$ and $p(\mathbb{U}) \subset q(\mathbb{U})$. Many results based on subordinations concerning univalent, starlike and convex functions can be found in various works (see for example, Miller and Mocanu ([62]), Obradović and Owa ([69]) and Owa and Srivastava ([74])). Motivated by these works, we aim to use subordinations in order to study the properties of our classes, as follows:

Theorem 3.3.1. If $f(z) \in \mathcal{A}_n$ is given by

$$f(z) = z + mz^n \qquad (n = 2, 3, 4, \cdots) \tag{3.3.11}$$

for some real $0 < m < 1$, then

$$\frac{1 - mnr^{n-1}}{1 - mr^{n-1}} \leqq \text{Re}\left(\frac{zf'(z)}{f(z)}\right) \leqq \frac{1 + mnr^{n-1}}{1 + mr^{n-1}} \tag{3.3.12}$$

for $z = re^{i\theta} \in \mathbb{U}$ and

$$\frac{1 - mn}{1 - m} \leqq \text{Re}\left(\frac{zf'(z)}{f(z)}\right) \leqq \frac{1 + mn}{1 + m} \qquad (z \in \mathbb{U}). \tag{3.3.13}$$

Proof. It follows from (3.3.11)

$$\frac{zf'(z)}{f(z)} = n - \frac{n-1}{1 + mz^{n-1}}. \tag{3.3.14}$$

Letting $z = re^{i\theta} \in \mathbb{U}$, we have that

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) = n - \frac{(n-1)(1 + mr^{n-1}\cos(n-1)\theta)}{1 + m^2r^{2(n-1)} + 2mr^{n-1}\cos(n-1)\theta}. \tag{3.3.15}$$

If we write

$$g(t) = \frac{1 + mr^{n-1}t}{1 + m^2r^{2(n-1)} + 2mr^{n-1}t} \qquad (t = \cos(n-1)\theta), \tag{3.3.16}$$

then

$$g'(t) = \frac{mr^{n-1}(m^2r^{2(n-1)} - 1)}{(1 + m^2r^{2(n-1)} + 2mr^{n-1}t)^2} < 0. \tag{3.3.17}$$

This shows us the inequality (3.3.12). Further, letting $r \to 1^-$ in (3.3.12), we obtain (3.3.13).

From Theorem 3.3.1, we easily say that

Theorem 3.3.2. If $f(z) \in \mathcal{A}_n$ is given by

$$f(z) = z + \frac{m}{n}z^n \qquad (n = 2, 3, 4, \cdots) \tag{3.3.18}$$

for some real m $(0 < m < 1)$, then

$$\frac{1 - mnr^{n-1}}{1 - mr^{n-1}} \leqq \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \leqq \frac{1 + mnr^{n-1}}{1 + mr^{n-1}} \tag{3.3.19}$$

for $z = re^{i\theta} \in \mathbb{U}$ and

$$\frac{1 - mn}{1 - m} \leqq \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \leqq \frac{1 + mn}{1 + m} \qquad (z \in \mathbb{U}). \tag{3.3.20}$$

Remark 3.3.3. Since $1 - mn \geqq 0$ for $0 < m \leqq \frac{1}{n}$, $f(z)$ given by (3.3.11) belongs to the class $\mathcal{S}_n^*\left(\frac{1+mn}{1+m}\right)$ for $0 < m \leqq \frac{1}{n}$ and $f(z)$ given by (3.3.18) belongs to the class $\mathcal{K}_n\left(\frac{1+mn}{1+m}\right)$ for $0 < m \leqq \frac{1}{n}$.

To discuss our next result, we have to recall here the following lemma due to Miller and Mocanu ([63]) (also due to Jack [49]).

Lemma 3.3.4. Let $w(z)$ be analytic in $\mathbb{U}$ with $w(0) = 0$. Then, if $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \mathbb{U}$, then we have

$$z_0w'(z_0) = mw(z_0) \tag{3.3.21}$$

and

$$\operatorname{Re}\left(1 + \frac{z_0w''(z_0)}{w'(z_0)}\right) \geqq m, \tag{3.3.22}$$

where $m \geqq 1$.

With the help of Lemma 3.3.4, we derive

Theorem 3.3.5. If $f(z) \in \mathcal{A}_n$ satisfies

$$\frac{zf'(z)}{f(z)} \prec \frac{\alpha(1+z^{n-1})}{\alpha+(2-\alpha)z^{n-1}} \qquad (z \in \mathbb{U}) \tag{3.3.23}$$

for some real $\alpha > 1$, then

$$\left|\frac{zf'(z)}{f(z)} - \frac{\alpha}{2}\right| < \frac{\alpha}{2} \qquad (z \in \mathbb{U}), \tag{3.3.24}$$

thus $f(z) \in \mathcal{S}_n^*(\alpha)$.

Proof. It follows from (3.3.23) that there exists an analytic function $w(z)$ in $\mathbb{U}$ with $w(0) = 0$, $|w(z)| < 1$ $(z \in \mathbb{U})$, and

$$\frac{zf'(z)}{f(z)} = \frac{\alpha(1+w(z)^{n-1})}{\alpha+(2-\alpha)w(z)^{n-1}} \qquad (z \in \mathbb{U}). \tag{3.3.25}$$

If we write that

$$F(z) = \frac{zf'(z)}{f(z)}, \tag{3.3.26}$$

then we obtain that

$$|w(z)^{n-1}| = \left|\frac{\alpha(F(z)-1)}{\alpha-(2-\alpha)F(z)}\right| < 1 \qquad (z \in \mathbb{U}). \tag{3.3.27}$$

Since (3.3.27) gives us that

$$2|F(z)|^2 - \alpha(F(z) + \overline{F(z)}) < 0 \qquad (z \in \mathbb{U}), \tag{3.3.28}$$

we obtain (3.3.24). Noting that (3.3.24) implies that

$$0 < \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < \alpha \qquad (z \in \mathbb{U}), \tag{3.3.29}$$

we say that $f(z) \in \mathcal{S}_n^*(\alpha)$.

For the class $\mathcal{K}_n(\alpha)$, we have

Theorem 3.3.6. If $f(z) \in \mathcal{A}_n$ satisfies

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{\alpha(1+z^{n-1})}{\alpha+(2-\alpha)z^{n-1}} \qquad (z \in \mathbb{U}) \tag{3.3.30}$$

for some real $\alpha > 1$, then

$$\left|1 + \frac{zf''(z)}{f'(z)} - \frac{\alpha}{2}\right| < \frac{\alpha}{2} \qquad (z \in \mathbb{U}), \tag{3.3.31}$$

thus $f(z) \in \mathcal{K}_n(\alpha)$.

Next, we consider

Theorem 3.3.7. If $f(z) \in \mathcal{A}_n$ satisfies

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)<\frac{(n-1)}{2(\alpha-1)} \qquad (z\in\mathbb{U}) \tag{3.3.32}$$

for some real $1<\alpha\leqq 2$ and

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)<\frac{(n-1)(\alpha-1)}{2} \qquad (z\in\mathbb{U}) \tag{3.3.33}$$

for some real $\alpha>2$, then

$$\left|\frac{zf'(z)}{f(z)}-\frac{\alpha}{2}\right|<\frac{\alpha}{2} \qquad (z\in\mathbb{U}), \tag{3.3.34}$$

therefore, $f(z)\in\mathcal{S}_n^*(\alpha)$.

Proof. Let us consider a function $w(z)$ which is analytic in $\mathbb{U}$, $w(0)=0$, and given by

$$\frac{zf'(z)}{f(z)}=\frac{\alpha(1+w(z)^{n-1})}{\alpha+(2-\alpha)w(z)^{n-1}} \qquad (z\in\mathbb{U}) \tag{3.3.35}$$

for $f(z)$ satisfying (3.3.32) or (3.3.33). It follows from (3.3.35) that

$$1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}=(n-1)\frac{zw'(z)}{w(z)}\left\{\frac{w(z)^{n-1}}{1+w(z)^{n-1}}-\frac{(2-\alpha)w(z)^{n-1}}{\alpha+(2-\alpha)w(z)^{n-1}}\right\}. \tag{3.3.36}$$

Let us suppose that there exists a point $z_0\in\mathbb{U}$ such that

$$\max_{|z|\leq|z_0|}|w(z)|=|w(z_0)|=1. \tag{3.3.37}$$

Then, Lemma 3.3.4 says that

$$z_0w'(z_0)=mw(z_0) \qquad (m\geqq 1). \tag{3.3.38}$$

Letting $w(z_0)=e^{i\theta}$, we have $(k>1)$:

$$\operatorname{Re}\left(1+\frac{z_0f''(z_0)}{f'(z_0)}-\frac{z_0f'(z_0)}{f(z_0)}\right)=(n-1)k\operatorname{Re}\left\{\frac{e^{i(n-1)\theta}}{1+e^{i(n-1)\theta}}-\frac{(2-\alpha)e^{i(n-1)\theta}}{\alpha+(2-\alpha)e^{i(n-1)\theta}}\right\} \tag{3.3.39}$$

$$=(n-1)k\left\{\frac{1}{2}-\frac{(2-\alpha)(2-\alpha+\alpha\cos(n-1)\theta)}{\alpha^2+(2-\alpha)^2+2\alpha(2-\alpha)\cos(n-1)\theta}\right\}.$$

Let us define the function $g(t)$ by

$$g(t)=\frac{2-\alpha+\alpha t}{\alpha^2+(2-\alpha)^2+2\alpha(2-\alpha)t} \qquad (t=\cos(n-1)\theta). \tag{3.3.40}$$

Then

$$g'(t) = \frac{4\alpha(\alpha-1)}{(\alpha^2+(2-\alpha)^2+2\alpha(2-\alpha)t)^2} > 0. \tag{3.3.41}$$

Since $g(t)$ is increasing for $t = \cos(n-1)\theta$, (n fixed), we obtain

$$\mathrm{Re}\left(1+\frac{z_0 f''(z_0)}{f'(z_0)} - \frac{z_0 f'(z_0)}{f(z_0)}\right) \geqq \frac{(n-1)k}{2(\alpha-1)} \geqq \frac{n-1}{2(\alpha-1)}, \tag{3.3.42}$$

for $1 < \alpha \leqq 2$ and

$$\mathrm{Re}\left(1+\frac{z_0 f''(z_0)}{f'(z_0)} - \frac{z_0 f'(z_0)}{f(z_0)}\right) \geqq \frac{(n-1)(\alpha-1)k}{2} \geqq \frac{(n-1)(\alpha-1)}{2} \tag{3.3.43}$$

for $\alpha > 2$. This contradicts the conditions (3.3.32) and (3.3.33). Therefore, we say that there is no $w(z)$ such that $w(0) = 0$ and $|w(z_0)| = 1$ for $z_0 \in \mathbb{U}$. This means that $|w(z)| < 1$, for all $z \in \mathbb{U}$. From the above, we have

$$|w(z)^{n-1}| = \left|\frac{\alpha\left(\dfrac{zf'(z)}{f(z)}-1\right)}{\alpha-(2-\alpha)\dfrac{zf'(z)}{f(z)}}\right| < 1 \qquad (z \in \mathbb{U}) \tag{3.3.44}$$

and $f(z) \in \mathcal{S}_n^*(\alpha)$.

Further, we can derive some description of our classes, by giving coefficients estimates, as follows:

Theorem 3.3.8. If $f(z) \in \mathcal{A}_n$ satisfies

$$\sum_{k=n}^{\infty}(|2k-\alpha|+\alpha)|a_k| \leqq \alpha - |2-\alpha| \tag{3.3.45}$$

for some real $\alpha > 1$, then $f(z) \in \mathcal{S}_n^*(\alpha)$. The equality in (3.3.45) is attained for

$$f(z) = z + \sum_{k=n}^{\infty}\frac{(\alpha-|2-\alpha|)n\epsilon}{k(k+1)(|2k-\alpha|+\alpha)}z^k \qquad (|\epsilon| = 1). \tag{3.3.46}$$

Proof. We know that if $f(z) \in \mathcal{A}_n$ satisfies

$$\left|\frac{zf'(z)}{f(z)} - \frac{\alpha}{2}\right| < \frac{\alpha}{2} \qquad (z \in \mathbb{U}) \tag{3.3.47}$$

for $\alpha > 1$, then $f(z) \in \mathcal{S}_n^*(\alpha)$. The inequality (3.3.47) is equivalent to

$$|2zf'(z) - \alpha f(z)| < \alpha|f(z)|, \tag{3.3.48}$$

that is,

$$\left|(2-\alpha)+\sum_{k=n}^{\infty}(2k-\alpha)a_k z^{k-1}\right| < \alpha\left|1+\sum_{k=n}^{\infty}a_k z^{k-1}\right| \tag{3.3.49}$$

for $z \in \mathbb{U}$. Therefore, if $f(z)$ satisfies

$$|2-\alpha|+\sum_{k=n}^{\infty}|2k-\alpha||a_k| \leqq \alpha-\alpha\sum_{k=n}^{\infty}|a_k|, \tag{3.3.50}$$

then $f(z) \in \mathcal{S}_n^*(\alpha)$. The inequality (3.3.50) is equivalent to (3.3.45). Further, if we consider a function $f(z)$ given by (3.3.46), then we have

$$\sum_{k=n}^{\infty}\left(|2k-\alpha|+\alpha\right)|a_k| = \sum_{k=n}^{\infty}\frac{(\alpha-|2-\alpha|)n}{k(k+1)} \tag{3.3.51}$$

$$= \sum_{k=n}^{\infty}(\alpha-|2-\alpha|)\,n\left(\frac{1}{k}-\frac{1}{k+1}\right) = \alpha-|2-\alpha|.$$

This implies that $f(z)$ given by (3.3.46) satisfies the equality in (3.3.45).

For the class $\mathcal{K}_n(\alpha)$, we have

Theorem 3.3.9. If $f(z) \in \mathcal{A}_n$ satisfies

$$\sum_{k=n}^{\infty}k(|2k-\alpha|+\alpha)|a_k| \leqq \alpha-|2-\alpha| \tag{3.3.52}$$

for some real $\alpha > 1$, then $f(z) \in \mathcal{K}_n(\alpha)$. The equality in (3.3.52) is attained for $f(z)$ given by

$$f(z) = z+\sum_{k=n}^{\infty}\frac{(\alpha-|2-\alpha|)n\epsilon}{k^2(k+1)(|2k-\alpha|+\alpha)}z^k \qquad (|\epsilon|=1). \tag{3.3.53}$$

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