HABILITATION THESIS

Topics in Variational Analysis via Monotonicity and Convexity

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Preface

This thesis is based on some significant results achieved by the author after obtaining his Ph.D. degree in Mathematics at Babeş-Bolyai University, in 2011. The work contains four chapters consisting of twelve thematically linked sections. Every section is based on a published paper of the author. Therefore, for the readers convenience, every section can be read as it, without the previous read of any other section. We treat some of the most important problems of the variational analysis, that is, optimization, variational inequalities, equilibrium problems and minimax problems. Further, our results rely on the core concepts of variational analysis, namely, monotonicity and its generalizations and convexity and its extensions, respectively.

Recall, that the concept of monotonicity for operators defined on a Banach space into its dual has been introduced some fifty years ago by the celebrated works of Browder and Minty (see, for example, [69, 70], [157, 158]. This notion (often called *Minty-Browder monotonicity*) have shown to be a cornerstone for the development of variational analysis, due to the fact that convexity of a proper, lower semicontinuous function can be characterized by the maximal monotonicity of its subdifferential (see, for instance, [188]).

During the last decades, the concept of classical monotonicity has imposed itself, due to its importance, and influenced some other branches of mathematics, such as differential equations or image processing, as well as economics, engineering, management science, probability theory and other applied sciences. Due to these interactions the concepts of monotonicity alongside with convexity were subjects of a dynamical evolution reflected in a number of new concepts - extensions of the classical assumption of monotonicity and convexity without the loss of valuable properties (see, for instance, [71], [105], [114], [161], [117] and the references therein).

We emphasize here the utility of the monotonicity property in the areas of ODE and that of global injectivity problems. Indeed, it ensures the backwards uniqueness for the so-

lutions of an autonomous systems of ODE $\dot{x} = f(x)$, where the monotonicity is required for $f : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ [109]. The stronger monotonicity requirement of δ-monotonicity ($0 < \delta \leq 1$) on $f : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$, ensure the uniqueness of the local solutions for the Cauchy problem $\dot{x} = f(x)$, $x(0) = x_0$ [201]. It also ensures the convexity of the inverse images and therefore the global injectivity of monotone local homeomorphisms [124]. In order to get the global injectivity results out of monotonicity of local homeomorphisms, we do not really need the monotonicity assumption on the whole source set. Instead, we only need the local monotonicity on the complements of subsets whose overlaps with segments, having their endpoints outside, is countable [123].

Consider the nonlinear equations $T(u) = v$, where $T : H \longrightarrow H$ is a (generalized) monotone operator, H is a Hilbert space and $v \in H$. For instance [94] deals with the boundary value problems $-u" + g(u) = f$ on $I = (0,1), u(0) = u(1) = 0$, respectively, with a nonlinear second order ordinary differential equation in divergent form, $\frac{d}{dx}[a(x,u(x),u'(x))] + b(x,u(x),u'(x)) = f(x)$ with Dirichlet boundary conditions on $I = (0,1), u(0) = u(1) = 0$, where $f \in L_2(I)$ is a given function. These problems can be written as $T(u) = v$, where the nonlinear operator T has a strongly monotone part. The study of injectivity of the operator T, among others, is motivated by the fact that the injectivity of T assures the uniqueness of the solution of the nonlinear equation $T(u) = v$.

Let us mention that several injectivity conditions for operators that are monotone in some sense were obtained recently in [122, 123] and [185]. These results were applied then to obtain some injectivity/univalency results for complex functions of one complex variable. Recall that one of the most known results that provides the univalency of a holomorphic function $f : D \subseteq \mathbb{C} \longrightarrow \mathbb{C}$, $f = u + iv$, is $\operatorname{Re} f'(z) > 0$ for all $z \in D$. However, it can easily be shown (see for instance [123]) that this condition is equivalent to the strict monotonicity of the vector function $f = (u,v)$, and it is well known that strictly monotone operators are injective. On the other hand the mentioned univalency condition is a particular case of Alexander-Noshiro-Warschawski and Wolff theorem (for $\gamma = 0$, see [160,175,205,206]), and the latter cannot be deduced by using the classical strict monotonicity concept of an operator. Nevertheless, by using a generalized monotonicity concept we will obtain a result which contains the Alexander-Noshiro-Warschawski and Wolff theorem as a particular case.

Beside the injectivity results mentioned before we use the concepts of monotonicity and convexity in order to show solution existence of several variational inequalities and some scalar and vector equilibrium problems. Further, by using these concepts, we obtain some new minimax results on dense sets and via discrete and continuous dynamical systems we approximate the minimizer of an objective function having a complex structure.

Chapter 1 deals with local (generalized) monotonicities and local (generalized) convexities on special dense sets. Further some injectivity results are also obtained.

In the first section we continue the investigations started in [123]: G. Kassay, C. Pintea, S. László, *Monotone operators and closed countable sets*, Optimization **60**, 1059-1069 (2011), and improve the results obtained there. We manage to prove here that the local monotonicity of a single-valued operator assumed just on residual subsets of the source set is enough to guarantee the global monotonicity of that operator and the convexity of the inverse images therefore. As a consequence one can deduce that the local monotonicity on residual subsets of local homeomorphisms ensures their global injectivity. We pay some special attention to the residual sets arising as complements of σ-affine sets, σ-compact sets and σ-algebraic varieties. We also show that the obtained results cannot be extended to the set-valued monotone operators. However, in case of generalized monotonicities, much stronger results than those obtained in [133]: S. László, *Generalized monotone operators, generalized convex functions and closed countable sets*, J. Convex Anal. **18**, 1075-1091 (2011), can be provided even in the set-valued case. Indeed, in the second section we show that the local generalized monotonicity of a lower semicontinuous set-valued operator on some certain type of dense sets ensures the global generalized monotonicity of that operator. We achieve this goal gradually by showing at first that the lower semicontinuous set-valued functions of one real variable, which are locally generalized monotone on a dense subsets of their domain are globally generalized monotone. Then, these results are extended to the case of set-valued operators on arbitrary Banach spaces. We close this section with some results on the global generalized convexity of a real valued function, which is obtained out of its local counterpart on some dense sets. In the last section we provide sufficient conditions that ensure the convexity of the inverse images of an operator, monotone in some sense. Further, conditions that ensure the monotonicity and the local injectivity of an operator are also obtained. Combining the conditions that provide the local injectivity and the convexity of the inverse images of an operator, we are able to obtain some global injectivity results. As applications some new analytical conditions that assure the injectivity and univalency, respectively, of a complex function of one complex variable are obtained. We also show that some classical results, such as Alexander-Noshiro-Warschawski and Wolff theorem or Mocanu theorem are particular instances of our results.

In Chapter 2 we deal with the minimization problem of the sum of two functions, both in convex and nonconvex setting.

In the first section we propose a forward-backward proximal-type algorithm with inertial/memory effects for minimizing the sum of a nonsmooth function with a smooth one in the nonconvex setting. Every sequence of iterates generated by the algorithm converges to a critical point of the objective function provided an appropriate regularization of the objective satisfies the Kurdyka-Łojasiewicz inequality, which is for instance fulfilled for semi-algebraic functions. We illustrate the theoretical results by considering two numerical experiments: the first one concerns the ability of recovering the local optimal solutions of nonconvex optimization problems, while the second one refers to the restoration of a noisy blurred image. Further, in the second section, we consider a second order dynamical system of the from $\ddot{x}(t)+\gamma(t)\dot{x}(t)+x(t)-J_{\lambda(t)A}(x(t)-\lambda(t)D(x(t))-\lambda(t)\beta(t)B(x(t)))=0$, where $A:\mathcal{H}\rightrightarrows\mathcal{H}$ is a maximal monotone operator, $J_{\lambda(t)A}:\mathcal{H}\longrightarrow\mathcal{H}$ is the resolvent operator of $\lambda(t)A$, $D,B:\mathcal{H}\to\mathcal{H}$ are cocoercive operators defined on a real Hilbert space $\mathcal{H}$, $\lambda,\beta:[0,+\infty)\to[0,+\infty)$ are relaxation functions and $\gamma:[0,+\infty)\to[0,+\infty)$ a damping function, all depending on time. We show the existence and uniqueness of strong global solutions in the framework of the Cauchy-Lipschitz-Picard Theorem and prove ergodic asymptotic convergence for the generated trajectories to a zero of the operator $A+D+N_C$, where $C=\mathrm{zer}(B)$ and N_C is the normal cone operator, by using Lyapunov analysis combined with the celebrated Opial Lemma in its ergodic continuous version. Furthermore, we show the strong convergence of trajectories to the unique zero of $A+D+N_C$ in case A is a strongly monotone operator. The framework allows to address as particular case the minimization of the sum of a nonsmooth convex function with a smooth convex one and allows us to recover and improve several results from the literature.

Chapter 3 deals with the solution existence of variational inequalities. We treat both the cases where the operator involved is single valued and set valued. We apply these results in order to obtain new (unknown) coincidence point results for two operators and new fixed point results, respectively.

In the first section we introduce some new type of operators that generalize the notion of operator of type ql ([132]), that can be view as an extension of the monotonicity property of real valued functions. An example of operator belonging to this class, that is not of type ql, is also provided. Further, we give some sufficient conditions that ensure the existence of the solutions for an extended general variational inequality. We also show by an example that these results fail outside of the class of operators introduced in this section. Finally, as application, based on the existence results of the solutions for the extended general variational inequalities established before, we obtain a coincidence point result in Hilbert spaces. While existence

results of the solution for the classical Stampacchia variational inequalities were abundant in the last years (see for instance [34, 40, 42, 79, 155]), this is not the case of general variational inequality, respectively of multivalued variational inequality. Some variants of the general variational inequality problem, respectively the multivalued variational inequality problem (see [207]) have also been studied in [132, 134] and [137] in a Banach space context. In these papers several existence results of the solution for these problems were established in the case when one of the operators involved is of type ql and the operators involved possess some continuity properties. Moreover, it has been shown by examples that the existence results of solution for these problems, obtained in the papers mentioned above, fail outside of the class of ql type operators. The second and third section of this chapter are strongly connected. In the second section we obtain several existence results of the solution for general variational inequalities of Stampacchia type. These results will be used for providing some unknown coincidence point results in Hilbert spaces. Also here, as corollaries, several fixed point theorems are obtained. In the third section we obtain some existence results of the solution for general variational inequalities without assuming that the operators involved are of type ql. We do not assume any continuity property of the operators involved, instead we work with some sequential conditions imposed on these operators. We use these results to obtain some new coincidence point results in Hilbert spaces. The fourth section deals with several multivalued inequality problem, both of Stampachia and Minty type. First we state and prove a useful adaptation of KKM principle in Banach spaces. Further, we obtain some results concerning on existence of solution for these multivalued variational inequalities. By an example we show, that our results are the best possible in some sense, that is, if we drop the assumption that one of the operators involved is of type ql, in the hypothesis of our main theorems, then their conclusion fail. Finally, as applications of the results obtained, we provide some coincidence point results in Hilbert spaces.

Chapter 4 deals with scalar and vector equilibrium problems on dense sets. Further, several new minimax results on dense sets are provided.

In the first section, we deal with set-valued equilibrium problems, for which we provide sufficient conditions for the existence of a solution. The conditions, that we consider, are imposed not on the whole domain, but rather on a self segment-dense subset of it, a special type of dense subset. As an application, we obtain a generalized Debreu-Gale-Nikaïdo-type theorem, with a considerably weakened Walras law in its hypothesis. Furthermore, we consider a non-cooperative n-person game and prove the existence of a Nash equilibrium, under assumptions that are less restrictive than the classical ones. Further, in section two, we pro-

vide sufficient conditions, that ensure the existence of the solution of some vector equilibrium problems in Hausdorff topological vector spaces ordered by a cone. Also here, the conditions, that we consider, are imposed not on the whole domain of the operators involved, but rather on a self-segment-dense subset of it, a special type of dense subset. We apply the obtained results to vector optimization and vector variational inequalities. In the last section we provide conditions that assure the infimum of a proper, lower semicontinuous and convex function on a dense subset of its domain is equal to the global infimum of that function. We also obtain conditions for the coincidence of two convex functions that are equal on a dense subset of their common domain. Then, we apply these results in order to obtain some minimax results on dense sets. Also here, by an example we show that the extension of Fan's and Sion's minimax result to usual dense sets is impossible. Finally, based on our minimax results, we obtain conditions that assure the denseness of several family of functionals in the function spaces $C(K)$ and $B(K)$, respectively. This setting allows us to give an alternative proof to the famous reflexivity result of James.

Keywords

Minty-Browder monotonicity; generalized monotonicity; generalized convexity; locally monotone operator; locally increasing function; first Baire category set; σ-affine set; σ-compact set; σ-algebraic variety; residual set; perfect set; locally generalized monotone functions; generalized monotone operators; self segment-dense sets; segment-dense set; generalized convex functions; monotone operator; injective operator; complex function; univalent function; conjugate function; maximal monotone operators; Fitzpatrick function; nonsmooth optimization; limiting subdifferential; Kurdyka-Łojasiewicz inequality; Bregman distance; inertial proximal algorithm; second order dynamical system; Cauchy-Lipschitz-Picard Theorem; Lyapunov analysis; Opial Lemma; cocoercive operator; generalized variational inequality; KKM mapping; Minty's lemma; Ky Fan's lemma; fixed point; coincidence point; operator of type g-ql; extended general variational inequality; GKKM mapping; set-valued equilibrium problem; Debreu-Gale-Nikaïdo-type theorem; Nash equilibrium; vector equilibrium problem; vector optimization; vector variational inequality; minimax theorem; dense family of functionals

1

Monotonicity and convexity on special sets. Injectivity results.

1.1 Monotone Operators and first category sets

In this section we show that the local monotonicity in the sense of Minty and Browder for a single valued operator on some residual sets assure the global monotonicity and the convexity of the inverse images. We achieve this goal gradually by showing at first that the continuous real valued functions of one real variable, which are locally increasing on sets whose complements have no nonempty perfect subsets, are globally increasing. Then we extend these results to single-valued operators that are locally monotone on some residual set. We pay some special attention to the residual sets arising as complements of some special first Baire category sets, namely the σ-affine sets, the σ-compact sets and the σ-algebraic varieties. Further we obtain some global injectivity results for operators that satisfy some certain analytical conditions using the following scheme. The convexity of the inverse images combined with their discreteness, in the case of local injective operators, ensure the global injectivity. Note that the global monotonicity and the local injectivity of regular enough operators is guaranteed by the positive definiteness of the symmetric part of their Gâteaux differentials on the involved residual sets. Finally we turn our attention on the global convexity of a real valued function, which is obtained out of its local counterpart on some residual sets. We close this section by showing that our results cannot be extended to the set-valued case. Let us mention that a part of the results from this section has been published in [122]:[G. Kassay, C. Pintea, S. László, *Monotone operators and first category sets*, Positivity **16**, 565-577 (2012)].

1.1.1 Special examples of first category sets

In this paragraph we define two special types of first Baire category sets, namely σ-*affine sets* and σ-*algebraic varieties* and recall that the σ-*compact sets* are of first category as well.

Let X be a Banach space and let $V \subseteq X$ be a subset of first category (in the sense of Baire), i.e. $V = \bigcup_{n=1}^{\infty} V_n$, where $V_n \subseteq X$ are nowhere dense sets ($\operatorname{int}(\operatorname{cl}(V_n)) = \emptyset$).

Remark 1.1.1. By using Baire's theorem (see for instance [191]), the following properties are immediate:

1. $X \setminus V \neq \emptyset$.

2. $\operatorname{int}(V) = \emptyset$. Indeed, $U_n := X \setminus \operatorname{cl}(V_n)$ are open and dense sets and, by using the Baire's theorem, the set $X \setminus \bigcup_{n=1}^{\infty} \operatorname{cl}(V_n) = \bigcap_{n=1}^{\infty} U_n$ is dense, hence $\operatorname{int}(V) = \emptyset$ as $\operatorname{int}(V) = \operatorname{int}\left(\bigcup_{n=1}^{\infty} V_n\right) \subseteq \operatorname{int}\left(\bigcup_{n=1}^{\infty} \operatorname{cl}(V_n)\right) = \emptyset$.

Remark 1.1.2. The following special types of sets are of first category, and they will be useful in the sequel:

1. σ-*affine sets*. We say, that $A \subseteq X$ is $\sigma-$*affine* if $A = \bigcup_{n=1}^{\infty} A_n$, where $A_n \subseteq X$ are closed, proper affine subsets for all $n \in \mathbb{N}$, i.e. $tA_n + (1-t)A_n \subseteq A_n$, for all $t \in \mathbb{R}$. In fact one can easily show that any proper (not necessarily closed) affine subset of a normed space X has empty interior.

2. σ-*compact sets*. Recall that a subset K of X is called $\sigma-$*compact*, if K can be written as a countable union of compact sets. If $K = \bigcup_{n=1}^{\infty} K_n$ is a $\sigma-$compact subset of the infinite dimensional Banach space X, then the set $A = \bigcup_{n=1}^{\infty} \operatorname{aff}(K_n)$ is $\sigma-$affine. Indeed, $\operatorname{aff}(K_n)$ is a finite dimensional affine set [67, Theorem 6.1] and it is therefore closed and proper. Note that if X is finite dimensional, then $\operatorname{aff}(K_n)$ might not be proper.

3. σ-*algebraic varieties*. Assume that V_n are proper algebraic varieties of some Euclidean space $\mathbb{R}^p$, i.e. each V_n is the zero set of a finite family of nonzero polynomial functions from $\mathbb{R}[x_1, \ldots, x_p]$. We say that $V \subseteq X$ is a σ-*algebraic variety* if $V = \bigcup_{n=1}^{\infty} V_n$, where $V_n \subseteq \mathbb{R}^p$ are proper algebraic varieties. A σ-algebraic variety of $\mathbb{R}^2$ will be called σ-*algebraic curve*. Note that V_n are closed with respect to the Euclidean topology and nowhere dense subsets of $\mathbb{R}^p$, which shows that $V$is of first category, hence $\operatorname{int}(V) = \emptyset$. The nowhere denseness of a proper algebraic variety is due to the fact that once it contains a nondegenerate segment, it must contain the whole support straight line of that segment. In fact one can say that the set of common points of a straight line with a proper algebraic variety is either finite or the line itself.

Remark 1.1.3. In the finite dimensional case, a σ-compact set $K = \bigcup_{n=1}^{\infty} K_n$ produces the σ-affine set $A = \bigcup_{n=1}^{\infty} \operatorname{aff}(K_n)$, whenever $\dim \operatorname{aff}(K_n) < p$, for all $n \in \mathbb{N}$, where p is the dimension of the ambient Euclidian space. Note that, in the finite dimensional case, any affine set is an algebraic variety.

1.1.2 Monotone real valued functions with one real variable and perfect sets

In this paragraph we present a condition which ensures that the local increasing monotonicity of a real valued function of one real variable on the complement of a set which has no nonempty perfect subsets implies its global counterpart. We also show that in some sense this is the best result that can be obtained, since we give an example of a continuous function which is locally increasing on the complement of the Cantor set (which is perfect), but not globally increasing. Our result implies, in particular, that the local increasing monotonicity of a real valued function on the complement of a countable set implies its global increasing monotonicity. This result extend and improve the results obtained in [123] and [124].

Let $I \subseteq \mathbb{R}$, be an interval, $J \subseteq I$ and $f : I \longrightarrow \mathbb{R}$ be a function. Recall that f is said to be *locally increasing* on J if for every $x \in J$ there exists an open interval $J_x \subseteq I$, such that $x \in J_x$ and the restriction $f|_{J_x}$ is increasing.

In order to continue our analysis we need the following result from [123].

Theorem 1.1.1. *([123]) Let $I \subseteq \mathbb{R}$ be an open interval and $f : I \longrightarrow \mathbb{R}$ be a continuous function. If $L \subseteq I$ is a countable set, closed relative to I such that f is locally increasing on $I \setminus L$, then f is increasing on I.*

Notice that the proof of Theorem 1.1.1 uses effectively the relative closedness of L with respect to I. In what follows we provide an extension of Theorem 1.1.1 by relaxing the assumptions on L in two ways:

- first, we get rid of the closedness condition on L reducing, however, the proof of the new statement to the situation when L is closed,
- second, we release the countability assumption on L by supposing only the lack of its nonempty perfect subsets.

Recall that a set $P \subseteq \mathbb{R}$ is called perfect, if $P = P'$, where P' is the set of all accumulation points of P. According to Cantor-Bendixson theorem [162, p. 66] any closed subset of $\mathbb{R}$ can be written as the disjoint union of a perfect subset and a countable subset.

Theorem 1.1.2. *Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \longrightarrow \mathbb{R}$ be a continuous function, and consider $J \subseteq I$ a subset such that $I \setminus J$ has no nonempty perfect subsets. If f is locally increasing on J, then f is increasing on I.*

Proof. For every $x \in J$ consider an open interval $J_x \subseteq I$, such that $x \in J_x$ and f is increasing on J_x. Observe that $J \subseteq U \subseteq I$ and U is open, where U stands for $\bigcup_{x \in J} J_x$. Hence $\mathrm{cl}(I) \setminus U$ is closed, which shows that $L := I \setminus U$ is closed relative to I. Note that f is locally increasing on U. We show that $\mathrm{cl}(I) \setminus U$ is countable. Indeed, if one assumes that $\mathrm{cl}(I) \setminus U$ is uncountable, it follows, according to Cantor-Bendixson theorem, that $\mathrm{cl}(I) \setminus U$ contains a nonempty perfect set, say P. Thus $P \subseteq \mathrm{cl}(I) \setminus U \subseteq \mathrm{cl}(I) \setminus J$ and we claim that there exist $\alpha, \beta \in I$, $\alpha < \beta$ such that $[\alpha, \beta] \cap P$ is uncountable. Indeed, otherwise, for every sequences $(a_n), (b_n) \subset I$ such that $a_n < b_n$ for all $n \geq 1$ and $a_n \searrow a := \inf(I)$, $b_n \nearrow b := \sup(I)$, one gets the countability of $[a_n, b_n] \cap P$ for all $n \geq 1$, which shows that their union

$$\bigcup_{n=1}^{\infty} (P \cap [a_n, b_n]) = P \cap \left(\bigcup_{n=1}^{\infty} [a_n, b_n] \right) = P \cap I = P \setminus \{a, b\},$$

is countable too, a contradiction to the uncountability of P. Thus, the set $[\alpha, \beta] \cap P$, which is obviously contained in $I \setminus U = I \cap (\mathrm{cl}(I) \setminus U) \subseteq I \cap (\mathrm{cl}(I) \setminus J) = I \setminus J$, being a closed uncountable subset of $I \setminus J$. According to the Cantor-Bendixson Theorem the set $[\alpha, \beta] \cap P$ has a nonempty perfect subset, which is a nonempty perfect set of $I \setminus J$ as well, a contradiction to the hypothesis. Thus, the set $\mathrm{cl}(I) \setminus U$, alongside its subset $L = I \setminus U$, is countable. The countability of L accompanied by its closedness property relative to I combined with the local increasing monotonicity of f on $U = I \setminus (I \setminus U) = I \setminus L$, show, via Theorem 1.1.1, that f is increasing on I. □

Remark 1.1.4. If J is a subset of an interval $I \subseteq \mathbb{R}$ such that the complement $I \setminus J$ has no nonempty perfect subsets, then the set J is rather reach, as this absence of perfect subsets in $I \setminus J$ implies the denseness of J in I.

By means of Theorem 1.1.2 we obtain, in particular an improved version of Theorem 1.1.1 (without the closedness assumption of the set L, as claimed before):

Corollary 1.1.1. *Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ be a continuous function and $L \subseteq I$ be a countable set. If f is locally increasing on $I \setminus L$, then f is increasing on I.*

Proof. Observe that L does not contain nonempty perfect subsets since it is countable. The result follows from Theorem 1.1.2 for $J = I \setminus L$. □

Remark 1.1.5. Obviously, if $L = \emptyset$ in Corollary 1.1.1, then we get that the local and global monotonicity of the continuous function f on the open interval I are equivalent.

The result below provides a sufficient condition for global monotonicity in case of locally C^1-smooth functions.

Corollary 1.1.2. *Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \longrightarrow \mathbb{R}$ be a continuous function, and consider $J \subseteq I$ such that $I \setminus J$ has no nonempty perfect subsets. Assume that f is locally C^1-smooth on J, i.e. for every $x \in J$ there exists an open interval, say $J_x \subseteq I$, such that $x \in J_x$ and the restriction $f|_{J_x}$ is C^1-smooth. If $(f|_{J_x})' \geq 0$ for all $x \in J$, then f is increasing on I.*

Remark 1.1.6. If we assume in Corollary 1.1.2 that $f|_{J_x}$ is C^1-smooth and $(f|_{J_z})' > 0$ for all $z \in J$, then, besides the global increasing monotonicity of f, one gets that f is injective, i.e. f is strictly increasing. Indeed, if we assume that $f(x) = f(y)$ for some $x, y \in I$, $x < y$, then one obtains, via the global monotonicity of f, that $f|_{[x,y]}$ is actually constant. Since the interval $[x,y]$ is a nonempty perfect subset of I, it follows that $J \cap [x,y] \neq \emptyset$, as the complement $I \setminus J$ of J has no nonempty perfect sets. On the other hand, according to our assumption, $f|_{J_z}$ is C^1-smooth and $(f|_{J_z})' > 0$ for all $z \in J \cap [x,y]$, which shows that f is strictly increasing on the nondegenerate interval $J_z \cap [x,y]$ for every $z \in J \cap [x,y]$, a contradiction with the constancy of $f\big|_{J_z \cap [x,y]}$ obtained above.

The next example shows that Theorem 1.1.2 cannot be weakened, in the sense that there exist continuous, locally increasing functions on the complement of a nonempty perfect set which are not globally increasing; even more: they are not locally increasing at the points of the perfect set. Recall that the Cantor function $c : [0,1] \longrightarrow [0,1]$ is defined recursively as follows. Let $c_0(x) = x$. Then, for every integer $n \geq 0$ the function $c_{n+1}(x)$ will be defined as

$$c_{n+1}(x) = \begin{cases} \frac{1}{2} \cdot c_n(3x), & \text{if} \quad 0 \leq x < \frac{1}{3}, \\ \\ \frac{1}{2}, & \text{if} \quad \frac{1}{3} \leq x < \frac{2}{3}, \\ \\ \frac{1}{2} + \frac{1}{2} \cdot c_n(3x-2), & \text{if} \quad \frac{2}{3} \leq x \leq 1. \end{cases}$$

It can be shown that the sequence $c_n(x)$ converges uniformly, and c will denote its limit. Hence, c is continuous and increasing, but not constant, on the interval $[0,1]$. Actually it is constant on the intervals $\left(\frac{1}{3}, \frac{2}{3}\right)$, $\left(\frac{1}{9}, \frac{2}{9}\right)$, $\left(\frac{7}{9}, \frac{8}{9}\right)$, ..., and every point not in the Cantor set is in one of these intervals, which shows that $-c$ is locally increasing on the complement of the Cantor set, but not increasing on $[0,1]$.

Therefore we have the following:

Example 1.1.1. Let $f:(0,1)\longrightarrow\mathbb{R}$, $f(x)=-c(x)$. Then f is continuous on $(0,1)$ and obviously is locally increasing on a subset of $(0,1)$ having complement a nonempty perfect set. But as we have seen f is not increasing on $(0,1)$.

1.1.3 Locally monotone operators on residual sets

In this paragraph we extend the results of the previous subsection to monotone operators in the sense of Minty and Browder defined on a subset of a Banach space to its dual. We show that a hemicontinuous operator which is locally monotone on a certain type of residual set (the complement of a first category set) is globally monotone. As immediate consequences we obtain, that a local monotone operator on some special residual sets, such as the complement of a $\sigma-$affine set, the complement of a σ-compact set and the complement of a σ-algebraic variety, is globally monotone. Finally, we consider sufficient conditions, in terms of Gâteaux differential and its matrix representation, guaranteeing the local monotonicity.

Let X be a Banach space, X^* its topological dual and $D\subseteq X$. Consider further $S:D\longrightarrow X^*$ an operator. Recall, that S is called monotone (in the sense of Minty and Browder), if for all $x,y\in D$ one has $\langle S(x)-S(y),x-y\rangle\geq 0$, where $\langle x^*,x\rangle$ denotes the duality pairing, that is the value of the linear and continuous functional $x^*\in X^*$ at $x\in X$.

Definition 1.1.1. Let $D\subseteq X$ be open and $S:D\longrightarrow X^*$ be an operator. We say that S is *locally monotone* on $C\subseteq D$, if every $x\in C$ admits an open neighborhood $U_x\subseteq D$, such that the restriction $S|_{U_x}$ is monotone.

Let us recall the following continuity concept:

Definition 1.1.2. We say that S is *hemicontinuous* at $x\in X$, if for all $(t_n)_{n\in\mathbb{N}}\subset\mathbb{R}$, $t_n\longrightarrow 0$, $n\longrightarrow\infty$ and $y\in X$, such that $x+t_ny\in D$, we have $S(x+t_ny)\rightharpoonup^* Sx$, $n\longrightarrow\infty$, where " $\rightharpoonup^*$ " denotes the convergence with respect to the *weak** topology of X^*. We say that S is hemicontinuous on D if it has this property at every $x\in D$.

Let $V\subseteq X$ be a set of first category. Consider the following assumption:

(HR) $\forall x,y\in X\setminus V$, the set $[x,y]\cap V$ does not contain nonempty perfect subsets, where $[x,y]$ stands for the line segment $\{(1-t)x+ty : t\in[0,1]\}$.

Remark 1.1.7. The σ-algebraic varieties and the σ-affine sets satisfy (HR). Indeed, when we study the overlaps between a proper algebraic variety with a given line one can observe

that the latter is either contained in the algebraic variety or their intersection reduces to a finite set. This shows that the σ-algebraic varieties satisfy (HR). If X is finite dimensional, then any σ-affine set satisfies (HR), being a special kind of σ-algebraic variety. When X is infinite dimensional the intersection between a proper affine set and a given line, is either the line itself or at most one point. Thus, the σ-affine sets satisfy the (HR) property.

The next result gives sufficient conditions for global monotonicity in terms of local monotonicity on residual sets.

Theorem 1.1.3. *Let X be a Banach space, $D \subseteq X$ be an open and convex set and let $V \subseteq X$ be a first category set satisfying (HR). Let $S : D \longrightarrow X^*$ be a hemicontinuous operator, locally monotone on $D \setminus V$. Then S is monotone on D.*

Proof. According to Remark 1.1.1(2) V has empty interior, and for all $x, y \in X \setminus V$ we have $[x,y] \cap V$ contains no nonempty perfect subsets, which shows that $[x,y] \cap V$ contains no nonempty perfect subsets, for all $x, y \in D \setminus V$. If $x, y \in D \setminus V$, consider an open interval $I \supseteq [0,1]$ such that $x + t(y-x) \in D$ for all $t \in I$. Let $L = \{t \in I : x + t(y-x) \in V\}$. Then obviously L contains no nonempty perfect subsets, and the function $\gamma : I \longrightarrow \mathbb{R}$, $\gamma(t) = \langle S(x+t(y-x)), y-x\rangle$ is locally increasing on $I \setminus L$ and, by the hemicontinuity of S, is continuous on I. According to Theorem 1.1.2, the function γ is increasing on I, which in particular shows that $\langle S(y), y-x\rangle = \gamma(1) \geq \gamma(0) = \langle S(x), y-x\rangle$, hence S is monotone on $D \setminus V$.

In the general case of arbitrary $x, y \in D$ consider some sequences $(u_n), (v_n) \subset D \setminus V$ such that $u_n = x + t_n z$ and $v_n = y + s_n w$, where $z, w \in X$, t_n, $s_n \in \mathbb{R}$, $n \geq 1$, with t_n, $s_n \longrightarrow 0, n \longrightarrow \infty$. Such sequences exist since V has empty interior. According to the first part of the proof, $\langle S(v_n) - S(u_n), v_n - u_n\rangle \geq 0$ for all $n \geq 1$, which shows, by the hemicontinuity of the operator S, that $\langle S(y) - S(x), y-x\rangle \geq 0$. □

As immediate consequences we have the following results:

Corollary 1.1.3. *Let X be a Banach space, $D \subseteq X$ be an open and convex set, $A \subseteq X$ be a $\sigma-$affine set, $K \subseteq X$ be a $\sigma-$compact set and let $S : D \longrightarrow X^*$ be a hemicontinuous operator.*

1. *If S is a locally monotone operator on $D \setminus A$, then S is monotone on D.*

2. *If S is a locally monotone operator on $D \setminus K$, then S is monotone on D.*

Proof. 1. Follows immediately from Theorem 1.1.3.
2. Consider the set $A = \bigcup_{n=1}^{\infty} \mathrm{aff}(K_n)$ which according to Remark 1.1.2(2) is $\sigma-$affine. Since $D \setminus A \subseteq D \setminus K$ one gets that S is locally increasing on $D \setminus A$. The statement follows from the previous item 1. □

Remark 1.1.8. Let $D \subseteq \mathbb{R}^p$ be an open and convex set, $V \subset \mathbb{R}^p$ be a σ-algebraic algebraic variety and let $S : D \longrightarrow \mathbb{R}^p$ be a hemicontinuous, locally monotone operator on $D \setminus V$. Taking into account Remark 1.1.7 and Theorem 1.1.3, it follows that S is monotone on D.

Corollary 1.1.4. *Let X be a Banach space, $D \subseteq X$ be an open and convex set and let $V \subseteq X$ be a first Baire category set satisfying (HR). Assume that $S : D \to X^*$ is a hemicontinuous operator which is also locally Gâteaux differentiable on $D \setminus V$, i.e for every $x \in D \setminus V$ there exists an open neighbourhood $U_x \subseteq D$ of x such that the restriction $S|_{U_x}$ is Gâteaux differentiable. If $\langle dS(x;y), y\rangle \geq 0$ for all $x \in D \setminus V$ and all $y \in X$, then S is a monotone operator, where $dS(x;\cdot)$ stands for the Gâteaux differential at $x \in D \setminus V$.*

Proof. Consider two vectors $u, v \in D \setminus V$ and the function $\gamma : I \supset [0,1] \longrightarrow \mathbb{R}$ given by $\gamma(t) = \langle S(u + t(v-u), v-u\rangle$, where I is a suitable open interval. Observe that γ is locally C^1-smooth on the set $I \setminus L$, where $L := \{t \in I : u + t(v-u) \in V\}$. Note that L contains no nonempty perfect subsets. Moreover, for every point $t \in I \setminus L$ there exists an interval, say $J_t \subseteq [0,1]$ such that $\gamma\big|_{J_t}$ is C^1-smooth and, additionally, $(\gamma\big|_{J_t})' \geq 0$ for all $t \in I \setminus L$, as $\gamma'(t) = \langle dS(u + t(v-u); v-u), v-u\rangle$. By using Corollary 1.1.2 one obtains that γ is increasing on I and therefore $\langle S(u), v-u\rangle = \gamma(0) \leq \gamma(1) = \langle S(v), v-u\rangle$. □

Corollary 1.1.5. *Let X be an infinite dimensional Banach space, $D \subseteq X$ be an open and convex set, $A \subseteq X$ be a $\sigma-$affine set, $K \subseteq X$ be a $\sigma-$compact set and let $S : D \longrightarrow X^*$ be a hemicontinuous operator.*

1. *If S is locally Gâteaux differentiable on $D \setminus A$ and $\langle dS(x;y), y\rangle \geq 0$ for all $x \in D \setminus A$ and all $y \in X$, then S is a monotone operator.*

2. *If S is locally Gâteaux differentiable on $D \setminus K$ and $\langle dS(x;y), y\rangle \geq 0$ for all $x \in D \setminus K$ and all $y \in X$, then S is a monotone operator.*

Proof. 1. Follows immediately from Corollary 1.1.4, taking into account that the σ-affine sets satisfy the (HR) condition.
2. Consider the set $A = \bigcup_{n=1}^{\infty} \operatorname{aff}(K_n)$ which according to Remark 1.1.2(2) is $\sigma-$affine. Since $D \setminus A \subseteq D \setminus K$ one gets that S is locally increasing on $D \setminus A$. The statement follows from the previous item 1. □

Corollary 1.1.6. *Let $S : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$, $S = (S_1, \ldots, S_n)$ be a continuous operator which is locally Gâteaux differentiable on $D \setminus V$, where $D \subseteq \mathbb{R}^n$ is a convex open set and $V \subseteq \mathbb{R}^n$ is a σ-algebraic variety. If all minors of the symmetric part $(JS)_x + (JS)_x^T$ of the Jacobian matrix $(JS)_x$ are non-negative, for all $x \in D \setminus V$, then S is a monotone operator.*

Proof. By using [212, Theorem 7.2] one gets the positive semidefiniteness of the the symmetric operators $dS(x;\cdot)+dS(x;\cdot)^*$, $x\in D\setminus V$, as the matrix representation of $dS(x;\cdot)^*$ with respect to the canonical basis is $(JS)^T$ [112, p. 294]. This shows that

$$\langle dS(x;y),y\rangle \geq 0, \text{ for all } x\in D\setminus V \text{ and all } y\in\mathbb{R}^n, \tag{1. 1}$$

as the positive semidefiniteness of the the symmetric operators $dS(x;\cdot)+dS(x;\cdot)^*$, $x\in D\setminus V$ is equivalent to (1. 1). The statement follows now from Corollary 1.1.4, taking into account that the σ-algebraic varieties satisfy the (HR) condition. □

In what follows $\mathbb{C}$ denotes the field of complex numbers.

Corollary 1.1.7. *Let $D\subseteq\mathbb{C}$ be a convex open set and $C\subseteq\mathbb{C}$ be a σ-algebraic curve. If $f:D\longrightarrow\mathbb{C}$, $f=u+iv$ is a continuous function, locally locally C^1-smooth on $D\setminus C$ and $Re(f_z)\geq|f_{\bar z}|$, u_x, $u_y+v_x\geq 0$ on $D\setminus C$, then f is monotone.*

Proof. Indeed, we have successively

$$\begin{aligned} Re(f_z(w))\geq|f_{\bar z}(w)| \iff & 4u_x(w)v_y(w)\geq(u_y(w)+v_x(w))^2\\ \iff & \det((Jf)_w+(Jf)_w^T)\geq 0, \end{aligned}$$

which shows that $v_y\geq 0$ on $D\setminus C$ as well, since $u_x|_{D\setminus C}\geq 0$. Consequently, all minors of $(Jf)_w+(Jf)_w^T$, i.e. u_x v_y, u_y+v_x and $\det((Jf)_w+(Jf)_w^T)$, are greater than or equal to zero on $D\setminus C$ and the monotonicity follows now via Corollary 1.1.6. □

1.1.4 A global injectivity condition

In this paragraph we refine the well-known injectivity result expressed in terms of positive definiteness of the symmetric part of all Fréchet differentials of C^1 smooth operators. Indeed, we manage to get the global injectivity out of local Gâteaux differentiability on some residual sets and the positive definiteness of the symmetric parts of all Gâteaux differentials at the points of the involved residual sets.

Theorem 1.1.4. *Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space, $D\subseteq H$ be open and convex and let $V\subseteq H$ be a first category set which contains no segments and satisfies (HR). If $S:D\to H$ is continuous on D and locally Gâteaux differentiable on $D\setminus V$ such that*

$$\langle dS(x;y),y\rangle > 0, \textit{ for all } x\in D\setminus V \textit{ and all } y\in H\setminus\{0\}, \tag{1. 2}$$

then S is injective on D.

Proof. According to Corollary 1.1.4, S is monotone on D. Thus, by means of [124, Corollary 3.2], one gets that the inverse images of S are all convex. We claim that S is locally injective on $D \setminus V$. Indeed, for every $x \in D \setminus V$, there exists $r_x > 0$ such that $S\big|_{B(x,r_x)}$ is Gâteaux differentiable. Consider $u, v \in B(x, r_x)$, $u \neq v$ and the function $\gamma : I \supset [0,1] \longrightarrow \mathbb{R}$, $\gamma(t) = \langle S(u+t(v-u)), v-u\rangle$, where $I \subseteq \mathbb{R}$ is a suitable such that $u+t(v-u) \in B(x, r_x)$, for all $t \in I$, where $B(x, r_x)$ is the open ball centered at x of radius r_x. Observe that γ is a differentiable function and $\gamma'(t) = \langle (dS)(u+t(v-u), v-u), v-u\rangle > 0$, for all $t \in I$. This shows that γ is strictly increasing on $[0,1]$ and therefore

$$\gamma(0) < \gamma(1), \text{ i.e. } \langle S(v) - S(u), v-u\rangle > 0, \text{ hence } S(u) \neq S(v).$$

Next, let $x, y \in D$ and assume that $S(x) = S(y) =: z$. Since $S^{-1}(z)$ is convex and $x, y \in S^{-1}(z)$, it follows that $[x,y] \subseteq S^{-1}(z)$, namely $S|_{[x,y]}$ is constant. Moreover $[x,y] \subseteq V$, as the relation $[x,y] \cap (D \setminus V) \neq \emptyset$ would imply that $[x,y] \cap B(w, r_w)$ contains a nondegenerate segment, for every $w \in [x,y] \cap (D \setminus V)$, on which S is obviously nonconstant. Since V contains no segments, one gets that $x = y$. □

By Theorem 1.1.4 one obtains immediately the following injectivity result related to σ-compact sets.

Corollary 1.1.8. *Let $(H, \langle\cdot,\cdot\rangle)$ be an infinite dimensional Hilbert space, let $D \subseteq H$ be open and convex and let $K \subseteq D$ be a $\sigma-$compact set which contains no segments. Let $S : D \longrightarrow H$ be continuous on D and locally Gâteaux differentiable on $D \setminus K$. If*

$$\langle dS(x;y), y\rangle > 0, \textit{ for all } x \in D \setminus K \textit{ and all } y \in H \setminus \{0\}, \tag{1. 3}$$

then S is injective on D.

Proof. Consider the σ-affine set $A = \bigcup_{n=1}^{\infty} \operatorname{aff}(K_n)$, which obviously contains K, i.e. $D \setminus A \subseteq D \setminus K$. Thus, the requirement (1. 3) is also satisfied on $D \setminus A$. According to Corollary 1.1.5 (1), S is monotone, which shows that its inverse images are all convex. The local injectivity of S on $D \setminus K$ alongside its global injectivity can be performed as in the proof of Theorem 1.1.4, in which the role of V is played by K. □

The next result deals with injectivity using σ-algebraic varieties.

Corollary 1.1.9. *Let V be a σ-algebraic variety of some Euclidean space $\mathbb{R}^p$ which contains no segments and let $D \subseteq \mathbb{R}^p$ be a convex open set. Let $S : D \longrightarrow \mathbb{R}^p$ be continuous on D and locally Gâteaux differentiable on $D \setminus V$. If*

1. *If $\langle dS(x;y), y\rangle > 0$, $\forall x \in \mathbb{R}^p \setminus V$, $y \in \mathbb{R}^p \setminus \{0\}$, then S is injective on $\mathbb{R}^p$.*
2. *If the leading principal minors of the Jacobian matrices $(JS)_x$, $x \in \mathbb{R}^p \setminus V$ are all strictly positive, i.e.*

$$\det\left[\frac{\partial S_i}{\partial x_j}(x) + \frac{\partial S_j}{\partial x_i}(x)\right]_{1\le i,j\le k} > 0, \textit{ for all } 1 \le k \le n \textit{ and all } x \in \mathbb{R}^p \setminus V,$$

then S is injective.

Proof. (1) Follows from Theorem 1.1.4.
(2) By using the Sylvester's criterion one gets the positive definiteness of the symmetric operators $dS(x;\cdot) + dS(x;\cdot)^*$, $x \in D \setminus V$, as the matrix representation of $dS(x;\cdot)^*$ with respect to the standard canonical basis of $\mathbb{R}^p$ is $(JS)^T$ [112, p. 294]. This shows, in its turn that

$$\langle dS(x;y), y\rangle > 0, \text{ for all } x \in D \setminus V \text{ and all } y \in \mathbb{R}^p \setminus \{0\}, \tag{1. 4}$$

as the positive definiteness of the symmetric operators $dS(x;\cdot) + dS(x;\cdot)^*$, $x \in D \setminus V$ is equivalent to (1. 4). At this stage we only need to apply Theorem 1.1.4 in order to get the global injectivity of S. □

Remark 1.1.9. The requirement on the sets K and V in Corollary 1.1.8, Theorem 1.1.4 and Corollary 1.1.9 respectively, not to contain segments, can be replaced by the hypothesis

$$f|_K \text{ or } f|_V \text{ to be light, i.e. } \dim(f|_K)^{-1}(z) \le 0 \text{ or } \dim(f|_V)^{-1}(z) \le 0 \text{ respectively.}$$

Corollary 1.1.10. *Let $D \subseteq \mathbb{C}$ be a convex open set and $C \subseteq \mathbb{C}$ be a σ-algebraic curve which contains no segments. If $f : D \longrightarrow \mathbb{C}$ is a continuous function, locally C^1-smooth on $D \setminus C$ and satisfies the inequality $Re(f_z) > |f_{\bar{z}}|$ on $D \setminus C$, then f is injective.*

Proof. Indeed, we have successively

$$\begin{aligned} Re(f_z(w)) > |f_{\bar{z}}(w)| \iff & 4u_x(w)v_y(w) > (u_y(w) + v_x(w))^2 \\ \iff & \det((Jf)_w + (Jf)_w^T) > 0. \end{aligned}$$

We therefore have the following two possible situations:

1. u_x, v_y, $\det((Jf)_w + (Jf)_w^T) > 0$ on $D \setminus C$,

2. u_x, $v_y < 0$, $\det((Jf)_w + (Jf)_w^T) > 0$ on $D \setminus C$.

Note that the second group of inequalities is equivalent to

$$2'.\ -u_x, -v_y > 0, \det((J(-f))_w + (J(-f))_w^T) = \det((Jf)_w + (Jf)_w^T) > 0 \text{ on } D \setminus C.$$

Since the leading principal minors of the Jacobian matrix $(Jf)_w$ are u_x and the determinant $\det((Jf)_w + (Jf)_w^T)$, it follows, by using the Sylvester's criterion, that the symmetric part $(Jf)_w + (Jf)_w^T$ of the Jacobian matrix $(Jf)_w$ is positive definite in the first situation and negative definite in the second one. This shows, in its turn, that $\langle (df)_w(z), z\rangle > 0$, $\forall w \in D \setminus C$, $z \in \mathbb{C} \setminus \{0\}$ in the first situation and $\langle (d(-f))_w(z), z\rangle > 0$, $\forall w \in D \setminus C$, $z \in \mathbb{C} \setminus \{0\}$ in the second one. According to Theorem 1.1.4, one gets that f is injective in both situations. □

1.1.5 A global convexity condition

In this paragraph we use the global monotonicity results, obtained before, in order to get some global convexity results out of their local counterparts on some residual sets.

Let D be an open subset of a locally convex space. A real-valued function $f : D \longrightarrow \mathbb{R}$ is said to be *locally convex* on a subset E of D if every point $x \in E$ has a convex open neighborhood $U_x \subseteq D$ such that the restriction $f\big|_{U_x}$ is convex. If f is locally convex on the subset E of D, then f is locally convex on $\bigcup_{x \in E} U_x$.

Theorem 1.1.5. *Let X be a Banach space, $D \subseteq X$ be a convex open set and let $V \subseteq X$ be a first category set which satisfies (HR). If $f : D \longrightarrow \mathbb{R}$ is C^1-smooth on D and locally convex on $D \setminus V$, then f is globally convex.*

Proof. For every $x \in D \setminus V$ let $D_x \subseteq D$ be a convex open set such that $f\big|_{D_x}$ is convex. It follows that the restriction $(\nabla f)\big|_{D_x}$ of the gradient operator ∇f of f is monotone, that is ∇f is locally monotone on $D \setminus V$. Consequently ∇f is, according to Theorem 1.1.3, globally monotone on D. This shows that f is globally convex on D. □

Remark 1.1.10. 1. Let X be an infinite dimensional Banach space, $D \subseteq X$ be a convex open set, $A \subseteq X$ be a σ-affine set, $K \subseteq X$ be a σ-compact set and let $f : D \longrightarrow \mathbb{R}$ be a C^1-smooth function.

   (a) If f is locally convex on $D \setminus A$, then f is globally convex.

(b) If f is locally convex on $D \setminus K$, then f is globally convex.

2. Let $D \subseteq \mathbb{R}^p$ be a convex open set, $V \subset \mathbb{R}^p$ be a σ-affine variety and let $f : D \to \mathbb{R}$ be a C^1-smooth function. If f is locally convex on $D \setminus V$, then f is globally convex.

Theorem 1.1.6. *Let $\big(H, \langle \cdot, \cdot \rangle\big)$ be an infinite dimensional Hilbert space, $D \subseteq H$ be a convex open set and let $V \subseteq H$ be a first category set which satisfies (HR). Assume that $f : D \longrightarrow \mathbb{R}$ is a C^1-smooth function on D which is locally C^2-smooth on $D \setminus V$, i.e. each $x \in D \setminus V$ has an open neighbourhood U_x such that $f|_{U_x}$ is C^2-smooth. If the second-order differential $(d^2 f)_x$ is positive semidefinite for all $x \in D \setminus V$, namely $(d^2 f)_x(y,y) \geq 0$ for all $x \in D \setminus V$ and all $y \in H$, then f is globally convex.*

Proof. We first observe that $(d^2 f)_p(x,y) = \big\langle y, \big(d(\nabla f)\big)_p(x)\big\rangle \geq 0$ for all $x \in D \setminus V$ and all $y \in H$, which shows, according to Corollary 1.1.4, that the gradient operator ∇f is monotone. Thus, f is globally convex. □

Remark 1.1.11. 1. Let $\big(H, \langle \cdot, \cdot \rangle\big)$ be an infinite dimensional Hilbert space, $D \subseteq H$ be a convex open set, $A \subseteq H$ be a σ-affine set, $K \subseteq H$ be a σ-compact set and let $f : D \longrightarrow \mathbb{R}$ be a C^1-smooth function.

(a) If f is locally C^2-smooth on $D \setminus A$ and $(d^2 f)_x$ is positive semidefinite for all $x \in D \setminus A$, then f is globally convex.

(b) If f is locally C^2-smooth on $D \setminus K$ and $(d^2 f)_x$ is positive semidefinite for all $x \in D \setminus K$, then f is globally convex.

2. Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ be a C^1-smooth function, where D is a convex open set. If $A \subseteq \mathbb{R}^n$ is a σ-algebraic variety such that f is C^2-smooth on $D \setminus A$ and its Hessian matrix $\mathrm{Hess}_x(f)$ is positive semidefinite for every $x \in D \setminus A$, then f is a convex function. This statement can be proved by means of similar arguments to those in the proof of Theorem 1.1.6. Since the Hessian matrix is symmetric one can apply [212, Theorem 7.2] to test its positive (semi)definiteness.

1.1.6 On the set-valued case

One may ask whether the technics used in the proof of Theorem 1.1.3 can be adapted for set-valued monotone operators? We show in what follows that this is not possible. However, in the next section we show that more general results are valid for generalized set-valued operators, such as strictly quasimonotone or pseudomonotone operators. In order to continue

our analysis we need the following notions. Let X_1, X_2 be Hausdorff topological spaces and let $T : X_1 \rightrightarrows X_2$ be a set-valued operator with nonempty values. Recall that T is said to be upper semicontinuous at $x \in X_1$ if for every open set $N \subseteq X_2$ containing $T(x)$, there exists a neighborhood $M \subseteq X_1$ of x such that $T(M) \subseteq N$. T is said to be lower semicontinuous at $x \in X_1$ if for every open set $N \subseteq X_2$ satisfying $T(x) \cap N \neq \emptyset$, there exists a neighborhood $M \subseteq X_1$ of x such that for every $y \in M$ one has $T(y) \cap N \neq \emptyset$. T is upper semicontinuos (lower semicontinuous) on X_1 if is upper semicontinuos (lower semicontinuous) at every $x \in X_1$.

Remark 1.1.12. It may be easily proved by direct arguments the following (see [4]):

If T is compact-valued, then T is upper semicontinuous if and only if, for every net $(x_i) \subseteq X_1$ such that $x_i \longrightarrow x \in X_1$ and for every $z_i \in T(x_i)$, there exist $z \in T(x)$ and a subnet (z_{i_j}) of (z_i) such that $z_{i_j} \longrightarrow z$.

T is lower semicontinuous if and only if, for every net $(x_i) \subseteq X_1$ such that $x_i \longrightarrow x \in X_1$ and for every $z \in T(x)$ there exist $z_i \in T(x_i)$ and a subnet (z_{i_j}) of (z_i) such that $z_{i_j} \longrightarrow z$.

Let X be a real Banach space with its dual denoted by X^*, and let $T : X \rightrightarrows X^*$ be a set-valued operator. We denote by $D(T) = \{x \in X : T(x) \neq \emptyset\}$ its domain and by $R(T) = \bigcup_{x \in D(T)} T(x)$ its range. The graph of the operator T is the set $G(T) = \{(x,u) \in X \times X^* : u \in T(x)\}$. Recall that T is called monotone if $\langle u - v, x - y \rangle \geq 0$ for all $(x,u),(y,v) \in G(T)$, while T is called quasimonotone if for every $x, y \in D(T)$ and for every $u \in T(x)$ satisfying $\langle u, y - x \rangle > 0$ one has $\langle v, y - x \rangle \geq 0$ for every $v \in T(y)$. Note that every monotone operator is also quasimonotone but the converse does not hold. T is called locally monotone if every $x \in D(T)$ has an open neighborhood U such that the restriction of the operator $T|_U$ is monotone.

Next we provide an example of upper semicontinuous set-valued operator which is locally monotone on its open and convex domain, excepting one point. On the other hand the operator is not even quasimonotone globally.

Example 1.1.2. Consider the operator $F : \mathbb{R} \rightrightarrows \mathbb{R}$, $F(x) := \begin{cases} [-1,1], \text{ if } x = 0, \\ 1, \text{ otherwise.} \end{cases}$ Let $V := \{0\}$. Then, F is upper semicontinuous and (locally) monotone on $\mathbb{R} \setminus V$, but is not even quasimonotone on $\mathbb{R}$.

Indeed, since $F \equiv 1$ on $\mathbb{R} \setminus V$, it is obvious that there is monotone. It can be easily verified that F is upper semicontinuous on $\mathbb{R}$. But, for $x = -1$ and $y = 0$, and $u = 1 \in F(-1)$ and $v = -1 \in F(0)$, we get $\langle u, y - x \rangle = 1 > 0$ and $\langle v, y - x \rangle = -1 < 0$, which shows that F is not quasimonotone.

We show next, that the extension of Theorem 1.1.3 to the lower semi-continuous case is pointless.

Lemma 1.1.1. *Let $I \subseteq \mathbb{R}$ be an open interval and let $F : I \rightrightarrows \mathbb{R}$ be a lower semicontinuous set-valued map. Then F is single valued.*

Proof. Let $x_1, x_2 \in I$, $x_1 < x_2$. Since F is monotone we have $x_1^* \leq x_2^*$ for every $x_1^* \in F(x_1)$, $x_2^* \in F(x_2)$.

Assume that F is multi-valued at $x \in I$ and let $u^*, v^* \in F(x)$, $u^* < v^*$. Let $r > 0$ such that $v^* \notin (u^* - r, u^* + r)$. Since F is lower semicontinuous at x, there exists an $\varepsilon > 0$ such that for every $y \in (x - \varepsilon, x + \varepsilon)$ we have $F(y) \cap (u^* - r, u^* + r) \neq \emptyset$. But then for $y > x$ and $y^* \in F(y)$ we obtain $v^* > y^*$, contradiction. □

The next theorem shows that a lower semicontinuous, monotone map is single-valued on every open and convex subset of its domain. Actually we show that a monotone operator is single valued in the points where is lower semicontinuous. This result is known in the literature as Kenderov's Theorem [125], however we give here an original proof.

Theorem 1.1.7. *Let X be a real Banach space, let X^* be its topological dual and consider the lower semicontinuous, monotone operator $T : D \rightrightarrows X^*$. Assume further that the set $D \subseteq X$ is open and convex. Then T is single valued.*

Proof. Let $x, y \in D$, $x \neq y$. We show first, that the multi-function $F_{x,y} : (0,1) \rightrightarrows \mathbb{R}$

$$F_{x,y}(t) = \{\langle u^*, y - x\rangle : u^* \in T(x + t(y - x))\}$$

is also monotone and lower semicontinuous.

Let $t_1, t_2 \in (0,1)$, $t_1 < t_2$ and let $\langle u_1^*, y - x\rangle \in F_{x,y}(t_1)$, $\langle u_2^*, y - x\rangle \in F_{x,y}(t_2)$. We have to show that $\langle u_1^*, y - x\rangle \leq \langle u_2^*, y - x\rangle$. Since $\langle u_2^* - u_1^*, y - x\rangle = \frac{1}{t_2 - t_1}\langle u_2^* - u_1^*, (x + t_2(y - x)) - (x + t_2(y - x))\rangle$, according to the monotonicity of T the right side is nonnegative. Hence, $F_{x,y}$ is monotone.

Let $t \in (0,1)$ and $\langle u^*, y - x\rangle \in F_{x,y}(t)$. We show that for every $\{t_n\} \subseteq (0,1)$ converging to t there exist a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ and a sequence $\{\langle u_k^*, y - x\rangle\}$ with $\langle u_k^*, y - x\rangle \in F_{x,y}(t_{n_k})$ for every $k \in \mathbb{N}$, converging to $\langle u^*, y - x\rangle$.

Let $\{t_n\} \subseteq (0,1)$ converging to t and $< u^*, y - x\rangle \in F_{x,y}(t)$. Then $u_n = x + t_n(y - x)$ converges to $u = x + t(y - x)$ and $u^* \in T(x + t(y - x))$, hence according to the lower semicontinuity of T there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and a sequence $u_k^* \in T(u_{n_k})$ such that $u_k^* \longrightarrow u^*$. But then $u_{n_k} = x + t_{n_k}(y - x)$ and $\langle u_k^*, y - x\rangle \in F_{x,y}(t_{n_k})$. It is also obvious that $\langle u_k^*, y - x\rangle \longrightarrow \langle u^*, y - x\rangle$. Hence , $F_{x,y}$ is lower semicontinuous.

According to Lemma 1.1.1, $F_{x,y}$ is single valued. Due to the fact that D is convex and open, for every $z \in D$ there exists $r > 0$ such that $B(z, r) \subseteq D$. Suppose that there exists

$u_1^*, u_2^* \in T(z)$, $u_1^* \neq u_2^*$. Then for $x, y \in B(z,r)$ such that $z \in (x,y)$ we have $F_{x,y}$ single valued and this shows that $\langle u_2^* - u_1^*, y - x\rangle = 0$. We show next, that $\langle u_2^* - u_1^*, u\rangle = 0$, for every $u \in X$. If $u = 0$ this is obvious, otherwise consider $x = z - \frac{r}{\|u\|+1}u$, $y = z + \frac{r}{\|u\|+1}u$. Then $x, y \in B(z,r)$ and $z \in (x,y)$, hence $\langle u_2^* - u_1^*, y - x\rangle = 0$, or equivalently $\frac{2r}{\|u\|+1}\langle u_2^* - u_1^*, u\rangle = 0$. The later equality shows that $u_2^* - u_1^* = 0$, contradiction. □

1.2 Generalized monotone operators on dense sets

In this section we provide sufficient conditions which ensure that a lower semicontinuous set-valued operator satisfying a generalized monotonicity property locally is actually globally generalized monotone. We are concerned on qusimonotone, strictly quasimonotone, pseudomonotone as well as strictly pseudomonotone set-valued operators, and these results extend and improve the results obtained in the single-valued case in [123, 133] and [122].

We study generalized monotonicity properties on certain dense subsets that we call self-segment-dense. By an example we show that this new concept differs from that of a segment-dense set introduced by Dinh The Luc [149] in the context of densely quasimonotone, respectively densely pseudomonotone operators.

Two counterexamples circumscribe the area of our research. We show that the local (generalized) monotonicity of an upper semicontinuous set-valued operator does not imply its global counterpart (Example 1.2.3). Then we also provide an example of lower semicontinuous set-valued operator, which is locally quasimonotone on its domain but is not globally quasimonotone (Example 1.2.2). Hence, we deal with locally strictly quasimonotone, locally pseudomonotone, respectively locally strictly pseudomonotone lower semicontinuous set-valued operators.

Our analysis starts with the one dimensional case ($X = \mathbb{R}$) in which we show that a lower semicontinuous set-valued operator that is locally generalized monotone on a dense subset of its domain must be globally generalized monotone. These results are then extended to the case of arbitrary Banach spaces using the concept of a self-segment-dense subset. Finally the previous results are applied to obtain the global generalized convexity of locally generalized convex functions under mild assumptions. Also an example of a locally quasiconvex continuously differentiable function which is not globally quasiconvex is given.

Let us mention that a part of the results from this section has been published in [143]:[S. László, A. Viorel, *Generalized monotone operators on dense sets*, Numerical Functional Analysis and Optimization **36**, 901-929 (2015)].

1.2.1 Local generalized monotonicities

In what follows X denotes a real Banach space and X^* denotes the topological dual of X. For $x \in X$ and $x^* \in X^*$ we denote by $\langle x^*, x\rangle$ the scalar $x^*(x)$.

We will also often use the following notations for the open, respectively closed, line segments in X with the endpoints x and y

$$\begin{aligned}(x,y) &= \big\{z \in X : z = x + t(y-x), t \in (0,1)\big\},\\ [x,y] &= \big\{z \in X : z = x + t(y-x), t \in [0,1]\big\}.\end{aligned}$$

For a non-empty set $D \subseteq X$, we denote by $\mathrm{int}(D)$ its interior and by $\mathrm{cl}(D)$ its closure. We say that $P \subseteq D$ is dense in D if $D \subseteq \mathrm{cl}(P)$, and that $P \subseteq X$ is closed regarding D if $\mathrm{cl}(P) \cap D = P \cap D$.

Let $T : X \rightrightarrows X^*$ be a set-valued operator. We denote by $D(T) = \{x \in X : T(x) \neq \emptyset\}$ its domain and by $R(T) = \bigcup\limits_{x \in D(T)} T(x)$ its range. The graph of the operator T is the set $G(T) = \{(x,x^*) \in X \times X^* : x^* \in T(x)\}$.

Recall that T is said to be *upper semicontinuous* at $x \in D(T)$ if for every open set $N \subseteq X^*$ containing $T(x)$, there exists a neighborhood $M \subseteq X$ of x such that $T(M) \subseteq N$. T is said to be *lower semicontinuous* at $x \in D(T)$ if for every open set $N \subseteq X^*$ satisfying $T(x) \cap N \neq \emptyset$, there exists a neighborhood $M \subseteq X$ of x such that for every $y \in M \cap D(T)$ one has $T(y) \cap N \neq \emptyset$. T is upper semicontinuous (lower semicontinuous) on $D(T)$ if it is upper semicontinuous (lower semicontinuous) at every $x \in D(T)$.

It can easily be observed that the definition of lower semicontinuity is equivalent to the following (see [30] Def. 1.4.2).

Remark 1.2.1. T is lower semicontinuous at $x \in D(T)$, if and only if, for every sequence $(x_n) \subseteq D(T)$ such that $x_n \longrightarrow x$ and for every $x^* \in T(x)$ there exists a sequence of elements $x_n^* \in T(x_n)$ such that $x_n^* \longrightarrow x^*$.

Obviously, when T is single-valued then upper semicontinuity and also lower semicontinuity become the usual notion of continuity.

Definition 1.2.1. Let $T : X \longrightarrow X^*$ be a single valued operator. We say that T is hemicontinuous at $x \in X$, if for all $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}$, $t_n \longrightarrow 0$, $n \longrightarrow \infty$ and $y \in X$, we have $T(x + t_n y) \rightharpoonup^* T(x)$, $n \longrightarrow \infty$, where " $\rightharpoonup^*$ " denotes the convergence with respect to the *weak** topology of X^*.

In what follows we recall the definitions of several monotonicity concepts (see, for instance [106, 107, 152, 181]). The operator $T : X \rightrightarrows X^*$ is called:

(1) *monotone* if for all $(x,x^*),(y,y^*) \in G(T)$ one has the following

$$\langle x^* - y^*, x - y\rangle \geq 0;$$

(2) *pseudomonotone* if for all $(x,x^*),(y,y^*) \in G(T)$, the following implication holds

$$\langle x^*, y - x\rangle \geq 0 \Longrightarrow \langle y^*, y - x\rangle \geq 0,$$

or equivalently, for all $(x,x^*),(y,y^*) \in G(T)$, one has

$$\langle x^*, y - x\rangle > 0 \Longrightarrow \langle y^*, y - x\rangle > 0;$$

(3) *strictly pseudomonotone* if for all $(x,x^*),(y,y^*) \in G(T)$, $x \neq y$, the following implication holds

$$\langle x^*, y - x\rangle \geq 0 \Longrightarrow \langle y^*, y - x\rangle > 0;$$

(4) *quasimonotone* if for every $(x,x^*),(y,y^*) \in G(T)$ the following implication holds

$$\langle x^*, y - x\rangle > 0 \Longrightarrow \langle y^*, y - x\rangle \geq 0;$$

(5) *strictly quasimonotone*, if T is quasimonotone, and for all $x,y \in D(T)$, $x \neq y$ there exist $z \in (x,y)$ and $z^* \in T(z)$ such that$\langle z^*, y - x\rangle \neq 0$.

The relation among these concepts is shown bellow.

$$\begin{array}{ccc} & & T \text{ is monotone} \\ & & \Downarrow \\ T \text{ is strictly pseudomonotone} & \Longrightarrow & T \text{ is pseudomonotone} \\ \Downarrow & & \Downarrow \\ T \text{ is strictly quasimonotone} & \Longrightarrow & T \text{ is quasimonotone.} \end{array}$$

The concepts of local monotonicity can be defined as follows.

Definition 1.2.2. We say that the operator T is locally monotone, (respectively, locally pseudomonotone, locally strict pseudomonotone, locally quasimonotone, locally strict quasimonotone), on its domain $D(T)$, if every $x \in D(T)$ admits an open neighborhood $U_x \subseteq X$

such that the restriction of the operator T on $U_x \cap D(T)$, $T\big|_{U_x \cap D(T)}$ is monotone, (respectively, pseudomonotone, strictly pseudomonotone, quasimonotone, strictly quasimonotone).

According to [123], respectively [133], local monotonicity, respectively local generalized monotonicity of a real valued function of one real variable in general implies its global counterpart. More precisely the following hold.

Proposition 1.2.1. *Let $I \subseteq \mathbb{R}$ be an open interval and let $f : I \longrightarrow \mathbb{R}$ be a function.*

(a) (see Lemma 2.1, [123]) *If f is locally increasing on I, then f is globally increasing on I.*

(b) (see Theorem 2.2, [133]) *If f is locally strictly quasimonotone on I, then f is globally strictly quasimonotone on I.*

(c) (see Theorem 2.3, [133]) *If f is locally pseudomonotone on I, then f is globally pseudomonotone on I.*

(d) (see Theorem 2.4, [133]) *If f is locally strictly pseudomonotone on I, then f is globally strictly pseudomonotone on I.*

However, local quasimonotonicity does not imply global quasimonotonicity as the next example shows.

Example 1.2.1. (Example 2.6, [133]) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$,

$$f(x) = \begin{cases} -x-1, \text{ if } x < -1 \\ 0, \text{ if } x \in [-1,1] \\ -x+1, \text{ if } x > 1. \end{cases}$$

We can see that f is continuous and locally quasimonotone on $\mathbb{R}$ but f is not globally quasimonotone.

Next we extend the statements $(a)-(d)$ in Proposition 1.2.1 to the case of set-valued operators. We need the following notion.

Let Z and Y be two arbitrary sets. A single valued selection of the set-valued map $F : Z \rightrightarrows Y$ is the single valued map $f : Z \longrightarrow Y$ satisfying $f(z) \in F(z)$ for all $z \in Z$.

Proposition 1.2.2. *Let $I \subseteq \mathbb{R}$ be an interval and let $F : I \rightrightarrows \mathbb{R}$ be a set-valued function. Assume that F is locally strictly quasimonotone, (respectively locally monotone, locally pseudomonotone, locally strictly pseudomonotone) on I. Then F is strictly quasimonotone, (respectively monotone, pseudomonotone, strictly pseudomonotone) on I.*

Proof. We treat the case when F is strictly quasimonotone, the other cases can be treated similarly. Observe first that due to the local strict quasimonotonicity property of F, for every $x, y \in I$, $x \neq y$, there exists $z \in (x, y)$ and $z^* \in F(z)$ such that $\langle z^*, y - x\rangle \neq 0$.

Let $x, y \in \text{int}(I)$. Suppose the contrary, that is, $\langle x^*, y - x\rangle > 0$ for some $x^* \in F(x)$ and $\langle y^*, y - x\rangle < 0$, for some $y^* \in F(y)$.

Let f be a single valued selection of F, which is also locally strictly quasimonotone, and assume that $f(x) = x^*$ and $f(y) = y^*$. According to Proposition 1.2.2 (b) we get that f is globally strictly quasimonotone on I, hence $\langle f(y), y - x\rangle = \langle y^*, y - x\rangle \geq 0$, contradiction. Thus, F is globally strictly quasimonotone on $\text{int}(I)$.

If I is closed from left assume that x is the left endpoint of I and $\langle x^*, y - x\rangle > 0$ for some $x^* \in F(x)$ and $y \in I$. Then obviously $x^* > 0$. Assume further that there exists $y^* \in F(y)$ such that $\langle y^*, y - x\rangle < 0$, i.e. $y^* < 0$. Since F is locally strictly quasimonotone on I, there exists an open neighborhood U_x of x, such that $F|_{U_x \cap I}$ is strictly quasimonotone, which in particular shows that for some $u \in \text{int}(I) \cap U_x$ there exists $u^* \in F(u)$, $u^* > 0$. On the other hand, it can be deduced in the same manner, that there exists an open neighborhood U_y of y, such that $F|_{U_y \cap I}$ is strictly quasimonotone, which in particular shows that for some $v \in \text{int}(I) \cap U_y$ there exists $v^* \in F(v)$, $v^* < 0$. Obviously we can assume $u < v$. But then, $\langle u^*, v - u\rangle > 0$ and $\langle v^*, v - u\rangle < 0$ which is in contradiction with the fact that F is strictly quasimonotone on $\text{int}(I)$.

The case when I is closed from right can be treated similarly and we omit it. □

Remark 1.2.2. Note that for the case when F is locally monotone, locally pseudomonotone, respectively locally strictly pseudomonotone, one may use Proposition 1.2.2 (a), (c) and (d) respectively.

The next example shows, that locally quasimonotone set-valued operators are not globally quasimonotone in general. Even more, our operator is also lower semicontinuous, hence searching for conditions that ensure global quasimonotonicity of a lower semicontinuous set-valued operator, based on its local quasimonotonicity, is meaningless.

Example 1.2.2. Consider the operator

$$F : \mathbb{R} \rightrightarrows \mathbb{R}, F(x) := \begin{cases} [0, -x-1], \text{ if } x < -1, \\ 0, \text{ if } x \in [-1, 1], \\ [-x+1, 0], \text{ if } x > 1. \end{cases}$$

Then F is lower semicontinuous and locally quasimonotone on $\mathbb{R}$, but is not globally quasimonotone.

It is easy to check that F is locally quasimonotone and lower semicontinuous on $\mathbb{R}$. On the other hand, for $x = -2, y = 2$ and $x^* = 1 \in F(x)$ we have $\langle x^*, y - x\rangle = 4 > 0$, but for $y^* = -1 \in F(y)$ we have $\langle y^*, y - x\rangle = -4 < 0$, which shows that F is not quasimonotone.

Next we define the local generalized monotonicity of an operator on a subset of its domain.

Definition 1.2.3. Let $T : X \rightrightarrows X^*$ be a set-valued operator and let $D \subseteq D(T)$. We say that T is locally quasimonotone, (respectively, locally monotone, locally pseudomonotone, locally strictly pseudomonotone, locally strictly quasimonotone), on D, if every $x \in D$ admits an open neighborhood U_x, such that the restriction $T|_{U_x \cap D}$ is quasimonotone, (respectively, monotone, pseudomonotone, strictly pseudomonotone, strictly quasimonotone).

Remark 1.2.3. There are other ways to define local generalized monotonicity of an operator on a subset of its domain. Indeed, one may assume that:

(i) every $x \in D(T)$ admits an open neighborhood U_x, such that the restriction $T|_{U_x \cap D}$ has the appropriate (generalized) monotonicity property, respectively,

(ii) every $x \in D$ admits an open neighborhood U_x, such that the restriction $T|_{U_x \cap D(T)}$ has the appropriate (generalized) monotonicity property.

However, it can be observed that the condition given in Definition 1.2.3 is the weakest among these conditions. Note that if $D(T)$ and also D are open, then the above definitions coincide.

1.2.2 Global generalized monotonicity results for single-valued operators

Having Definition 1.2.3 in mind, an important question is related to which properties must the subset $D \subseteq D(T)$ have such that the local generalized monotonicity on D implies the global monotonicity of the same kind. The following results concerning single-valued operators, established in [123], respectively [133], will serve as a starting point for our investigations in Section 3.

Proposition 1.2.3. (Theorem 3.4, [123]) *Let X be a real Banach space and let $D \subseteq X$ be open and convex. Let $C \subseteq D$ be a set closed regarding D, with empty interior, satisfying the following condition*

$$\forall x, y \in D \setminus C \text{ the set } [x, y] \cap C \text{ is countable, possibly empty.}$$

Assume that $T : D \longrightarrow X^*$ is a single-valued operator. If T is continuous and the restriction $T|_{D\setminus C}$ is locally monotone, then T is monotone on D

Proposition 1.2.4. *Let X be a real Banach space and let $D \subseteq X$ be open and convex. Let $C \subseteq D$ be a set closed regarding D, with empty interior, satisfying the following condition*

$$\forall x, y \in D \setminus C \text{ the set } [x,y] \cap C \text{ is countable, possibly empty.}$$

Assume that $T : D \longrightarrow X^*$ is a hemicontinuous operator with the property that $\langle T(z), y-x\rangle \neq 0$ for all $z \in [x,y] \cap C$, $x, y \in D$, $x \neq y$. Then the following hold.

(a) (Theorem 3.6, [133]) *If $T|_{D\setminus C}$ is locally strictly quasimonotone, then T is strictly quasimonotone on D.*

(b) (Theorem 3.7, [133]) *If $T|_{D\setminus C}$ is locally pseudomonotone, then T is pseudomonotone on D.*

(c) (Theorem 3.8, [133]) *If $T|_{D\setminus C}$ is locally strictly pseudomonotone, then T is strictly pseudomonotone on D.*

Recall that a set $V \subseteq X$ is of first category in the sense of Baire, if $V = \bigcup_{n=1}^{\infty} V_n$, where $V_n \subseteq X$, $n \in \mathbb{N}$ are nowhere dense sets i. e. $\operatorname{int}(\operatorname{cl}(V_n)) = \emptyset$.

Note that condition

$$\forall x, y \in D \setminus C \text{ the set } [x,y] \cap C \text{ is countable, possibly empty}$$

imposed on the set C in Proposition 1.2.4, has been weakened in [122]. More precisely, in [122] it was established the following result.

Proposition 1.2.5. (Theorem 3, [122]) *Let X be a real Banach space and let $V \subseteq X$ be a set of first category with the following property*

$$\forall x, y \in D \setminus V, \text{ the set } [x,y] \cap V \text{ is countable, possibly empty.}$$

Assume that $T : D \longrightarrow X^*$ is hemicontinuous and locally monotone on $D \setminus V$. Then T is monotone on D.

Remark 1.2.4. Note that the conclusion of Proposition 1.2.5 remains valid if the condition $\forall x, y \in D \setminus V$, the set $[x,y] \cap V$ is countable, possibly empty is replaced by the condition

$$\forall x, y \in D \setminus V, \text{ the set } [x,y] \cap V \text{ contains no nonempty perfect subsets.}$$

Remark 1.2.5. It is worth mentioning that Proposition 1.2.5 cannot be extended to the set-valued case since, according to a theorem of Kenderov (see [125], Proposition 2.6), a monotone operator that is lower semicontinuous at a point of its domain must be single-valued at that point.

Furthermore, Proposition 1.2.5 does not remain valid for set-valued upper semicontinuous operators (see [122], Example 2). Next we also provide an example of an upper semicontinuous set-valued operator which is strictly pseudomonotone on a dense subset of its open and convex domain, but which is not even quasimonotone on its domain.

Example 1.2.3. Consider the operator

$$F : \mathbb{R} \rightrightarrows \mathbb{R},\ F(x) := \begin{cases} x+2, \text{ if } x \in \mathbb{R} \setminus \{0\}, \\ [-1,3], \text{ if } x = 0. \end{cases}$$

Let $V := \{0\}$. Then, F is upper semicontinuous on $D(F) = \mathbb{R}$ and strictly pseudomonotone on $D(F) \setminus V$, but is not even quasimonotone on $D(F)$.
Obviously $D(F) \setminus V = \mathbb{R} \setminus \{0\}$ is dense in $D(F) = \mathbb{R}$. Since $F(x) = x+2$ on $D(F) \setminus V$, it is obvious that there is strictly pseudomonotone. It can easily be verified that F is upper semicontinuous on $\mathbb{R}$. But, for $(x,x^*) = (-1,1) \in G(F)$ and $(y,y^*) = (0,-1) \in G(F)$, we get $\langle x^*, y-x \rangle = 1 > 0$ and $\langle y^*, y-x \rangle = -1 < 0$, which shows that F is not quasimonotone on $\mathbb{R}$.

Remark 1.2.6. According to Example 1.2.2, Example 1.2.3 and Remark 1.2.5 we can obtain global generalized monotonicity results for a set-valued operator, based on its local appropriate generalized monotonicity property, only in the case when the operator is lower semicontinuous and the mentioned monotonicity property is one of the following: strict quasimonotonicity, pseudomonotonicity or strict pseudomonotonicity, respectively.

1.2.3 Self-segment-dense sets

In [149], Definition 3.4, The Luc has introduced the notion of a so-called *segment-dense* set. Let $V \subseteq X$ be a convex set. One says that the set $U \subseteq V$ is segment-dense in V if for each $x \in V$ there can be found $y \in U$ such that x is a cluster point of the set $[x,y] \cap U$.

In what follows we introduce a denseness notion, slightly different from the concept of The Luc presented above, but which is sufficient for the conditions in Proposition 1.2.3, Proposition 1.2.4, respectively Proposition 1.2.5.

Definition 1.2.4. Consider the sets $U \subseteq V \subseteq X$ and assume that V is convex.

We say that U is self-segment-dense in V if U is dense in V and

$$\forall x, y \in U, \text{ the set } [x,y] \cap U \text{ is dense in } [x,y].$$

Remark 1.2.7. Obviously in one dimension the concepts of a segment-dense set respectively a self-segment-dense set are equivalent to the concept of a dense set.

Assume that C, D, respectively V are sets as in Proposition 1.2.3, Proposition 1.2.4, respectively Proposition 1.2.5. Then the set $D \setminus C$, respectively the set $D \setminus V$ is self-segment-dense in D.

In what follows we provide some examples of self-segment-dense sets.

Example 1.2.4. (see also Example 3.1, [146]) Let V be the two dimensional Euclidean space $\mathbb{R}^2$ and define U to be the set

$$U := \{(p,q) \in \mathbb{R}^2 : p \in \mathbb{Q}, q \in \mathbb{Q}\},$$

where $\mathbb{Q}$ denotes the set of all rational numbers. Then, it is clear that U is dense in $\mathbb{R}^2$. On the other hand U is not segment-dense in $\mathbb{R}^2$, since for $x = (0, \sqrt{2}) \in \mathbb{R}^2$ and for every $y = (p,q) \in U$, one has $[x,y] \cap U = \{y\}$.

It can easily be observed that U is also self-segment-dense in $\mathbb{R}^2$, since for every $x, y \in U$ $x = (p,q)$, $y = (r,s)$ we have $[x,y] \cap U = \{(p + t(r-p), q + t(s-q)) : t \in [0,1] \cap \mathbb{Q}\}$, which is obviously dense in $[x,y]$.

Example 1.2.5. Let $V^0([0,1], \mathbb{R})$ be the space of piecewise constant $\mathbb{R}$-valued functions.

Obviously $V^0([0,1], \mathbb{Q})$ is dense in $V^0([0,1], \mathbb{R})$. So, it follows from the density of $V^0([0,1], \mathbb{R})$ in $L^2([0,1], \mathbb{R})$ that

$$V^0([0,1], \mathbb{Q}) \text{ is dense in } L^2([0,1], \mathbb{R}).$$

Moreover, $V^0([0,1], \mathbb{Q})$ is self-segment-dense in $L^2([0,1], \mathbb{R})$, and for any $u, v \in V^0([0,1], \mathbb{Q})$ the segment $[u,v]$ is not contained in $V^0([0,1], \mathbb{Q})$. However, $V^0([0,1], \mathbb{Q})$ is not segment-dense (in the sense of The Luc) in $L^2([0,1], \mathbb{R})$, as can be seen by taking $u(x) = \sqrt{2}$ for all $x \in [0,1]$ and for any $v \in V^0([0,1], \mathbb{Q})$ we have that $[u,v] \cap V^0([0,1], \mathbb{Q}) = \{v\}$.

The proof of the fact that $V^0([0,1], \mathbb{R})$ is dense in $L^2([0,1], \mathbb{R})$ relies on the following classical arguments.

Let $\varepsilon > 0$ be given. We have that $C_c^\infty([0,1],\mathbb{R})$ is dense in $L^2([0,1],\mathbb{R})$ (see for example [65]), so for a given $u \in L^2([0,1],\mathbb{R})$ the exists $u^\varepsilon \in C_c^\infty([0,1],\mathbb{R})$ such that

$$\|u - u^\varepsilon\|_2 < \varepsilon/2.$$

Now since u^ε is continuous on the compact set $[0,1]$, uniform continuity guarantees that there exists $\delta > 0$ such that for any $x, y \in [0,1]$ with $|x-y| < \delta$ we have

$$|u^\varepsilon(x) - u^\varepsilon(y)| < \varepsilon/2.$$

If we take a regular grid (partition) with step h small enough ($h < \delta$) and choose $u^h \in V^0([0,1],\mathbb{R})$ such that $u^h\left(\left(i-\frac{1}{2}\right)h\right) = u^\varepsilon\left(\left(i-\frac{1}{2}\right)h\right)$ for $i := 0,1,\ldots\frac{1}{h}$ then

$$\left\|u^h - u^\varepsilon\right\|_\infty < \varepsilon/2.$$

Hence

$$\begin{aligned}\left\|u - u^h\right\|_2 &\leq \|u - u^\varepsilon\|_2 + \left\|u^\varepsilon - u^h\right\|_2 \\ &\leq \varepsilon/2 + \varepsilon/2.\end{aligned}$$

In the next section we show that a lower semicontinuous set-valued operator that possesses a local generalized monotonicity property on a self-segment-dense subset of its domain is actually globally generalized monotone. According to the previous remark, this result not only extends but also improves the results presented in Proposition 1.2.4.

Remark 1.2.8. Note that every dense convex subset of a Banach space is self-segment-dense. In particular dense subspaces and dense affine subsets are self-segment-dense. Other interesting properties of a self-segment-dense set will be emphasized in Chapter 4.

1.2.4 Locally quasiomonotone and strictly quasimonotone operators

In this paragraph we study the local quasimonotonicity, respectively the local strict quasimonotonicity of set-valued operators on dense sets. As it was expected some additional conditions are needed in order to assure that local generalized monotonicity on a dense subset of a set-valued operator implies its global counterpart. Such conditions will be given

and by an example we show that our conditions are essential in obtaining these results. As consequences we obtain some results that improve the results stated in [133].

Let $F : I \subseteq \mathbb{R} \rightrightarrows \mathbb{R}$ be a set-valued function. Let us introduce the following notations.

$$F(x) \geq 0 \Leftrightarrow x^* \geq 0, \forall x^* \in F(x), \text{ and } F(x) > 0, \text{ if } F(x) \geq 0 \text{ and } 0 \notin F(x).$$

Obviously the inequalities $F(x) \leq 0$, respectively $F(x) < 0$ can be introduced analogously.

Lemma 1.2.1. *Let $I \subseteq \mathbb{R}$ be an interval and consider the set-valued function $F : I \rightrightarrows \mathbb{R}$. Let $J \subseteq I$ dense in I and assume that F is locally quasimonotone on J and F is lower semicontinuous on $I \setminus J$. Then, for any fixed $x \in I \setminus J$ one has $F(x) \geq 0$, or $F(x) \leq 0$.*

Proof. Let $x \in I \setminus J$, and suppose, that there exist $x_1^*, x_2^* \in F(x)$, $x_1^* > 0$, $x_2^* < 0$. Consider the intervals $V_1 = (x_1^* - \varepsilon, x_1^* + \varepsilon)$, respectively $V_2 = (x_2^* - \varepsilon, x_2^* + \varepsilon)$, where $\varepsilon > 0$ such that $x_1^* - \varepsilon > 0$ and $x_2^* + \varepsilon < 0$. By the lower semicontinuity of F at x, there exists an open neighborhood U_x of x, such that for every $y \in U_x \cap I$ one has $F(y) \cap V_1 \neq \emptyset$ and $F(y) \cap V_2 \neq \emptyset$. Let $y \in U_x \cap J$. Then, according to the hypothesis of the lemma, there exists an open neighborhood U_y of y such that $F|_{U_y \cap J}$ is quasimonotone.

Let $U = U_x \cap U_y \cap J$ and let $u, v \in U$, $u \neq v$. Then, there exist $u_1^*, u_2^* \in F(u)$, $v_1^*, v_2^* \in F(v)$ such that $u_1^* > 0, v_1^* > 0,$ and $u_2^* < 0, v_2^* < 0$.

Hence,

$$\langle u_1^*, v - u \rangle > 0, \text{ or } \langle u_2^*, v - u \rangle > 0,$$

on the other hand

$$\langle v_1^*, v - u \rangle < 0, \text{ or } \langle v_2^*, v - u \rangle < 0,$$

which contradicts the fact that F is quasimonotone in U.

Consequently, for any $x \in I \setminus J$ one has $F(x) \geq 0$, or $F(x) \leq 0$. □

Now we are able to state and prove the following result concerning on the strict quasimonotonicity of a set-valued function.

Theorem 1.2.1. *Let $I \subseteq \mathbb{R}$ be an interval and consider the set-valued function $F : I \rightrightarrows \mathbb{R}$. Let $J \subseteq I$ dense in I and assume that F is locally strictly quasimonotone on J and lower semicontinuous on $I \setminus J$. Further, assume that $0 \notin F(x)$, for all $x \in I \setminus J$. Then F is strictly quasimonotone on I.*

Proof. First we show, that for all $u, v \in I$, $u < v$ there exists $z \in (u, v)$ and $z^* \in F(z)$, such that $\langle z^*, v - u \rangle \neq 0$. Indeed, let $u, v \in U_x \cap I$, $u < v$. Then, by the denseness of J in I, we obtain

that there exists $w \in (u,v) \cap J$, and according to the hypothesis of the theorem w admits an open neighborhood U_w such that $F|_{U_w \cap J}$ is strictly quasimonotone. This in particular shows, that there exists $z \in (u,v)$ and $z^* \in F(z)$, such that $\langle z^*, v-u \rangle \neq 0$. Hence, in order to show the local strict quasimonotonicity of F on I, it is enough to show that every $x \in I$ admits an open neighbourhood U_x such that $F|_{U_x \cap I}$ is quasimonotone.

Let $x \in I \setminus J$. According to Lemma 1.2.1 one has $F(x) \geq 0$, or $F(x) \leq 0$. Note that according to the hypothesis of the theorem one has $F(x) > 0$, or $F(x) < 0$. Assume that $F(x) > 0$ the other case can be treated analogously. Obviously for $x^* \in F(x)$ one has $x^* > 0$. Let $V = (x^* - \varepsilon, x_1^* + \varepsilon)$ be an open interval, where $\varepsilon > 0$ such that $x^* - \varepsilon > 0$. Since F is lower semicontinuous at x, there exists an open neighborhood U_x of x, such that for every $y \in U_x \cap I$ we have $F(y) \cap V \neq \emptyset$. We show that F is quasimonotone on $U_x \cap I$.

Indeed, assume that there exist $u, v \in U_x \cap I$ and $u^* \in F(u)$, $v^* \in F(v)$ such that

$$\langle u^*, v-u \rangle > 0 \text{ and } \langle v^*, v-u \rangle < 0.$$

Obviously, we can assume that $u < v$. It follows, that $u^* > 0$ and $v^* < 0$. But then $v \in J$, since otherwise, according to Lemma 1.2.1 and the hypothesis of the theorem we have $F(v) < 0$, which contradicts the fact $F(v) \cap V \neq \emptyset$.

Hence, $v \in J$, and by the hypothesis of the theorem, there exists an open neighbourhood U_v of v such that $F|_{U_v \cap J}$ is quasimonotone.

Let $w \in U_v \cap J \cap U_x$ such that $w < v$. Then $F(w) \cap V \neq \emptyset$, hence there exists $w^* \in F(w)$ with $w^* > 0$. But then, $\langle w^*, v-w \rangle > 0$ and $\langle v^*, v-w \rangle < 0$ which contradicts the quasimonotonicity of F on U_v. Hence F is quasimonotone on $U_x \cap I$ for all $x \in I \setminus J$.

If $x \in J$ there exists an open neighborhood U_x of x such that $F|_{U_x \cap J}$ is strictly quasimonotone. Assume that there exist $u, v \in U_x \cap I$ and $u^* \in F(u)$, $v^* \in F(v)$ such that

$$\langle u^*, v-u \rangle > 0 \text{ and } \langle v^*, v-u \rangle < 0.$$

Obviously, we can assume that $u < v$. It follows, that $u^* > 0$ and $v^* < 0$.

If $u \in I \setminus J$ then by the lower semicontinuity of F at u and by the denseness of J in I we have that there exists $w \in (u,v) \cap J$ and $w^* \in F(w)$ such that $w^* > 0$. But then $v \in I \setminus J$, since otherwise one has $\langle w^*, v-w \rangle > 0$ and $\langle v^*, v-w \rangle < 0$ which contradicts the strict quasimonotonicity of F on $U_x \cap J$.

On the other hand if $v \in I \setminus J$ then by the lower semicontinuity of F at v and by the denseness of J in I we have that there exists $z \in (w,v) \cap J$ and $z^* \in F(z)$ such that $z^* < 0$.

But then, $\langle w^*, z-w \rangle > 0$ and $\langle z^*, z-w \rangle < 0$ which contradicts the strict qusimonotonicity

of F on $U_x \cap J$.

If $u \in J$ then $v \in I \setminus J$, and analogously we obtain that there exists $z \in (u,v) \cap J$ and $z^* \in F(z)$ such that $z^* < 0$ which contradicts the strict quasimonotonicity of F on $U_x \cap J$.

Hence F is quasimonotone on $U_x \cap I$ for all $x \in J$.

We have shown that F is locally quasimonotone on I, therefore according to the first part of the proof, F is locally strictly quasimonotone on I.

Consequently, in virtue of Proposition 1.2.2 F is strictly quasimonotone on I. □

In what follows we extend Theorem 1.2.1 for set-valued operators that are locally strictly quasimonotone on a self-segment-dense subset of their convex domain.

Theorem 1.2.2. *Let X be a real Banach space and let X^* be its topological dual. Let $T : X \rightrightarrows X^*$ be an operator with convex domain $D(T)$. Let $D \subseteq D(T)$ self-segment-dense in $D(T)$ and assume that T is locally strictly quasimonotone on D and lower semicontinuous on $D(T) \setminus D$. Assume further, that the following condition holds:*

$$\forall x, y \in D(T), x \neq y, z \in [x,y] \cap (D(T) \setminus D) \text{ and } z^* \in T(z) \text{ one has } \langle z^*, y-x \rangle \neq 0. \tag{1. 5}$$

Then T is strictly quasimonotone.

Proof. First of all observe, that for every $x, y \in D(T)$, $x \neq y$ there exists $z \in (x,y)$ and $z^* \in T(z)$ such that $\langle z^*, y-x \rangle \neq 0$. Indeed, if $[x,y] \cap D$ is dense in $[x,y]$ then the statement follows from the local strict quasimonotonicity property of T on D. Otherwise, there exists $z \in (x,y)$, $z \notin D$. But then the statement follows from condition (1. 5). Hence, it is enough to show, that for all $(x,x^*), (y,y^*) \in G(T)$, such that $\langle x^*, y-x \rangle > 0$ we have $\langle y^*, y-x \rangle \geq 0$.

We divide the proof into two cases.

If $x, y \in D$ then consider the real set-valued function

$$F_{x,y} : [0,1] \rightrightarrows \mathbb{R}, F_{x,y}(t) = \{\langle z^*, y-x \rangle : z^* \in T(x+t(y-x))\}.$$

Since T is locally strict quasimonotone on D, obviously T is locally strict quasimonotone on $[x,y] \cap D$. On the other hand D is self-segment-dense in $D(T)$, hence $[x,y] \cap D$ is dense in $[x,y]$. Let

$$J = \{t \in [0,1] : x + t(y-x) \in [x,y] \cap D\}.$$

Obviously J is dense in $[0,1]$ and $F_{x,y}$ is locally strict quasimonotone on J and $F_{x,y}$ is lower semicontinuous on $[0,1] \setminus J$. On the other hand, from (1.5) we obtain, that $0 \notin F_{x,y}(s)$ for every $s \in [0,1] \setminus J$. Hence, according to Theorem 1.2.1, $F_{x,y}$ is strictly quasimonotone on

$[0,1]$. In particular, the latter shows, that for $t^* \in F_{x,y}(0)$ one has $t^* > 0$ then $s^* \geq 0$ for all $s^* \in F_{x,y}(1)$, or equivalently if $\langle x^*, y-x\rangle > 0$ for some $x^* \in T(x)$ then $\langle y^*, y-x\rangle \geq 0$ for all $y^* \in T(y)$.

For $x \in D(T) \setminus D$ and $y \in D$ assume that $\langle x^*, y-x\rangle > 0$. Since T is lower semicontinuous at x according to Remark 1.2.1, for every sequence $(x_n) \subseteq D(T)$ such that $x_n \longrightarrow x$ there exists a sequence of elements $x_n^* \in T(x_n)$ such that $x_n^* \longrightarrow x^*$. Let us fix $y^* \in T(y)$. Since D is dense in $D(T)$, one can consider $(x_n) \subseteq D$. Obviously $\langle x_n^*, y-x_n\rangle \longrightarrow \langle x^*, y-x\rangle > 0$, hence $\langle x_n^*, y-x_n\rangle > 0$ if n is big enough. According to the first part of the proof $\langle y^*, y-x_n\rangle \geq 0$. By taking the limit $n \longrightarrow \infty$ one obtains $\langle y^*, y-x\rangle \geq 0$.

Let now $x, y \in D(T) \setminus D$ arbitrary and assume that $\langle x^*, y-x\rangle > 0$. We show that $\langle y^*, y-x\rangle \geq 0$, for all $y^* \in T(y)$. Since T is lower semicontinuous at x and y, according to Remark 1.2.1, for every sequence $(x_n) \subseteq D(T)$ such that $x_n \longrightarrow x$ there exists a sequence of elements $x_n^* \in T(x_n)$ such that $x_n^* \longrightarrow x^*$, respectively for every sequence $(y_n) \subseteq D(T)$ such that $y_n \longrightarrow y$ and for every $y^* \in T(y)$ there exists a sequence of elements $y_n^* \in T(y_n)$ such that $y_n^* \longrightarrow y^*$. Let us fix $y^* \in T(y)$. Since D is dense in $D(T)$, one can consider $(x_n), (y_n) \subseteq D$. Obviously $\langle x_n^*, y_n - x_n\rangle \longrightarrow \langle x^*, y-x\rangle > 0$, hence $\langle x_n^*, y_n - x_n\rangle > 0$ if n is big enough. According to the first part of the proof $\langle y_n^*, y_n - x_n\rangle \geq 0$. By taking the limit $n \longrightarrow \infty$ one obtains $\langle y^*, y-x\rangle \geq 0$. □

According to the next corollary the previous result improves Theorem 3.6 from [133].

Corollary 1.2.1. *Let $T : D(T) \longrightarrow X^*$ be an operator with convex domain $D(T)$. Let $D \subseteq D(T)$ self-segment-dense in $D(T)$ and assume that T is locally strictly quasimonotone on D and continuous on $D(T) \setminus D$. Assume further, that the following condition holds:*

$$\forall x, y \in D(T),\, x \neq y,\, z \in [x,y] \cap (D(T) \setminus D) \text{ one has } \langle T(z), y-x\rangle \neq 0.$$

Then T is strictly quasimonotone.

1.2.5 Locally pseudomonotone and strictly pseudomonotone operators

In what follows we study the local pseudomonotonicity, respectively the local strict pseudomonotonicity of set-valued operators on self-segment-dense sets. First we prove the following result that ensures the pseudomonotonicity of a locally pseudomonotone set-valued real function on a dense subset.

Theorem 1.2.3. *Let $I \subseteq \mathbb{R}$ be an interval and consider the set-valued function $F : I \rightrightarrows \mathbb{R}$. Let $J \subseteq I$ dense in I and assume that F is locally pseudomonotone on J and lower semicontinuous on $I \setminus J$. Further, assume that $0 \notin F(x)$, for all $x \in I \setminus J$. Then F is pseudomonotone on I.*

Proof. We show that F is locally pseudomonotone on I. According to Lemma 1.2.1, for any $x \in I \setminus J$ one has $F(x) \geq 0$, or $F(x) \leq 0$. Note that according to the hypothesis of the theorem, for any $x \in I \setminus J$ one has $F(x) > 0$, or $F(x) < 0$.

Let $x \in I \setminus J$ and assume that $F(x) > 0$, the other case can be treated analogously. Obviously for $x^* \in F(x)$ one has $x^* > 0$. Let $V = (x^* - \varepsilon, x_1^* + \varepsilon)$, an open interval, where $\varepsilon > 0$ such that $x^* - \varepsilon > 0$. Since F is lower semicontinuous at x, there exists an open neighborhood U_x of x, such that for every $y \in U_x \cap I$ we have $F(y) \cap V \neq \emptyset$. But then, according to Lemma 1.2.1 and the hypothesis of the theorem $F(y) > 0$, for all $y \in U_x \cap I \setminus J$. We show that F is pseudomonotone on $U_x \cap I$. Suppose the contrary, that is, there exist $u, v \in U_x \cap I$ and $u^* \in T(u)$, $v^* \in T(v)$ such that

$$\langle u^*, v - u \rangle \geq 0 \text{ and } \langle v^*, v - u \rangle < 0.$$

One can assume that $v > u$. Hence, $v^* < 0$ which shows that $v \in J$. But then, there exists an open neighbourhood U_v of v such that $F|_{U_v \cap J}$ is pseudomonotone. Since J is dense in I, there exists $w \in U_x \cap I$ such that $w \in U_v \cap J$ and $w < v$. But then $F(w) \cap V \neq \emptyset$. Let $w^* \in F(w)$, $w^* > 0$.

Then $\langle w^*, v - w \rangle > 0$ and $\langle v^*, v - w \rangle < 0$ which contradicts the pseudomonotonicity of F on $U_v \cap J$. We have shown so far that F is locally pseudomonotone on $I \setminus J$.

Let now $x \in J$. In this case there exists an open neighborhood U_x of x such that $F|_{U_x \cap J}$ is pseudomonotone. We show that F is pseudomonotone on $U_x \cap I$. Suppose the contrary, that is, there exist $u, v \in U_x \cap I$, $u < v$ and $u^* \in T(u)$, $v^* \in T(v)$ such that

$$\langle u^*, v - u \rangle \geq 0 \text{ and } \langle v^*, v - u \rangle < 0.$$

We have $u^* \geq 0$, respectively $v^* < 0$ and observe that u or v must be in $I \setminus J$.

Assume that $u \in I \setminus J$. Then according to Lemma 1.2.1 and the hypothesis of the theorem $F(u) > 0$. Hence, by the lower semicontinuity of F at u and by the denseness of J in I, there exists $w \in (u, v) \cap U_x$, $w \in J$ such that $w^* > 0$ for some $w^* \in F(w)$. But in this case v is also in $I \setminus J$, otherwise we have $\langle w^*, v - w \rangle > 0$ and $\langle v^*, v - w \rangle < 0$ which contradicts the pseudomonotonicity of F on $U_x \cap J$.

Also in this case by the lower semicontinuity of F at v and by the denseness of J in I, there exists $z \in (w, v) \cap U_x$, $z \in J$ such that $z^* < 0$ for some $z^* \in F(z)$. But then, we have

$\langle w^*, z-w\rangle > 0$ and $\langle z^*, z-w\rangle < 0$ which contradicts the pseudomonotonicity of F on $U_x \cap J$.

Hence, u must belong to J. But then v belongs to $I \setminus J$ and by similar argument we obtain that there exists $z \in (u,v) \cap U_x$, $z \in J$ such that $z^* < 0$ for some $z^* \in F(z)$. But then, we have $\langle u^*, z-u\rangle > 0$ and $\langle z^*, z-u\rangle < 0$ which contradicts the pseudomonotonicity of F on $U_x \cap J$.

We have shown, that F is locally pseudomonotone on I. According to Proposition 1.2.2, F is globally pseudomonotone on I and the proof is completely done. □

Now we are ready to state and prove one of the main results of this section concerning on pseudomonotonicity of a set-valued operator, locally pseudomonotone on a self-segment-dense subset of its domain.

Theorem 1.2.4. *Let X be a real Banach space and let X^* be its topological dual. Let $T : X \rightrightarrows X^*$ be a set-valued operator with convex domain $D(T)$. Let $D \subseteq D(T)$ be self-segment-dense in $D(T)$ and assume that T is locally pseudomonotone on D as well that T is lower semicontinuous on $D(T) \setminus D$. Assume further, that the following condition holds:*

(1. 6) $\quad \forall x, y \in D(T), x \neq y, z \in [x,y] \cap (D(T) \setminus D)$ *and* $z^* \in T(z)$ *one has* $\langle z^*, y-x\rangle \neq 0$.

Then T is pseudomonotone on $D(T)$.

Proof. For $x, y \in D$ consider the real set-valued function

$$F_{x,y} : [0,1] \rightrightarrows \mathbb{R},\ F_{x,y}(t) = \{\langle z^*, y-x\rangle : z^* \in T(x+t(y-x))\}.$$

Obviously T is lower semicontinuous on $[x,y] \setminus D$. Since T is locally pseudomonotone on D, obviously T is locally pseudomonotone on $[x,y] \cap D$. On the other hand $[x,y] \cap D$ is dense in $[x,y]$. Let

$$J = \{t \in [0,1] : x + t(y-x) \in [x,y] \cap D\}.$$

Obviously J is dense in $[0,1]$ and $F_{x,y}$ is locally pseudomonotone on J and lower semicontinuous on $[0,1] \setminus J$. On the other hand, from (1. 6) we obtain, that $0 \notin F_{x,y}(s)$ for every $s \in [0,1] \setminus J$. Hence, according to Theorem 1.2.2, $F_{x,y}$ is pseudomonotone on $[0,1]$. In particular, the latter shows, that for $t^* \in F_{x,y}(0)$ one has $t^* \geq 0$ then $s^* \geq 0$ for all $s^* \in F_{x,y}(1)$, or equivalently if $\langle x^*, y-x\rangle \geq 0$ for some $x^* \in T(x)$ then $\langle y^*, y-x\rangle \geq 0$ for all $y^* \in T(y)$. Hence, T is pseudomonotone on D.

Let $x \in D(T) \setminus D$, $y \in D(T) \setminus D$ and assume that $\langle x^*, y-x\rangle > 0$ for some $x^* \in T(x)$. We show that $\langle y^*, y-x\rangle > 0$, for all $y^* \in T(y)$. Since T is lower semicontinuous at x and y, according to Remark 1.2.1, for every sequence $(x_n) \subseteq D(T)$ such that $x_n \longrightarrow x$ there exists a

sequence of elements $x_n^* \in T(x_n)$ such that $x_n^* \longrightarrow x^*$, respectively for every sequence $(y_n) \subseteq D(T)$ such that $y_n \longrightarrow y$ and for every $y^* \in T(y)$ there exists a sequence of elements $y_n^* \in T(y_n)$ such that $y_n^* \longrightarrow y^*$. Let us fix $y^* \in T(y)$. Since D is dense in $D(T)$, one can consider $(x_n),(y_n) \subseteq D$. Obviously $\langle x_n^*, y_n - x_n\rangle \longrightarrow \langle x^*, y-x\rangle > 0$, hence $\langle x_n^*, y_n - x_n\rangle > 0$ if n is big enough. According to the first part of the proof $\langle y_n^*, y_n - x_n\rangle > 0$. By taking the limit $n \longrightarrow \infty$ one obtains $\langle y^*, y-x\rangle \geq 0$. Since $y \in D(T)\setminus D$ then by the assumption of the theorem $\langle y^*, y-x\rangle \neq 0$, hence $\langle y^*, y-x\rangle > 0$. Hence, T is pseudomonotone on $D(T)\setminus D$.

It remained to show the pseudomonotonicity when $x \in D$ and $y \in D(T)\setminus D$. Assume that $\langle x^*, y-x\rangle > 0$ for some $x^* \in T(x)$. We show that $\langle y^*, y-x\rangle > 0$, for all $y^* \in T(y)$. Since $\langle x^*, y-x\rangle > 0$ in virtue of the continuity of duality pairing, there exists an open neighbourhood U_y of y such that $\langle x^*, z-x\rangle > 0$ for all $z \in V_y \cap D(T)$. Since D is dense in $D(T)$, there exists a sequence $(y_n) \subseteq V_y \cap D$ converging to y. Let us fix $y^* \in T(y)$. By the lower semicontinuity of T at y, there exists a sequence $y_n^* \in T(y_n)$ converging to y^*. Thus $\langle x^*, y_n - x\rangle > 0$, and according to the first part of the proof $\langle y_n^*, y_n - x\rangle > 0$. Letting $n \longrightarrow \infty$ we get $\langle y^*, y-x\rangle \geq 0$. But $y \in D(T)\setminus D$, hence according to the hypothesis of the theorem $\langle y^*, y-x\rangle \neq 0$.

□

In single-valued case we have the following corollary which improves Theorem 3.7 from [133].

Corollary 1.2.2. *Let $T : D(T) \longrightarrow X^*$ be an operator with convex domain $D(T)$. Let $D \subseteq D(T)$ be self-segment-dense in $D(T)$ and assume that T is locally pseudomonotone on D and continuous on $D(T)\setminus D$. Assume further, that the following condition holds:*

$$\forall x, y \in D(T),\, x \neq y,\, z \in [x,y] \cap (D(T)\setminus D) \text{ one has } \langle T(z), y-x\rangle \neq 0.$$

Then T is pseudomonotone.

We have some similar result to those established in Theorem 1.2.3 for locally strict pseudomonotone set-valued functions.

Theorem 1.2.5. *Let $I \subseteq \mathbb{R}$ be an interval and consider the set-valued function $F : I \rightrightarrows \mathbb{R}$. Let $J \subseteq I$ be dense in I and assume that F is locally strictly pseudomonotone on J and lower semicontinuous on $I\setminus J$. Further, assume that $0 \notin F(x)$, for all $x \in I\setminus J$. Then F is strictly pseudomonotone on I.*

Proof. We prove that F is locally strictly pseudomonotone on I. Let $x \in I$ and U_x a neighborhood of x, such that F is locally strict pseudomonotone on $U_x \cap J$. Then, according to

Theorem 1.2.2 F is pseudomonotone on $U_x \cap I$. Assume that there exist $u, v \in U_x \cap I$, $u \neq v$ and $u^* \in F(u)$, $v^* \in F(v)$ such that $\langle u^*, v-u\rangle \geq 0$ and $\langle v^*, v-u\rangle \leq 0$. Since F is pseudomonotone on $U_x \cap I$, we must have $\langle v^*, v-u\rangle = 0$. Hence $v^* = 0$ and due to the hypothesis of the theorem we obtain $v \in J$. But $\langle v^*, u-v\rangle = 0$ and by the pseudomonotonicity of F on $U_x \cap I$ we get $\langle u^*, u-v\rangle \geq 0$. Hence $u^* = 0$ and arguing as before $u \in J$.

But F is strictly pseudomonotone on $U_x \cap J$, hence

$$\langle u^*, v-u\rangle = 0 \Rightarrow \langle v^*, v-u\rangle > 0,$$

contradiction.

Since F is locally strict pseudomonotone on I according to Proposition 1.2.2 F is strict pseudomonotone on I. □

In infinite dimension we have the following result concerning on strict pseudomonotonicity of a set-valued operator, locally strictly pseudomonotone on a self-segment-dense subset of its domain..

Theorem 1.2.6. *Let X be a real Banach space and let X^* be its topological dual. Let $T : X \rightrightarrows X^*$ be an operator with convex domain $D(T)$. Let $D \subseteq D(T)$ be self-segment-dense in $D(T)$ and assume that T is locally strictly pseudomonotone on D and lower semicontinuous on $D(T) \setminus D$. Assume further, that the following condition holds:*

(1. 7) $\quad \forall x, y \in D(T), x \neq y, z \in [x,y] \cap (D(T) \setminus D)$ *and* $z^* \in T(z)$ *one has* $\langle z^*, y-x\rangle \neq 0$.

Then T is strictly pseudomonotone.

Proof. For $x, y \in D$ the proof is similar to the proof of Theorem 1.2.4. If $x \in D(T)$, $y \in D(T) \setminus D$ then by (1. 7) one has $\langle y^*, y-x\rangle \neq 0$, for all $y^* \in T(y)$, hence the implication

$$\langle x^*, y-x\rangle \geq 0 \Rightarrow \langle y^*, y-x\rangle > 0$$

can be obtained as in the proof of Theorem 1.2.4. □

In single-valued case we have the following corollary which improves Theorem 3.8 from [133].

Corollary 1.2.3. *Let $T : D(T) \longrightarrow X^*$ be an operator with convex domain $D(T)$. Let $D \subseteq D(T)$ be self-segment-dense in $D(T)$ and assume that T is locally strictly pseudomonotone*

on D and continuous on $D(T)\setminus D$. Assume further, that the following condition holds:

$$\forall x,y\in D(T),\, x\neq y,\, z\in[x,y]\cap(D(T)\setminus D) \text{ one has } \langle T(z),y-x\rangle\neq 0.$$

Then T is strictly pseudomonotone.

Remark 1.2.9. The condition

$$\forall x,y\in D(T),\, z\in[x,y]\cap(D(T)\setminus D) \text{ and } z^*\in T(z) \text{ one has } \langle z^*,y-x\rangle\neq 0$$

in the hypothesis of Theorem 1.2.2, Theorem 1.2.4, respectively Theorem 1.2.6, in particular the condition $0\notin F(x)$, for all $x\in I\setminus J$ in the hypothesis of Theorem 1.2.1, Theorem 1.2.3, respectively Theorem 1.2.5, is essential as the next example shows.

Example 1.2.6. Let

$$F:\mathbb{R}\rightrightarrows\mathbb{R},\, F(x):=\begin{cases}(0,-x], \text{ if } x<0,\\ 0, \text{ if } x=0,\\ [-x,0), \text{ if } x>0.\end{cases}$$

Then F is lower semicontinuous on $\mathbb{R}$ and locally strictly pseudomonotone on $\mathbb{R}\setminus\{0\}$, but F is not even quasimonotone on $\mathbb{R}$.

It is easy to check that F is locally strictly pseudomonotone on $D=\mathbb{R}\setminus\{0\}$ and lower semicontinuous on $D(F)=\mathbb{R}$. Obviously $\mathbb{R}\setminus\{0\}$ is self-segment-dense in $\mathbb{R}$, hence all the assumptions in the hypothesis of Theorem 1.2.2, Theorem 1.2.4, respectively Theorem 1.2.6 are satisfied excepting the one that

$$\forall x,y\in D(F),\, x\neq y,\, z\in[x,y]\cap(D(F)\setminus D) \text{ and } z^*\in F(z) \text{ one has } \langle z^*,y-x\rangle\neq 0.$$

Consequently, their conclusion fail.

Indeed, for $(x,x^*)=(-1,1)\in G(F)$ and $(y,y^*)=(1,-1)\in G(F)$ we have $\langle x^*,y-x\rangle=2>0$, and $\langle y^*,y-x\rangle=-2<0$, which shows that F is not quasimonotone on $\mathbb{R}$.

Remark 1.2.10. The condition T is lower semicontinuous on $D(T)\setminus D$ in the hypothesis of Theorem 1.2.2, Theorem 1.2.4, respectively Theorem 1.2.6, is also essential as the next example shows.

Example 1.2.7. Let

$$F:\mathbb{R}\rightrightarrows\mathbb{R},\, F(x):=\begin{cases}(0,1], \text{ if } x<0,\\ \{-1,1\}, \text{ if } x=0,\\ [-1,0), \text{ if } x>0.\end{cases}$$

Then F is not lower semicontinuous at 0, (though is lower semicontinuous on $\mathbb{R}\setminus\{0\}$) and F is locally strictly pseudomonotone on $\mathbb{R}\setminus\{0\}$ which is dense in $\mathbb{R}$, but F is not even quasimonotone on $\mathbb{R}$.

It is easy to check that F is locally strictly pseudomonotone on $D=\mathbb{R}\setminus\{0\}$. Obviously $\mathbb{R}\setminus\{0\}$ is self-segment-dense in $\mathbb{R}$. It is obvious that the condition

$$\forall x,y\in D(F),\, x\neq y,\, z\in[x,y]\cap(D(F)\setminus D) \text{ and } z^*\in F(z) \text{ one has } \langle z^*,y-x\rangle\neq 0,$$

is satisfied. Hence all the assumptions in the hypothesis of Theorem 1.2.2, Theorem 1.2.4, respectively Theorem 1.2.6 are satisfied excepting the one that F is lower semicontinuous on $D(T)\setminus D=\{0\}$. Consequently, their conclusion fail.

Indeed, for $(x,x^*)=(-1,1)\in G(F)$ and $(y,y^*)=(1,-1)\in G(F)$ we have $\langle x^*,y-x\rangle=2>0$, and $\langle y^*,y-x\rangle=-2<0$, which shows that F is not quasimonotone on $\mathbb{R}$.

1.2.6 Generalized convex functions on dense subsets

In this paragraph we apply the results obtained so far to prove the generalized convexity of a locally generalized convex function on a self-segment-dense subset of its domain. In the sequel X denotes a real Banach space and X^* denotes its topological dual.

In order to continue our analysis we need the following concepts (see, for instance [75, 187]). Let $f:X\longrightarrow\overline{\mathbb{R}}=\mathbb{R}\cup\{-\infty,+\infty\}$ be a given function, and let $x\in X$ such that $f(x)\in\mathbb{R}$. Recall that the Clarke-Rockafellar generalized derivative of f at x in direction v is defined by

$$f^{\uparrow}(x,v)=\sup_{\varepsilon>0}\limsup_{\substack{(y,\alpha)\downarrow_f x\\ t\downarrow 0}}\inf_{u\in B(v,\varepsilon)}\frac{f(y+tu)-\alpha}{t},$$

where $(y,\alpha)\downarrow_f x$ means $y\longrightarrow x$, $\alpha\longrightarrow f(x)$, $\alpha\geq f(y)$ and $B(v,\varepsilon)$ denotes the open ball with center v and radius ε.

If f is lower semicontinuous at x the Clarke-Rockafellar generalized derivative of f at x

in direction v (see [187]) can be expressed as

$$f^{\uparrow}(x,v) = \sup_{\varepsilon>0} \limsup_{\substack{y\downarrow_f x \\ t\downarrow 0}} \inf_{u\in B(v,\varepsilon)} \frac{f(y+tu)-f(y)}{t},$$

where $y \downarrow_f x$ means $y \longrightarrow x$, $f(y) \longrightarrow f(x)$.

When f is locally Lipschitz $f^{\uparrow}$ coincides with the Clarke directional derivative (see [75]), i.e.

$$f^{C}(x,v) = \limsup_{\substack{y\to x, u\to v \\ t\downarrow 0}} \frac{f(y+tu)-f(y)}{t}.$$

The Clarke-Rockafellar subdifferential of f at x is given by

$$\partial^{\uparrow} f(x) = \{x^* \in X^* : \langle x^*, v\rangle \leq f^{\uparrow}(x,v), \forall v \in X\}.$$

In what follows we present some convexity notions of a real valued function $f : X \longrightarrow \mathbb{R}\cup\{\infty\}$ (see, for instance [31, 32, 107, 180, 181]). Recall that the domain of f is the set $\operatorname{dom} f = \{x \in X : f(x) \in \mathbb{R}\}$. In the sequel we assume that $\operatorname{dom} f$ is a convex subset of X. Recall that the function f is called:

(1) *convex*, if for all $x, y \in X$, $t \in [0,1]$ one has the following

$$f(x+t(y-x)) \leq f(x)+t(f(y)-f(x))\},$$

(2) *pseudoconvex*, if for all $x, y \in \operatorname{dom} f$, the following implication holds

$$f(y) < f(x) \Rightarrow \forall x^* \in \partial^{\uparrow} f(x) : \langle x^*, y-x\rangle < 0,$$

(3) *strictly pseudoconvex*, if for all $x, y \in \operatorname{dom} f$, the following implication holds

$$f(y) \leq f(x) \Rightarrow \forall x^* \in \partial^{\uparrow} f(x) : \langle x^*, y-x\rangle < 0,$$

(4) *quasiconvex*, if for all $x, y \in \operatorname{dom} f$, $t \in [0,1]$ one has the following

$$f(x+t(y-x)) \leq \max\{f(x), f(y)\},$$

(5) *strictly quasiconvex*, if for all $x, y \in \operatorname{dom} f$, $x \neq y$, $t \in (0,1)$ one has the following

$$f(x+t(y-x)) < \max\{f(x), f(y)\}.$$

The study of connection between the (generalized) convexity property of a real valued function and appropriate monotonicity of its Clarke-Rockafellar subdifferential has a rich literature (see for instance [31, 32, 80, 82, 128, 179, 180] and the references therein). In what follows we assume that $f : X \longrightarrow \mathbb{R} \cup \{\infty\}$ is locally Lipschitz. The following implications hold.
f is convex $\Rightarrow$ f is pseudoconvex $\Rightarrow$ f is quasiconvex.
f is strictly pseudoconvex $\Rightarrow$ f is strictly quasiconvex $\Rightarrow$ f is quasiconvex.
f is strictly pseudoconvex $\Rightarrow$ f is pseudoconvex.

Remark 1.2.11. Observe that de definition of pseudoconvexity and strict pseudoconvexity respectively, is equivalent to the following: for all $x, y \in \operatorname{dom} f$,

$$\exists x^* \in \partial^{\uparrow} f(x) : \langle x^*, y-x \rangle \geq 0 \Rightarrow f(y) \geq f(x),$$

respectively

$$\exists x^* \in \partial^{\uparrow} f(x) : \langle x^*, y-x \rangle \geq 0 \Rightarrow f(y) > f(x),$$

The following statement relates the quasiconvexity of a function to the quasimonotonicity of its Clarke-Rockafellar subdifferential (see [32, 179, 180]):

Proposition 1.2.6 (Theorem 4.1 [32])**.** *A lower semicontinuous function $f : X \longrightarrow \mathbb{R} \cup \{\infty\}$ is quasiconvex, if and only if, $\partial^{\uparrow} f$ is quasimonotone.*

The following statement holds (see [80]):

Proposition 1.2.7 (Theorem 4.1 [80])**.** *A locally Lipschitz function $f : X \longrightarrow \mathbb{R} \cup \{\infty\}$ is strictly quasiconvex, if and only if, $\partial^{\uparrow} f$ is strictly quasimonotone.*

The next result is well-known, see for instance [181].

Proposition 1.2.8 (Theorem 4.1 [181])**.** *A lower semicontinuous, radially continuous function $f : X \longrightarrow \mathbb{R} \cup \{\infty\}$ is pseudoconvex, if and only if, $\partial^{\uparrow} f$ is pseudomonotone.*

A similar result concerning on the strict pseudomonotonicity of the subdifferential of a strictly pseudoconvex function was established in [181].

Proposition 1.2.9 (Theorem 5.1 [181]). *A locally Lipschitz function $f : X \longrightarrow \mathbb{R} \cup \{\infty\}$ is strictly pseudoconvex, if and only if, $\partial^{\uparrow} f$ is strictly pseudomonotone.*

Definition 1.2.5. A real valued function $f : X \longrightarrow \mathbb{R} \cup \{\infty\}$ is said to be locally convex, (respectively, locally quasiconvex, locally strictly quasiconvex, locally pseudoconvex, locally strictly pseudoconvex) if every $x \in X$ possesses an open and convex neighborhood U_x such that the restriction of f on U_x, $f|_{U_x}$ is convex, (respectively, quasiconvex, strictly quasiconvex, pseudoconvex, strictly pseudoconvex).

Remark 1.2.12. It is known, that local convexity and also local generalized convexity of a differentiable function in general implies its global counterpart (see [122, 133]). However, according to the next example local quasiconvexity does not implies global quasiconvexity.

Example 1.2.8. [Example 4.9 [133]] Consider the function $F : \mathbb{R} \longrightarrow \mathbb{R}$,

$$F(x) = \begin{cases} -\dfrac{x^2}{2} - x, \text{ if } x < -1, \\ \dfrac{1}{2}, \text{ if } x \in [-1, 1], \\ -\dfrac{x^2}{2} + x, \text{ if } x > 1. \end{cases}$$

Then it can easily be verified that F is continuously differentiable and locally quasiconvex but is not quasiconvex globally.

Assume now, that the set $D \subseteq \operatorname{dom} f$ is self-segment dense in $\operatorname{dom} f$. We define the local generalized convexity of f on D as follows.

We say that the function f is:

(1) *locally pseudoconvex* on D, if every $z \in D$ admits an open neighborhood U_z such that for all $x, y \in U_z \cap D$ the following implication holds

$$f(y) < f(x) \Rightarrow \forall x^* \in \partial^{\uparrow} f(x) : \langle x^*, y - x \rangle < 0,$$

(2) *locally strictly pseudoconvex* on D, if every $z \in D$ admits an open neighborhood U_z such that for all $x, y \in U_z \cap D$ the following implication holds

$$f(y) \leq f(x) \Rightarrow \forall x^* \in \partial^{\uparrow} f(x) : \langle x^*, y - x \rangle < 0,$$

(3) *locally quasiconvex* on D, if every $z \in D$ admits an open neighborhood U_z such that for all $x, y \in U_z \cap D$ and $t \in [0,1]$ such that $x + t(y-x) \in D$ one has the following

$$f(x+t(y-x)) \leq \max\{f(x), f(y)\},$$

(4) *locally strictly quasiconvex* on D, if every $z \in D$ admits an open neighborhood U_z such that for all $x, y \in U_z \cap D$ and $t \in (0,1)$ such that $x + t(y-x) \in D$ one has the following

$$f(x+t(y-x)) < \max\{f(x), f(y)\}.$$

The following lemma will be useful in the sequel.

Lemma 1.2.2. *Let $f : X \longrightarrow \mathbb{R} \cup \{\infty\}$ be lower semicontinuous with convex domain, and let $D \subseteq \operatorname{dom} f$ self-segment-dense in $\operatorname{dom} f$. Assume that f is locally strictly quasiconvex on D. Then $\partial^{\uparrow} f$ is locally quasimonotone on D. If f is also locally Lipschitz, then $\partial^{\uparrow} f$ is locally strictly quasimonotone on D.*

Proof. Let $z \in D$ and consider U_z an open and convex neighborhood of z such that for all $x, y \in U_z \cap D$ and $t \in (0,1)$ with $x + t(y-x) \in D$ we have $f(x+t(y-x)) < \max\{f(x), f(y)\}$. We show that $\partial^{\uparrow} f$ is quasimonotone on $U_z \cap D$.

Indeed, let $x, y \in U_z \cap D$ and assume that $\langle x^*, y-x \rangle > 0$ for some $x^* \in \partial^{\uparrow} f(x)$. Then $f^{\uparrow}(x, y-x) > 0$, hence there exists $\varepsilon > 0$ and the sequences $x_n \longrightarrow x$, $t_n \searrow 0$ such that

$$\inf_{v \in B(y-x,\varepsilon)} \frac{f(x_n + t_n v) - f(x_n)}{t_n} > 0.$$

Since D is self-segment-dense in $\operatorname{dom} f$ we can assume that $(x_n), (x_n + t_n(y-x_n)) \subseteq D$. If n is big enough then $\|x_n - x\| < \varepsilon$, hence $y - x_n \in B(y-x, \varepsilon)$. Thus, $f(x_n + t_n(y-x_n) > f(x_n)$ and in virtue of locally strict quasiconvexity of f on D we get $f(x_n) < f(y)$. Using the lower semicontinuity of f we have

$$f(x) \leq \liminf_{n \longrightarrow \infty} f(x_n),$$

thus $f(x) \leq f(y)$.

Suppose that there exists $y^* \in \partial^{\uparrow} f(y)$ such that $\langle y^*, y-x \rangle < 0$. Then $\langle y^*, x-y \rangle > 0$ which leads to $f^{\uparrow}(y, x-y) > 0$ and using the same arguments as before, we conclude that $f(y) \leq f(x)$. Hence, we have $f(x) = f(y)$.

Since $\langle x^*, y-x \rangle > 0$, by the continuity property of the duality pairing we obtain that there exists an open neighborhood V of y such that $\langle x^*, y_1 - x \rangle > 0$ for all $y_1 \in V$. Let $y_1 \in$

$V \cap (x,y) \cap D$. Obviously $y_1 \in U_z$. Using the same argument as in te first part of the proof, we conclude that $f(x) \leq f(y_1)$. But then $f(y_1) \geq \max\{f(x), f(y)\}$ which contradicts the strict quasiconvexity of f on U_z.

We proved that $\partial^{\uparrow} f$ is locally quasimonotne on D. It remained to show, in the case when f is locally Lipschitz, that for all $x, y \in U_z \cap D$ there exists $u \in (x,y) \cap D$ such that $\langle u^*, y-x \rangle \neq 0$ for some $u^* \in \partial^{\uparrow} f(u)$. Let $x, y \in U_z \cap D$. Assume that $f(x) \neq f(y)$. This can be assumed since, otherwise, in virtue of locally strict quasiconvexity of f on D one can take $y' \in (x,y) \cap D$ such that $f(y') < \max\{f(x), f(y)\} \Rightarrow f(y') \neq f(x)$. Assume that $f(y) - f(x) > 0$, the case $f(y) - f(x) < 0$ can be treated similarly.

According to Lebourg mean value theorem (see [75]), there exists $u \in (x,y)$ and $u^* \in \partial^{\uparrow} f(u)$ such that $\langle u^*, y-x \rangle = f(y) - f(x) > 0$. □

In what follows we provide, in a Banach space context, sufficient conditions for strict quasiconvexity of a locally strictly quasiconvex functions.

Theorem 1.2.7. *Let $f : X \longrightarrow \mathbb{R} \cup \{\infty\}$ be a locally Lipschitz function, locally strictly quasiconvex on D, where $D \subseteq \operatorname{dom} f$ is self-segment-dense in $\operatorname{dom} f$. If $\partial^{\uparrow} f$ is lower semicontinuous on $\operatorname{dom} f \setminus D$ and has the property, that $\langle z^*, x-y \rangle \neq 0$ for all $z \in [x,y] \cap \operatorname{dom} f \setminus D$, $x, y \in \operatorname{dom} f$, $x \neq y$, $z^* \in \partial^{\uparrow} f(z)$ then f is globally strictly quasiconvex on X.*

Proof. According to Lemma 1.2.2, $\partial^{\uparrow} f$ is locally strictly quasimonotone on D. According to Theorem 1.2.2, $\partial^{\uparrow} f$ is strictly quasimonotone on $\operatorname{dom} f$. The conclusion follows from Proposition 1.2.7. □

Similar results to Theorem 1.2.7 hold for locally pseudoconvex, respectively locally strict pseudoconvex functions.

Theorem 1.2.8. *Let $f : X \longrightarrow \mathbb{R} \cup \{\infty\}$ be a locally Lipschitz function, locally pseudoconvex on D, where $D \subseteq X$ is self-segment-dense in $\operatorname{dom} f$. If $\partial^{\uparrow} f$ is lower semicontinuous on $\operatorname{dom} f \setminus D$ and has the property, that $\langle z^*, x-y \rangle \neq 0$ for all $z \in [x,y] \cap X \setminus D$, $x, y \in \operatorname{dom} f$, $x \neq y$, $z^* \in \partial^{\uparrow} f(z)$ then f is globally pseudoconvex on X.*

Proof. Let $z \in D$ and consider U_z an open and convex neighborhood of z such that for all $x, y \in U_z \cap D$ we have $f(y) < f(x) \Rightarrow \forall x^* \in \partial^{\uparrow} f(x) : \langle x^*, y-x \rangle < 0$. We show that $\partial^{\uparrow} f$ is pseudomonotone on $U_z \cap D$.

First of all, observe that f is quasiconvex on $U_z \cap D$. Indeed, let $x, y \in U_z \cap D$ and assume that $f(u) > \max\{f(x), f(y)\}$ for some $u \in (x,y) \cap D$. But then $u = x + t(y-x)$ for some $t \in (0,1)$ and by the pseudoconvexity of f on $U_z \cap D$ we obtain that $\forall u^* \in \partial^{\uparrow} f(u)$ we have

$\langle u^*, x-u\rangle < 0$ and $\langle u^*, y-u\rangle < 0$ or, equivalently $\langle u^*, -t(y-x)\rangle < 0$ and $\langle u^*, (1-t)(y-x)\rangle < 0$, impossible.

Assume $\partial^{\uparrow} f$ is not pseudomonotone on $U_z \cap D$ that is, there exists $x, y \in U_z \cap D$ such that $\langle x^*, y-x\rangle > 0$ for some $x^* \in \partial^{\uparrow} f(x)$ and $\langle y^*, y-x\rangle \leq 0$ for some $y^* \in \partial^{\uparrow} f(y)$. According to Remark 1.2.11 $\langle x^*, y-x\rangle > 0 \Rightarrow f(y) \geq f(x)$, and $\langle y^*, y-x\rangle \leq 0 \Rightarrow f(y) \leq f(x)$, hence $f(x) = f(y)$.

Since $\langle x^*, y-x\rangle > 0$ we have $f^{\uparrow}(x, y-x) > 0$, hence there exists $\varepsilon > 0$ and the sequences $x_n \longrightarrow x$, $t_n \searrow 0$ such that

$$\inf_{v \in B(y-x,\varepsilon)} \frac{f(x_n + t_n v) - f(x_n)}{t_n} > 0.$$

Since D is self-segment-dense in $\operatorname{dom} f$ we can assume that $(x_n), (x_n + t_n(y-x_n)) \subseteq D$. If n is big enough then $\|x_n - x\| < \frac{\varepsilon}{2}$, hence $y_1 - x_n \in B(y-x, \varepsilon)$ if $y_1 \in B\left(y, \frac{\varepsilon}{2}\right)$. Thus, $f(x_n + t_n(y_1 - x_n) > f(x_n)$ and in virtue of local quasiconvexity of f on D we get $f(x_n) < f(y_1)$ for all $y_1 \in B\left(y, \frac{\varepsilon}{2}\right) \cap D$. The latter relation combined with the pseudoconvexity of f on $U_z \cap D$ in particular shows that $0 \notin \partial^{\uparrow} f(y)$.

Using the continuity of f we have

$$f(y) = f(x) = \lim_{n \longrightarrow \infty} f(x_n) \leq f(y_1),$$

Hence, $f(y) \leq f(y_1)$ for all $y_1 \in B\left(y, \frac{\varepsilon}{2}\right) \cap D$ which shows that y is a local minimum on $U_z \cap D$. We show that y is a minimum on $B\left(y, \frac{\varepsilon}{2}\right) \cap \operatorname{dom} f$. Indeed, let $u \in B\left(y, \frac{\varepsilon}{2}\right) \cap \operatorname{dom} f$. Since D is dense in $\operatorname{dom} f$, there exists a sequence $u_n \in B\left(y, \frac{\varepsilon}{2}\right) \cap D$, $u_n \longrightarrow u$. Obviously $f(u_n) \geq f(y)$, and since f is continuous we have $f(u) = \lim_{n \longrightarrow \infty} f(u_n) \geq f(y)$. Hence y is a local minimum on $U_z \cap \operatorname{dom} f$, which implies $0 \in \partial^{\uparrow} f(y)$, contradiction.

Thus, $\partial^{\uparrow} f$ is locally pseudomonotone on D. According to Theorem 1.2.3, $\partial^{\uparrow} f$ is pseudomonotone on $\operatorname{dom} f$. In virtue of Proposition 1.2.8, f is pseudoconvex. □

Next we establish some conditions that ensure that a locally strictly pseudoconvex function on a self-segment-dense subset in its domain is strictly pseudoconvex.

Theorem 1.2.9. *Let $f : X \longrightarrow \mathbb{R} \cup \{\infty\}$ be a locally Lipschitz function, locally strictly pseudoconvex on D, where $D \subseteq X$ is self-segment-dense in* $\operatorname{dom} f$. *If $\partial^{\uparrow} f$ is lower semicontinuous on* $\operatorname{dom} f \setminus D$ *and has the property, that $\langle z^*, x-y\rangle \neq 0$ for all $z \in [x,y] \cap X \setminus D$, $x, y \in \operatorname{dom} f$, $x \neq y$, $z^* \in \partial^{\uparrow} f(z)$ then f is globally strictly pseudoconvex on X.*

Proof. Let $z \in D$ and consider U_z an open and convex neighborhood of z such that for all

$x, y \in U_z \cap D$ we have $f(y) \leq f(x) \Rightarrow \forall x^* \in \partial^{\uparrow} f(x) : \langle x^*, y-x \rangle < 0$. We show that $\partial^{\uparrow} f$ is strictly pseudomonotone on $U_z \cap D$.

Suppose the contrary, that is there exist $x, y \in U_z \cap D$ such that $\langle x^*, y-x \rangle \geq 0$ for some $x^* \in \partial^{\uparrow} f(x)$ and $\langle y^*, y-x \rangle \leq 0$ for some $y^* \in \partial^{\uparrow} f(y)$.

According to Remark 1.2.11

$$\langle x^*, y-x \rangle \geq 0 \Rightarrow f(y) > f(x),$$

while

$$\langle y^*, y-x \rangle \leq 0 \Rightarrow f(x) > f(y),$$

impossible.

Since $\partial^{\uparrow} f$ is strictly pseudomonotone on D, according to Theorem 1.2.4, $\partial^{\uparrow} f$ is strictly pseudomonotone on $\operatorname{dom} f$. In virtue of Proposition 1.2.9, f is strictly pseudoconvex. □

Remark 1.2.13. The assumptions imposed on $\partial^{\uparrow} f$ in the hypothesis of the previous theorems cannot be dropped as the next example shows.

Example 1.2.9. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $f(x) = -x - |y|$. Then f is locally strictly pseudoconvex on $D = \mathbb{R}^2 \setminus \mathbb{R} \times \{0\}$. We show that $\partial^{\uparrow} f$ is not lower semicontinuous at the points of $\mathbb{R} \times \{0\}$ and and that f is not even quasiconvex on $\mathbb{R}^2$.

Indeed, it can easily be verified that

$$\partial^{\uparrow} f(x,y) = \begin{cases} (-1,1), \text{ if } y < 0, \\ \{-1\} \times [-1,1], \text{ if } y = 0, \\ (-1,-1), \text{ if } y > 0. \end{cases}$$

Obviously $\partial^{\uparrow} f$ is not lower semicontinuous on $\mathbb{R}^2 \setminus D = \mathbb{R} \times \{0\}$. Let $u = (1,-1)\, v = (1,1)$. Then for $w = (1,0) \in (u,v)$ one has $f(w) = -1 > \max\{f(u), f(v)\} = -2$, which shows that f is not quasiconvex.

1.3 On injectivity of a class of monotone operators

In this section we deal with operators which are monotone relative to another operator. We obtain some sufficient (analytical) conditions that ensure this monotonicity property. We provide conditions that assure the convexity of inverse images for an operator having this monotonicity property. We also show that operators having this monotonicity property are injective

under some circumstances. By combining the mentioned results we obtain some analytical conditions that ensure injectivity of an operator. We also extend the well-known injectivity result expressed in terms of positive definiteness of the symmetric part of all Frèchet differentials of operators of class C^1 in finite dimensional spaces. Further, we obtain some new conditions that ensure the injectivity and the univalency of differentiable and holomorphic complex functions, respectively, of one complex variable.

Let us mention that a part of the results from this section has been published in [141]:[S. László, *On injectivity of a class of monotone operators with some univalency consequences*, Mediterranean J. Math. **13(2)**, 729-744 (2016)].

1.3.1 On convexity of preimages of a monotone operator

In this paragraph we provide some conditions that assure the convexity of the inverse images of an operator having some monotonicity property. In what follows, unless is otherwise specified, X denotes a real Banach space and X^* denotes its topological dual. Let $D \subseteq X$ and consider the operator $S : D \longrightarrow X^*$. Recall that S is called monotone (in Minty-Browder sense), if for all $x, y \in D$ one has $\langle S(x) - S(y), x - y\rangle \geq 0$, where $\langle x^*, x\rangle$ denotes the duality pairing, that is the value of the linear and continuous functional $x^* \in X^*$ at $x \in X$. S is called strictly monotone if S is monotone and $\langle S(x) - S(y), x - y\rangle = 0$ if and only if $x = y$. Let us denote the preimage (or inverse image) of $x^* \in X^*$ through S by $S^{-1}(x^*)$, that is $S^{-1}(x^*) = \{x \in D : S(x) = x^*\}$. Obviously S is injective if, and only if $S^{-1}(x^*)$ contains at most one point for all $x^* \in X^*$. Note that a strictly monotone operator is injective. Recall that a monotone operator is called maximal monotone if its graph is not included in the graph of any other monotone operator. It is well known that maximal monotone operators possess convex inverse images. Moreover, in a Hilbert space context, the monotonicity property of a continuous operator ensures the convexity of the inverse images of that operator (see [124]). More precisely the following holds.

Proposition 1.3.1. *[Theorem 3.5, [124]] If $(H, \langle\cdot,\cdot\rangle)$ is a Hilbert space, $D \subseteq H$ is a convex open set and $S : D \longrightarrow H$ is a continuous Minty-Browder monotone operator, then S has convex inverse images.*

Remark 1.3.1. Let us emphasize that in the view of the previous proposition, in a Hilbert space context, a locally injective, continuous, monotone operator with a convex and open domain is actually globally injective. Indeed, the operator has convex inverse images, which by the local injectivity of the operator contain at most one point.

It is well known, that in finite dimensional case, a sufficient condition that provides local injectivity of a continuously differentiable operator is the nonsingularity of its Jacobi matrix. However, note that local injectivity in general does not imply global injectivity. For instance the operator $f : \mathbb{C} \longrightarrow \mathbb{C}$, $f(z) = e^z$ is locally injective, having $f'(z) = e^z \neq 0$ for all $z \in \mathbb{C}$, but is not globally injective, being periodic with period $2\pi i$.

Definition 1.3.1. Let $D \subseteq X$ and consider the operators $S : D \longrightarrow X^*$ and $A : X \longrightarrow X$. We say that the operator S is monotone relative to A (or shortly $A-$monotone), if for all $x, y \in D$ one has

$$\langle S(x) - S(y), A(x) - A(y) \rangle \geq 0. \tag{1. 8}$$

S is called strictly monotone relative to A if in (1. 8) equality holds only for $A(x) = A(y)$.

Observe that when A is the identity operator, that is $A(x) = x$ for all $x \in X$, then the concept of $A-$monotonicity of S collapse into the concept of classical monotonicity of S.

In what follows we will extend and improve the result obtained in Proposition 1.3.1. More precisely we give a condition in a Banach space context, that provides the convexity of the preimages of an $A-$monotone operator without assuming the openness of its domain, or the continuity of the operator.

For $x, y \in X$ let us denote by (x, y) the open line segment with the endpoints x, respectively y, i.e.

$$(x, y) = \{x + t(y - x) : 0 < t < 1\}.$$

The next result provides sufficient conditions for the convexity of inverse images of an $A-$monotone operator.

Theorem 1.3.1. *Let D be a convex subset of X. Let $A : X \longrightarrow X$ be a linear operator and let $S : D \longrightarrow X^*$ be an $A-$monotone operator. Assume that for all $x, y \in D$, $x \neq y$ such that $S(x) = S(y)$ and for all $z \in (x, y)$, there exists $w \in X$ with $z + w \in D$ such that*

$$\langle S(x) - S(z + w), A(w) \rangle > 0. \tag{1. 9}$$

Then S has convex preimages.

Proof. Let $u \in X^*$ and assume that $S^{-1}(u)$ is not convex. Obviously in this case $S^{-1}(u) \neq \emptyset$. Let $x, y \in S^{-1}(u)$ and $z \in (x, y)$ such that $S(z) \neq u$. According to the hypothesis of theorem, there exists $w \in H$ with $z + w \in D$ such that

$$\langle S(x) - S(z + w), A(w) \rangle > 0.$$

From the $A-$monotonicity of S we obtain

$$\langle S(x)-S(z+w),A(x)-A(z+w)\rangle \geq 0$$

and

$$\langle S(y)-S(z+w),A(y)-A(z+w)\rangle \geq 0.$$

Taking into account that $S(y)=S(x)$, A is a linear operator and $z=x+t_0(y-x)$ for some $t_0\in(0,1)$, we obtain

$$\langle S(x)-S(z+w),A(x-y)\rangle \geq \frac{1}{t_0}(\langle S(x)-S(z+w),A(w)\rangle,$$

respectively

$$\langle S(x)-S(z+w),A(y-x)\rangle \geq \frac{1}{1-t_0}(\langle S(x)-S(z+w),A(w)\rangle.$$

By adding the above inequalities and using the linearity of A and (1. 9) we obtain

$$0\geq\left(\frac{1}{t_0}+\frac{1}{1-t_0}\right)\langle S(x)-S(z+w),A(w)\rangle>0,$$

contradiction. □

As an immediate consequence we have the following result.

Corollary 1.3.1. *Let D be a convex subset of X. Let $S:D\longrightarrow X^*$ be a monotone operator, and assume that for all $x,y\in D$, $x\neq y$ such that $S(x)=S(y)$ and for all $z\in(x,y)$, there exists $w\in X$ with $z+w\in D$ such that*

$$\langle S(x)-S(z+w),w\rangle>0.$$

Then S has convex preimages.

Remark 1.3.2. However, condition (1. 9) in the hypothesis of Theorem 1.3.1 seems hard to be verified. Fortunately, by using the same technique we are able to provide the convexity of the preimages of a hemicontinuous $A-$monotone operator.

Let us recall the following continuity concept:

Definition 1.3.2. We say that S is hemicontinuous at $x\in D$, if for all $(t_n)_{n\in\mathbb{N}}\subset\mathbb{R}$, $t_n\longrightarrow 0$, $n\longrightarrow\infty$ and $y\in X$, such that $x+t_ny\in D$, we have $S(x+t_ny)\rightharpoonup^* S(x)$, $n\longrightarrow\infty$, where " $\rightharpoonup^*$ "

denotes the convergence with respect to the *weak** topology of X^*. We say that S is hemicontinuous on D if it has this property at every $x \in D$.

Theorem 1.3.2. *Let D be an open and convex subset of X. Let $A : X \longrightarrow X$ be a surjective linear operator, i.e. $A(X) = X$ and let $S : D \longrightarrow X^*$ be a hemicontinuous $A-$monotone operator. Then S has convex preimages.*

Proof. Let $u \in X^*$ and assume that $S^{-1}(u)$ is not convex. Obviously we can take $S^{-1}(u) \neq \emptyset$. Then there exist $x, y \in S^{-1}(u)$ and $z \in (x, y)$ such that $S(z) \neq u$. But then $S(x) \neq S(z)$, hence in virtue of surjectivity of A there exists $v \in X$ such that $\langle S(x) - S(z), A(v)\rangle > 0$. Since D is open and $z \in D$ obviously there exists $\varepsilon_0 > 0$ such that for all $t \in [0, \varepsilon_0]$ we have $z + tv \in D$.

Consider the function $\varphi : [0, \varepsilon_0] \longrightarrow \mathbb{R}$, $\varphi(t) = \langle S(x) - S(z+tv), A(v)\rangle$. In virtue of hemicontinuity of S and the continuity of the duality pairing φ is continuous at 0, further $\varphi(0) > 0$, hence there exists $\varepsilon > 0$ such that $\varphi(t) > 0$ for all $t \in [0, \varepsilon)$. Let $t_0 \in (0, \varepsilon)$ and let $w = t_0 v$. Then

$$\langle S(x) - S(z+w), A(w)\rangle > 0. \tag{1. 10}$$

Sice S is monotone relative to A we have $\langle S(x) - S(z+w), A(x) - A(z+w)\rangle \geq 0$, respectively $\langle S(y) - S(z+w), A(y) - A(z+w)\rangle \geq 0$. Taking into account that $S(y) = S(x)$, A is a linear operator and $z = x + t_1(y - x)$ for some $t_1 \in (0, 1)$, we obtain $\langle S(x) - S(z+w), A(x - y)\rangle \geq \frac{1}{t_1}(\langle S(x) - S(z+w), A(w)\rangle$, respectively $\langle S(x) - S(z+w), A(y-x)\rangle \geq \frac{1}{1-t_1}(\langle S(x) - S(z+w), A(w)\rangle$. By adding the above inequalities and using the linearity of A and (1. 10) we obtain

$$0 \geq \left(\frac{1}{t_1} + \frac{1}{1-t_1}\right) \langle S(x) - S(z+w), A(w)\rangle > 0,$$

contradiction. □

As an immediate consequence we have the following result, an improvement of Proposition 1.3.1.

Corollary 1.3.2. *Let D be an open and convex subset of X and let $S : D \longrightarrow X^*$ be a hemicontinuous monotone operator. Then S has convex preimages.*

Remark 1.3.3. One may ask whether we can obtain some similar conditions to those stated in Theorem 1.3.1 that assure the convexity of the preimages of an operator having a known generalized monotonicity property. We emphasize in the following that formula 1. 9 can be extend to $\theta-$monotone operators, which is one of the most general monotonicity concept in the literature (see [135]).

Let $\theta : X \times X \longrightarrow \mathbb{R}$ be a function with the property that $\theta(x,y) = \theta(y,x)$ for all $x,y \in X$. Recall that the operator $S : D \subseteq X \longrightarrow X^*$ is called $\theta-$monotone if

$$\langle S(x) - S(y), x - y\rangle \geq \theta(x,y)\|x-y\| \text{ for all } x,y \in D.$$

According to [135], the concept of $\theta-$monotonicity generalizes several monotonicity concepts known in literature, such as the concept of Minty-Browder monotonicity ($\theta(x,y) = 0$ for all $x,y \in D$, see [69, 70, 157, 158]), the concept of strong monotonicity ($\theta(x,y) = r\|x-y\|$ for all $x,y \in D$ where $r \in \mathbb{R}_+ \setminus \{0\}$, see [204]), the concept of uniform monotonicity ($\theta(x,y) = f(\|x-y\|)$ for all $x,y \in D, x \neq y$ where $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is an increasing function, with $\lim\limits_{t\downarrow 0} f(t) = 0$, $\lim\limits_{t\to\infty} f(t) = \infty$, see [120]), the concept of $\varepsilon-$monotonicity ($\theta(x,y) = -\varepsilon$, $\varepsilon \in \mathbb{R}_+ \setminus \{0\}$ for all $x,y \in D$, see [116, 151]), the concept of m-relaxed monotonicity ($\theta(x,y) = -m\|x-y\|$, $m \in \mathbb{R}_+ \setminus \{0\}$ for all $x,y \in D$, see [204]), the concept of $\gamma-$paramonotonicity ($\theta(x,y) = -C\|x-y\|^{\gamma-1}$, $C \in \mathbb{R}_+ \setminus \{0\}$, $\gamma \in \mathbb{R}$, $\gamma > 1$ for all $x,y \in D$, see [118]), or the concept of premonotonicity ($\theta(x,y) = -\min\{\sigma(x), \sigma(y)\}$, for all $x,y \in D$ where $\sigma : X \longrightarrow \mathbb{R}_+ \setminus \{0\}$ is a given function, see [114]). Similar to the proof of Theorem 1.3.1 one may prove the following.

If $S : D \subseteq X \longrightarrow X^*$ is a $\theta-$monotone operator and we assume that for all $x,y \in D, x \neq y$ such that $S(x) = S(y)$ and for all $z = x + t_0(y-x) \in (x,y)$, $t_0 \in (0,1)$ there exists $w \in X$ with $z + w \in D$ such that

$$\langle S(x) - S(z+w), w\rangle + (1-t_0)\theta(x, z+w)\|z+w-x\| + t_0\theta(y, z+w)\|z+w-y\| > 0,$$

then S has convex preimages.

1.3.2 Analytical conditions for monotonicity and injectivity

In this paragraph we provide some analytical conditions that assure the $A-$monotonicity, respectively strict $A-$monotonicity of an operator. Moreover, we obtain some conditions that ensure the global injectivity of an operator. We pay some special attention to the finite dimensional case and we provide some formulas involving the Jacobi matrix of the operator that assure global injectivity.

Let H be a real Hilbert space identified with its topological dual and let $D \subseteq H$ be an open set. For a differentiable operator $S : D \longrightarrow H$, we denote by $dS_x(\cdot)$ its Frèchet differential at $x \in D$. In what follows we provide an analytical condition that ensures the monotonicity of an operator relative to another operator.

Proposition 1.3.2. *Let $D \subseteq H$ be an open and convex set, let $S : D \longrightarrow H$ be an operator of class C^1 and let $A : H \longrightarrow H$ be an operator. Assume that for all $x, y \in D$, with $A(x) \neq A(y)$ and $z \in (x, y)$ one has*

$$\langle dS_z(y-x), A(y)-A(x)\rangle > 0.$$

Then S is strictly monotone relative to A.

Proof. Let $x, y \in D$ such that $A(x) \neq A(y)$. We show that $\langle S(y)-S(x), A(y)-A(x)\rangle > 0$.

Consider the real function $\phi : [0,1] \longrightarrow \mathbb{R}$, $\phi(t) = \langle S(x+t(y-x)), A(y)-A(x)\rangle$. Then ϕ is continuous on $[0,1]$ and differentiable on $(0,1)$, hence according to mean value theorem, there exists $c \in (0,1)$ such that $\phi'(c) = \phi(1)-\phi(0)$. Equivalently, we can state

$$\langle dS_{x+c(y-x)}(y-x), A(y)-A(x)\rangle = \langle S(y)-S(x), A(y)-A(x)\rangle.$$

On the other hand $c \in (0,1)$ implies $x+c(y-x) \in (x,y)$ and by the hypothesis of theorem we have

$$\langle dS_{x+c(y-x)}(y-x), A(y)-A(x)\rangle > 0,$$

which shows that

$$\langle S(y)-S(x), A(y)-A(x)\rangle > 0.$$

□

Remark 1.3.4. It can analogously be proved that the condition $\langle dS_z(y-x), A(y)-A(x)\rangle \geq 0$ for all $x, y \in D$ and $z \in (x,y)$, ensures that S is monotone relative to A. Note that the condition $\langle dS_z(y-x), A(y)-A(x)\rangle \leq (<) 0$ for all $x, y \in D$, $x \neq y$ and $z \in (x,y)$ ensures that S is (strictly) monotone relative to $-A$.

For a linear operator $A : H \longrightarrow H$ we denote by $\ker A$ the set of zeroes of A, that is

$$\ker A = \{x \in H : A(x) = 0\}.$$

Corollary 1.3.3. *Let $D \subseteq H$ be an open and convex set, let $S : D \longrightarrow H$ be an operator of class C^1 and let $A : H \longrightarrow H$ be a linear operator. Assume that for all $x \in D$ and $y \in H \setminus \ker A$ one has*

$$\langle dS_x(y), A(y)\rangle > 0.$$

Then S is strictly monotone relative to A.

Proof. Indeed, let $u, v \in D$, with $A(u) \neq A(v)$. Take $y = v-u$ and $x = w \in (u,v)$. Since $A(u) \neq A(v)$ we have $A(y) \neq 0$, hence $y \in D \setminus \ker A$.

Consequently, the condition $\langle dS_x(y), A(y)\rangle > 0$ for all $y \in H \setminus \ker A$ becomes

$$\langle dS_w(v-u), A(v)-A(u)\rangle > 0, \forall u, v \in D, \text{ with } A(u) \neq A(v).$$

The conclusion follows from Proposition 1.3.2. □

Remark 1.3.5. If we assume that for all $x \in D$ and $y \in H \setminus \ker A$ one has $\langle dS_x(y), A(y)\rangle < 0$, we obtain that S is strictly monotone relative to $-A$.

Next we provide some conditions that ensure the injectivity of an operator which is monotone relative to an operator A.

Proposition 1.3.3. *An operator $S : D \subseteq H \longrightarrow H$ which is strictly monotone relative to $A : H \longrightarrow H$, is injective on $D \setminus \{x \in D : \exists y \in D, x \neq y, A(x) = A(y)\}$. If A is injective on D, that is, for all $x, y \in D$, $x \neq y$ one has $A(x) \neq A(y)$, then S is also injective on its whole domain.*

Proof. Indeed, for $u, v \in D \setminus \{x \in D : \exists y \in D, x \neq y, A(x) = A(y)\}$, $u \neq v$ one has $A(u) \neq A(v)$, hence

$$\langle S(u) - S(v), A(u) - A(v)\rangle > 0.$$

The latter relation shows that $S(u) \neq S(v)$.

If A is injective on D, then $\{x \in D : \exists y \in D, x \neq y, A(x) = A(y)\} = \emptyset$, hence S is injective on D. □

Combining the results obtained so far we get the following.

Theorem 1.3.3. *Let $D \subseteq H$ be an open and convex set, let $S : D \longrightarrow H$ be an operator of class C^1 and let $A : H \longrightarrow H$ be an operator injective on D. Assume that one of the following conditions hold.*

(a) For all $x, y \in D$ with $A(x) \neq A(y)$ and $z \in (x, y)$ one has

$$\langle dS_z(y-x), A(y) - A(x)\rangle > 0.$$

(b) A is linear and for all $x \in D$ and $y \in H \setminus \ker A$ one has

$$\langle dS_x(y), A(y)\rangle > 0.$$

Then S is injective.

Proof. The conclusion follows from Proposition 1.3.2 and Proposition 1.3.3, respectively from Corollary 1.3.3 and Proposition 1.3.3. □

Remark 1.3.6. Since A is injective on D if and only if $-A$ is injective on D, according to Remark 1.3.4, respectively Remark 1.3.5, the conditions

(a) For all $x, y \in D$, $A(x) \neq A(y)$ and $z \in (x, y)$ one has

$$\langle dS_z(y-x), A(y)-A(x)\rangle < 0,$$

respectively

(b) A is linear and for all $x \in D$ and $y \in H \setminus \ker A$ one has

$$\langle dS_x(y), A(y)\rangle < 0,$$

also assure the injectivity of S.

Remark 1.3.7. Note that Theorem 1.3.3 (b) extends the classical injectivity result of Gale and Nikaido [97], which follows by taking in the hypothesis of Theorem 1.3.3 (b), $H = \mathbb{R}^n$ and A the identity matrix.

Consider now $H = \mathbb{R}^n$ endowed with the usual euclidian scalar product, let $D \subseteq H$ be open and let $S : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a Frèchet differentiable operator. For $x^0 = (x_1^0, x_2^0, \ldots, x_n^0) \in D$ we denote by $J_S(x^0)$ the Jacobian matrix of S in x^0. Recall that S is called locally ijective, if every $x \in D$ posessess a neighbourhood $V_x \subseteq D$, such that the restriction of S on V_x is injective. In order to continue our analysis we need the following result from [113].

Theorem 1.3.4. *[Theorem 4, [113]] Let D be an open subset of $\mathbb{R}^n$ and let $S : D \longrightarrow \mathbb{R}^n$ be differentiable on D. Suppose that* $\det(J_S(x)) \neq 0$ *for all $x \in D$. Then S is locally injective on D.*

Next we provide some conditions that assures the global injectivity of a differentiable $A-$monotone operator. Note that we do not require that the operator involved to be of class C^1, as we did in Theorem 1.3.3.

Theorem 1.3.5. *Let $D \subseteq \mathbb{R}^n$ be open and convex, let $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a surjective linear operator and let $S : D \longrightarrow \mathbb{R}^n$ be a differentiable operator monotone relative to A. Assume further that* $\det(J_S(x)) \neq 0$ *for all $x \in D$. Then S is globally injective on D.*

Proof. According to Theorem 1.3.2 for every $u \in \mathbb{R}^n$ the inverse image of u through S, i.e. $S^{-1}(u)$, is convex. On the other hand by Theorem 1.3.4 S is locally injective, hence $S^{-1}(u)$ contains at most one point. □

As an immediate consequence we obtain the following result concerning the global injectivity of Minty-Browder monotone operators.

Corollary 1.3.4. *Let $D \subseteq \mathbb{R}^n$ be open and convex and let $S : D \longrightarrow \mathbb{R}^n$ be a differentiable Minty-Browder monotone operator. Assume that* $\det(J_S(x)) \neq 0$ *for all $x \in D$. Then S is globally injective on D.*

Remark 1.3.8. In Remark 1.3.3 we gave a very general formula that ensures the convexity of the preimages of a $\theta-$monotone operator. Hence, the following result holds. Let $D \subseteq \mathbb{R}^n$ be open and convex and let $S : D \longrightarrow \mathbb{R}^n$ be a differentiable $\theta-$monotone operator. Assume that for all $x, y \in D$, $x \neq y$ such that $S(x) = S(y)$ and for all $z = x + t_0(y - x) \in (x, y)$, $t_0 \in (0, 1)$ there exists $w \in \mathbb{R}^n$ with $z + w \in D$ such that

$$\langle S(x) - S(z+w), w\rangle + (1-t_0)\theta(x, z+w)\|z+w-x\| + t_0\theta(y, z+w)\|z+w-y\| > 0.$$

Assume further $\det(J_S(x)) \neq 0$ for all $x \in D$. Then S is globally injective on D. Note that by considering particular instances of θ one can easily derive global injectivity conditions for operators that are monotone in some sense.

Next we provide some analytical conditions that ensure global injectivity of an operator of class C^1. Note that the linear operator A can be identified with a real square matrix $(a_{ij})_{1\leq i,j\leq n}$. Let us denote by $A^\top$ the transpose of A. Recall that a matrix A is positive definite if $\langle Ax, x\rangle > 0$ for all $x \in \mathbb{R}^n$, $x \neq 0$. A matrix A is negative definite if for all $x \in \mathbb{R}^n$, $x \neq 0$ one has $\langle Ax, x\rangle < 0$. Sylvester's criterion states that a symmetric matrix, i.e. $A = A^\top$, is positive definite if and only if all of the leading principal minors are positive. A symmetric matrix is negative definite if and only if all k-th order leading principal minors are negative when k is odd, and positive when k is even. For a given square matrix B of order n we denote the submatrix obtained by deleting the last $n-m$ rows and the last $n-m$ columns by $(B)_{1\leq i,j\leq m}$.

Theorem 1.3.6. *Let $D \subseteq \mathbb{R}^n$ be an open and convex set, let $S : D \longrightarrow \mathbb{R}^n$ be an operator of class C^1 and let $A : R^n \longrightarrow \mathbb{R}^n$ be a linear operator with* $\det(A) \neq 0$. *Assume that for all $x \in D$ one of the following conditions hold.*

(a) $\det(A^\top J_S(x) + J_S^\top(x)A)_{1\leq i,j\leq m} > 0, \forall m \in \{1, 2, \ldots, n\}$.

(b) $(-1)^m \det(A^\top J_S(x) + J_S^\top(x)A)_{1\le i,j\le m} > 0, \forall m \in \{1,2,\dots,n\}$.

Then S is injective.

Proof. Let $x \in D$. Then we have $\langle dS_x(y), A(y)\rangle = \langle A^\top dS_x(y), y\rangle = y^\top A^\top J_S(x)y$. This shows that the positive definiteness, respectively negative definiteness of $A^\top J_S(x)$ is equivalent to

$$\langle dS_x(y), A(y)\rangle > 0, \forall y \in \mathbb{R}^n \setminus \{0\},$$

respectively

$$\langle dS_x(y), A(y)\rangle < 0, \forall y \in \mathbb{R}^n \setminus \{0\}.$$

On the other hand, $y^\top A^\top J_S(x)y = ((A^\top J_S(x))^\top y)^\top y = (J_S^\top(x)Ay)^\top y = y^\top((J_S^\top(x)Ay)^\top)^\top = y^\top J_S^\top(x)Ay$, hence

$$y^\top A^\top J_S(x)y = \frac{1}{2}y^\top\left(A^\top J_S(x) + J_S^\top(x)A\right)y.$$

Observe that

$$\det(A^\top J_S(x) + J_S^\top(x)A)_{1\le i,j\le m} > 0, \forall m \in \{1,2,\dots,n\}.$$

is actually Sylvester's criterion for positive definiteness of the symmetric matrix

$$A^\top J_S(x) + J_S^\top(x)A,$$

meanwhile the condition $(-1)^m \det(A^\top J_S(x) + J_S^\top(x)A)_{1\le i,j\le m} > 0, \forall m \in \{1,2,\dots,n\}$ is actually Sylvester's criterion for negative definiteness of the symmetric matrix

$$A^\top J_S(x) + J_S^\top(x)A.$$

But the latter relations are equivalent to the positive definiteness, respectively negative definiteness of $A^\top J_S(x)$. Hence, we have

$$\langle dS_x(y), A(y)\rangle > 0, \forall y \in \mathbb{R}^n \setminus \{0\},$$

respectively

$$\langle dS_x(y), A(y)\rangle < 0, \forall y \in \mathbb{R}^n \setminus \{0\}.$$

Since $\det(A) \neq 0$ we obtain that A is injective. The conclusion follows from Theorem 1.3.3, respectively Remark 1.3.6. □

1.3.3 Injective complex functions

In this paragraph we apply the results from previous paragraphs in order to obtain some new injectivity, respective univalency results involving complex functions. Let us emphasize that some celebrated results, such as Mocanu theorem, respectively Alexander-Noshiro-Warschawski and Wolff theorem are direct consequences of the main result of this section.

Let us denote by $\mathbb{C}$ the set of complex numbers, that is

$$\mathbb{C} = \{z = x + iy : x, y \in \mathbb{R},\, i^2 = -1\}.$$

For $z = x + iy \in \mathbb{C}$ we denote by $\operatorname{Re} z$, $\operatorname{Im} z$, $\bar{z}$, respectively $|z|$ the real part, imaginary part, conjugate and absolute value respectively, that is $\operatorname{Re} z = x$, $\operatorname{Im} z = y$, $\bar{z} = x - iy$ and $|z| = \sqrt{x^2 + y^2}$. Obviously $z\bar{z} = |z|^2$. Note that the real linear space $\mathbb{C}$ becomes a real Hilbert space with the inner product

$$\langle z, w \rangle = \operatorname{Re} z\overline{w}.$$

This real Hilbert space may be identified with the real Hilbert space $\mathbb{R}^2$ endowed with the euclidian scalar product, therefore we can identify $z \in \mathbb{C}$ by $(\operatorname{Re} z, \operatorname{Im} z) \in \mathbb{R}^2$.

Let $D \subseteq \mathbb{C}$ be open. For a complex function of one complex variable $f : D \longrightarrow \mathbb{C}$, $f(z) = u(x,y) + iv(x,y)$, $\forall z = x + iy \in D$ of class $C^1(D)$, we denote by $J_f(z_0)$ the Jacobian matrix of f in $z_0 = x_0 + iy_0$, i.e.

$$J_f(z_0) = \begin{pmatrix} u'_x(x_0,y_0) & u'_y(x_0,y_0) \\ v'_x(x_0,y_0) & v'_y(x_0,y_0) \end{pmatrix}.$$

If we consider f as the vector function (u,v) then its differential in $z_0 = x_0 + iy_0$ can be defined as

$$df_{(x_0,y_0)}(p,q) = J_f(z_0) \cdot \begin{pmatrix} p \\ q \end{pmatrix},$$

hence for $w = p + iq$ the differential of f in z_0 becomes

$$df_{z_0}(w) = (u'_x(x_0,y_0)p + u'_y(x_0,y_0)q) + i(v'_x(x_0,y_0)p + v'_y(x_0,y_0)q).$$

The partial derivatives of f are defined as:

$$\frac{\partial f}{\partial x}(z_0) = u'_x(x_0,y_0) + iv'_x(x_0,y_0),$$

respectively

$$\frac{\partial f}{\partial y}(z_0) = u'_y(x_0, y_0) + iv'_y(x_0, y_0).$$

Let us introduce the following notations:

$$\frac{\partial f}{\partial z}(z_0) = \frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) - i\frac{\partial f}{\partial y}(z_0)\right),$$

respectively

$$\frac{\partial f}{\partial \bar{z}}(z_0) = \frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) + i\frac{\partial f}{\partial y}(z_0)\right).$$

The main result of this section is the following general injectivity result.

Theorem 1.3.7. *Let $D \subseteq \mathbb{C}$ be open and convex and let $f : D \longrightarrow \mathbb{C}$ be a function of class C^1. Assume that there exist $w_1, w_2 \in \mathbb{C}$ such that $\operatorname{Re} w_1 \operatorname{Im} w_2 \neq \operatorname{Re} w_2 \operatorname{Im} w_1$ and for all $z \in D$ the following condition holds:*

$$\operatorname{Re}\left(\frac{\partial f}{\partial z}(z)w_1 + \frac{\partial f}{\partial \bar{z}}(z)\overline{w_1}\right) + \operatorname{Im}\left(\frac{\partial f}{\partial z}(z)w_2 + \frac{\partial f}{\partial \bar{z}}(z)\overline{w_2}\right) > \left|\frac{\partial f}{\partial z}(z)(w_2 - iw_1) + \frac{\partial f}{\partial \bar{z}}(z)\overline{w_2 + iw_1}\right|. \tag{1. 11}$$

Then f is injective.

Proof. One can assume that $w_1 = a + ib$, $w_2 = c + id$, $a, b, c, d \in \mathbb{R}$. It can easily be deduced, that (1.11) is equivalent to

$$(u'_x a + u'_y b) + (v'_x c + v'_y d) > \sqrt{\left((u'_x c + u'_y d) + (v'_x a + v'_y b)\right)^2 + \left((v'_x c + v'_y d) - (u'_x a + u'_y b)\right)^2}.$$

By taking the square of both sides we obtain

$$4(u'_x a + u'_y b)(v'_x c + v'_y d) > \left((u'_x c + u'_y d) + (v'_x a + v'_y b)\right)^2,$$

or equivalently

$$4\operatorname{Re} df_z(w_1) \cdot \operatorname{Im} df_z(w_2) > (\operatorname{Re} df_z(w_2) + \operatorname{Im} df_z(w_1))^2, \forall z \in D.$$

The latter relation can be written as

$$\det\begin{pmatrix} 2\operatorname{Re} df_z(w_1) & \operatorname{Re} df_z(w_2)+\operatorname{Im} df_z(w_1) \\ \operatorname{Re} df_z(w_2)+\operatorname{Im} df_z(w_1) & 2\operatorname{Im} df_z(w_2) \end{pmatrix} > 0, \forall z \in D. \tag{1. 12}$$

Let us denote by L the matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$. Then (1.12) becomes

$$\det(L^\top J_f(z) + J_f^\top(z)L) > 0, \forall z \in D.$$

We show next that $\operatorname{Re} df_z(w_1) > 0$ for all $z \in D$, or $\operatorname{Re} df_z(w_1) < 0$ for all $z \in D$. Observe that (1.12) assures that $\operatorname{Re} df_z(w_1) \neq 0$ for all $z \in D$. Assume for instance that $\operatorname{Re} df_{z_1}(w_1) > 0$ and $\operatorname{Re} df_{z_2}(w_1) < 0$ for some $z_1, z_2 \in D$. Then, the intermediate value theorem, applied to the function $g : D \longrightarrow \mathbb{R}$, $g(z) = \operatorname{Re} df_z(w_1)$, provides the existence of $z_3 \in D$ such that $\operatorname{Re} df_{z_3}(w_1) = 0$, contradiction.

In conclusion one of the following conditions is fulfilled.

(i) $\operatorname{Re} df_z(w_1) > 0$ and $\det(L^\top J_f(z) + J_f^\top(z)L) > 0$ for all $z \in D$, or

(ii) $\operatorname{Re} df_z(w_1) < 0$ and $\det(L^\top J_f(z) + J_f^\top(z)L) > 0$ for all $z \in D$.

Note that (i), respectively (ii) are equivalent to

(a) $\det(L^\top J_f(z) + J_f^\top(z)L)_{1\le i,j\le m} > 0$, $\forall m \in \{1,2\}$, respectively

(b) $(-1)^m \det(L^\top J_f(z) + J_f^\top(z)L)_{1\le i,j\le m} > 0$, $\forall m \in \{1,2\}$.

Since $\operatorname{Re} w_1 \operatorname{Im} w_2 \neq \operatorname{Re} w_2 \operatorname{Im} w_1$, we obtain that L is invertible, hence injective. According to Theorem 1.3.6 f is injective. □

Remark 1.3.9. One can easily deduce that for $z \in D$ and $w \in \mathbb{C}$ we have

$$df_z(w) = \frac{\partial f}{\partial z}(z)w + \frac{\partial f}{\partial \overline{z}}(z)\overline{w},$$

hence the condition (1.11) in the hypothesis of Theorem 1.3.7 can be replaced by

$$\operatorname{Re} df_z(w_1) + \operatorname{Im} df_z(w_2) > |df_z(w_2) - i df_z(w_1)|, \forall z \in D.$$

The next Corollary can be viewed as an extension of Mocanu's injectivity result.

Corollary 1.3.5. *Let $D \subseteq \mathbb{C}$ be open and convex and let $f : D \longrightarrow \mathbb{C}$ be a function of class C^1. Assume that there exist $\gamma \in \mathbb{R}$ such that, for all $z \in D$ it holds:*

$$\operatorname{Re}\left(\frac{\partial f}{\partial z}(z)e^{i\gamma}\right) > \left|\frac{\partial f}{\partial \bar{z}}(z)\right|.$$

Then f is injective.

Proof. Take $w_1 = e^{i\gamma}$ and $w_2 = ie^{i\gamma}$. Then an easy computation shows, that

$$2\operatorname{Re}\left(\frac{\partial f}{\partial z}(z)e^{i\gamma}\right) = \operatorname{Re}\left(\frac{\partial f}{\partial z}(z)w_1 + \frac{\partial f}{\partial \bar{z}}(z)\overline{w_1}\right) + \operatorname{Im}\left(\frac{\partial f}{\partial z}(z)w_2 + \frac{\partial f}{\partial \bar{z}}(z)\overline{w_2}\right).$$

On the other hand

$$2\left|\frac{\partial f}{\partial \bar{z}}(z)\right| = \left|\frac{\partial f}{\partial z}(z)(w_2 - iw_1) + \frac{\partial f}{\partial \bar{z}}(z)\overline{w_2 + iw_1}\right|.$$

Hence,

$$\operatorname{Re}\left(\frac{\partial f}{\partial z}(z)e^{i\gamma}\right) > \left|\frac{\partial f}{\partial \bar{z}}(z)\right|$$

is equivalent to

$$\operatorname{Re}\left(\frac{\partial f}{\partial z}(z)w_1 + \frac{\partial f}{\partial \bar{z}}(z)\overline{w_1}\right) + \operatorname{Im}\left(\frac{\partial f}{\partial z}(z)w_2 + \frac{\partial f}{\partial \bar{z}}(z)\overline{w_2}\right) >$$

$$\left|\frac{\partial f}{\partial z}(z)(w_2 - iw_1) + \frac{\partial f}{\partial \bar{z}}(z)\overline{w_2 + iw_1}\right|.$$

Since $\operatorname{Re} w_1 \operatorname{Im} w_2 - \operatorname{Re} w_2 \operatorname{Im} w_1 = \cos^2\gamma + \sin^2\gamma = 1$, obviously $\operatorname{Re} w_1 \operatorname{Im} w_2 \neq \operatorname{Re} w_2 \operatorname{Im} w_1$. The conclusion follows from Theorem 1.3.7. □

Remark 1.3.10. Note that for $\gamma = 0$ in Corollary 1.3.5, we obtain Mocanu's injectivity theorem, see [159].

Corollary 1.3.6. *Let $D \subseteq \mathbb{C}$ be open and convex and let $f : D \longrightarrow \mathbb{C}$ be a function of class C^1. Assume that there exist $\gamma \in \mathbb{R}$ such that, for all $z \in D$ it holds:*

$$\operatorname{Re}\left(\frac{\partial f}{\partial \bar{z}}(z)e^{i\gamma}\right) > \left|\frac{\partial f}{\partial z}(z)\right|.$$

Then f is injective.

Proof. Take $w_1 = e^{i\gamma}$ and $w_2 = -ie^{i\gamma}$. Then $\operatorname{Re} w_1 \operatorname{Im} w_2 - \operatorname{Re} w_2 \operatorname{Im} w_1 = -\cos^2\gamma - \sin^2\gamma =$

-1, hence $\operatorname{Re} w_1 \operatorname{Im} w_2 \neq \operatorname{Re} w_2 \operatorname{Im} w_1$. The rest of the proof is analogous to the proof of Corollary 1.3.5. □

Let D be open and connected. Recall that a function $f : D \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ is called holomorphic on D if f is derivable at every point of D. Note, that in the case when f is holomorphic on D, one has

$$\frac{\partial f}{\partial \bar{z}}(z) = 0, \text{ for all } z \in D,$$

and

$$df_z(w) = \frac{\partial f}{\partial z}(z)w = f'(z)w, \text{ for all } z \in D \text{ and } w \in \mathbb{C}.$$

Recall that a holomorphic function which is also injective is called univalent. The next result is an extension of the univalency result of Alexander-Noshiro-Warschawski and Wolff.

Corollary 1.3.7. *Let $D \subseteq \mathbb{C}$ be open and convex and let $f : D \longrightarrow \mathbb{C}$ be a holomorphic function. Assume that there exist $w_1, w_2 \in \mathbb{C}$ such that $\operatorname{Re} w_1 \operatorname{Im} w_2 \neq \operatorname{Re} w_2 \operatorname{Im} w_1$ and for all $z \in D$ the following condition holds:*

$$\operatorname{Re} f'(z)w_1 + \operatorname{Im} f'(z)w_2 > |f'(z)||w_2 - iw_1|.$$

Then f is univalent.

Proof. According to Remark 1.3.9 the condition

$$\operatorname{Re}\left(\frac{\partial f}{\partial z}(z)w_1 + \frac{\partial f}{\partial \bar{z}}(z)\overline{w_1}\right) + \operatorname{Im}\left(\frac{\partial f}{\partial z}(z)w_2 + \frac{\partial f}{\partial \bar{z}}(z)\overline{w_2}\right) >$$

$$\left|\frac{\partial f}{\partial z}(z)(w_2 - iw_1) + \frac{\partial f}{\partial \bar{z}}(z)\overline{w_2 + iw_1}\right|,$$

is equivalent to

$$\operatorname{Re} df_z(w_1) + \operatorname{Im} df_z(w_2) > |df_z(w_2) - idf_z(w_1)|.$$

On the other hand the latter relation is exactly

$$\operatorname{Re} f'(z)w_1 + \operatorname{Im} f'(z)w_2 > |f'(z)||w_2 - iw_1|.$$

The conclusion follows from Theorem 1.3.7. □

From the previous result one can easily obtain Alexander-Noshiro-Warschawski and Wolff univalency theorem (see [3] and [175, 205, 206]), that is:

Corollary 1.3.8. *Let $D \in \mathbb{C}$ be open and convex and let $f : D \longrightarrow \mathbb{C}$ be a holomorphic function. Assume that there exist $\gamma \in \mathbb{R}$ such that, for all $z \in D$ it holds:*

$$\operatorname{Re} f'(z)e^{i\gamma} > 0.$$

Then f is univalent.

Proof. Indeed, let $w_1 = e^{i\gamma}$, $w_2 = ie^{i\gamma}$. Then

$$\operatorname{Re} f'(z)w_1 + \operatorname{Im} f'(z)w_2 = 2\operatorname{Re} f'(z)e^{i\gamma}.$$

Obviously $|f'(z)||w_2 - iw_1| = 0$, hence $\operatorname{Re} f'(z)e^{i\gamma} > 0$ is equivalent to

$$\operatorname{Re} f'(z)w_1 + \operatorname{Im} f'(z)w_2 > |f'(z)||w_2 - iw_1|.$$

Note that $\operatorname{Re} w_1 \operatorname{Im} w_2 - \operatorname{Re} w_2 \operatorname{Im} w_1 = \cos^2\gamma + \sin^2\gamma = 1$, hence $\operatorname{Re} w_1 \operatorname{Im} w_2 \neq \operatorname{Re} w_2 \operatorname{Im} w_1$. The conclusion follows from Corollary 1.3.7. $\square$

2

On the minimization of the sum of two functions

2.1 On the sum of a smooth nonconvex and a nonsmooth nonconvex function

Proximal-gradient splitting methods are powerful techniques used in order to solve optimization problems where the objective to be minimized is the sum of a finite collection of smooth and/or nonsmooth functions. The main feature of this class of algorithmic schemes is the fact that they access each function separately, either by a gradient step if this is smooth or by a proximal step if it is nonsmooth.

In the convex case (when all the functions involved are convex), these methods are well understood, see for example [36], where the reader can find a presentation of the most prominent methods, like the forward-backward, forward-backward-forward and the Douglas-Rachford splitting algorithms.

On the other hand, the nonconvex case is less understood, one of the main difficulties coming from the fact that the proximal point operator is in general not anymore single-valued. However, one can observe a considerable progress in this direction when the functions in the objective have the *Kurdyka-Łojasiewicz property* (so-called *KL functions*), as it is the case for the ones with different analytic features. This applies for both the forward-backward algorithm (see [48], [22]) and the forward-backward-forward algorithm (see [60]). We refer the reader also to [20,21,77,95,111,177] for literature concerning proximal-gradient splitting methods in the nonconvex case relying on the *Kurdyka-Łojasiewicz property*.

A particular class of the proximal-gradient splitting methods are the ones with iner-

tial/memory effects. These iterative schemes have their origins in the time discretization of some differential inclusions of second order type (see [7, 9]) and share the feature that the new iterate is defined by using the previous two iterates. The increasing interest in this class of algorithms is emphasized by a considerable number of papers written in the last fifteen years on this topic, see [7–9, 28, 57–61, 72, 73, 76, 153, 154, 163, 182].

Recently, an inertial forward-backward type algorithm has been proposed and analyzed in [177] in the nonconvex setting, by assuming that the nonsmooth part of the objective function is convex, while the smooth counterpart is allowed to be nonconvex. In this section we introduce an inertial forward-backward algorithm in the full nonconvex setting and we study its convergence properties. The techniques for proving the convergence of the numerical scheme use the same three main ingredients, as other algorithms for nonconvex optimization problems involving KL functions. More precisely, we show a sufficient decrease property for the iterates, the existence of a subgradient lower bound for the iterates gap and, finally, we use the analytic features of the objective function in order to obtain convergence, see [22, 48]. The *limiting (Mordukhovich) subdifferential* and its properties play an important role in the analysis. The main result of this chapter shows that, provided an appropriate regularization of the objective satisfies the Kurdyka-Łojasiewicz property, the convergence of the inertial forward-backward algorithm is guaranteed. As a particular instance, we also treat the case when the objective function is semi-algebraic and present the convergence properties of the algorithm.

In the last subsection we consider two numerical experiments. The first one has an academic character and shows the ability of algorithms with inertial/memory effects to detect optimal solutions which are not found by the non-inertial versions (similar allegations can be found also in [177, Section 5.1] and [41, Example 1.3.9]). The second one concerns the restoration of a noisy blurred image by using a nonconvex misfit functional with nonconvex regularization.

Let us mention that the results from this section have been partially published in [62]:[R.I. Boţ, E.R. Csetnek, S. László, *An inertial forward-backward algorithm for minimizing the sum of two non-convex functions*, Euro Journal on Computational Optimization 4(1), 3-25 (2016)].

2.1.1 On Kurdyka-Łojasiewicz property

We recall some notions and results which are needed throughout this section. Let $\mathbb{N} = \{0, 1, 2, ...\}$ be the set of nonnegative integers. For $m \geq 1$, the Euclidean scalar product and the induced norm on $\mathbb{R}^m$ are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Notice that all the finite-

dimensional spaces considered in the section are endowed with the topology induced by the Euclidean norm.

The *domain* of the function $f : \mathbb{R}^m \to (-\infty, +\infty]$ is defined by $\operatorname{dom} f = \{x \in \mathbb{R}^m : f(x) < +\infty\}$. We say that f is *proper* if $\operatorname{dom} f \neq \emptyset$. For the following generalized subdifferential notions and their basic properties we refer to [161, 190]. Let $f : \mathbb{R}^m \to (-\infty, +\infty]$ be a proper and lower semicontinuous function. If $x \in \operatorname{dom} f$, we consider the *Fréchet (viscosity) subdifferential* of f at x as the set

$$\hat{\partial} f(x) = \left\{ v \in \mathbb{R}^m : \liminf_{y \to x} \frac{f(y) - f(x) - \langle v, y - x \rangle}{\|y - x\|} \geq 0 \right\}.$$

For $x \notin \operatorname{dom} f$ we set $\hat{\partial} f(x) := \emptyset$. The *limiting (Mordukhovich) subdifferential* is defined at $x \in \operatorname{dom} f$ by

$$\partial f(x) = \{v \in \mathbb{R}^m : \exists x_n \to x, f(x_n) \to f(x) \text{ and } \exists v_n \in \hat{\partial} f(x_n), v_n \to v \text{ as } n \to +\infty\},$$

while for $x \notin \operatorname{dom} f$, one takes $\partial f(x) := \emptyset$.

Notice that in case f is convex, these notions coincide with the *convex subdifferential*, which means that $\hat{\partial} f(x) = \partial f(x) = \{v \in \mathbb{R}^m : f(y) \geq f(x) + \langle v, y - x \rangle \ \forall y \in \mathbb{R}^m\}$ for all $x \in \operatorname{dom} f$.

Notice the inclusion $\hat{\partial} f(x) \subseteq \partial f(x)$ for each $x \in \mathbb{R}^m$. We will use the following closedness criteria concerning the graph of the limiting subdifferential: if $(x_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$ are sequences in $\mathbb{R}^m$ such that $v_n \in \partial f(x_n)$ for all $n \in \mathbb{N}$, $(x_n, v_n) \to (x, v)$ and $f(x_n) \to f(x)$ as $n \to +\infty$, then $v \in \partial f(x)$.

The Fermat rule reads in this nonsmooth setting as: if $x \in \mathbb{R}^m$ is a local minimizer of f, then $0 \in \partial f(x)$. Notice that in case f is continuously differentiable around $x \in \mathbb{R}^m$ we have $\partial f(x) = \{\nabla f(x)\}$. Let us denote by

$$\operatorname{crit}(f) = \{x \in \mathbb{R}^m : 0 \in \partial f(x)\}$$

the set of *(limiting)-critical points* of f. Let us mention also the following subdifferential rule: if $f : \mathbb{R}^m \to (-\infty, +\infty]$ is proper and lower semicontinuous and $h : \mathbb{R}^m \to \mathbb{R}$ is a continuously differentiable function, then $\partial(f + h)(x) = \partial f(x) + \nabla h(x)$ for all $x \in \mathbb{R}^m$.

We turn now our attention to functions satisfying the *Kurdyka-Łojasiewicz property*. This class of functions will play a crucial role when proving the convergence of the proposed inertial algorithm. For $\eta \in (0, +\infty]$, we denote by Θ_η the class of concave and continuous functions $\varphi : [0, \eta) \to [0, +\infty)$ such that $\varphi(0) = 0$, φ is continuously differentiable on $(0, \eta)$,

continuous at 0 and $\varphi'(s) > 0$ for all $s \in (0, \eta)$. In the following definition (see [21, 48]) we use also the *distance function* to a set, defined for $A \subseteq \mathbb{R}^m$ as $\operatorname{dist}(x, A) = \inf_{y \in A} \|x - y\|$ for all $x \in \mathbb{R}^m$.

Definition 2.1.1. (*Kurdyka-Łojasiewicz property*) Let $f : \mathbb{R}^m \to (-\infty, +\infty]$ be a proper and lower semicontinuous function. We say that f satisfies the *Kurdyka-Łojasiewicz (KL) property* at $\bar{x} \in \operatorname{dom} \partial f = \{x \in \mathbb{R}^m : \partial f(x) \neq \emptyset\}$ if there exist $\eta \in (0, +\infty]$, a neighborhood U of $\bar{x}$ and a function $\varphi \in \Theta_\eta$ such that for all x in the intersection

$$U \cap \{x \in \mathbb{R}^m : f(\bar{x}) < f(x) < f(\bar{x}) + \eta\}$$

the following inequality holds

$$\varphi'(f(x) - f(\bar{x})) \operatorname{dist}(0, \partial f(x)) \geq 1.$$

If f satisfies the KL property at each point in $\operatorname{dom} \partial f$, then f is called a *KL function*.

The origins of this notion go back to the pioneering work of Łojasiewicz [148], where it is proved that for a real-analytic function $f : \mathbb{R}^m \to \mathbb{R}$ and a critical point $\bar{x} \in \mathbb{R}^m$ (that is $\nabla f(\bar{x}) = 0$), there exists $\theta \in [1/2, 1)$ such that the function $|f - f(\bar{x})| \|\nabla f\|^{-1}$ is bounded around $\bar{x}$. This corresponds to the situation when $\varphi(s) = s^{1-\theta}$. The result of Łojasiewicz allows the interpretation of the KL property as a re-parametrization of the function values in order to avoid flatness around the critical points. Kurdyka [131] extended this property to differentiable functions definable in an o-minimal structure. Further extensions to the nonsmooth setting can be found in [21, 45–47].

One of the remarkable properties of the KL functions is their ubiquity in applications, according to [48]. To the class of KL functions belong semi-algebraic, real sub-analytic, semiconvex, uniformly convex and convex functions satisfying a growth condition. We refer the reader to [20–22, 45–48] and the references therein for more details regarding all the classes mentioned above and illustrating examples.

An important role in our convergence analysis will be played by the following uniformized KL property given in [48, Lemma 6].

Lemma 2.1.1. *Let $\Omega \subseteq \mathbb{R}^m$ be a compact set and let $f : \mathbb{R}^m \to (-\infty, +\infty]$ be a proper and lower semicontinuous function. Assume that f is constant on Ω and f satisfies the KL property at each point of Ω. Then there exist $\varepsilon, \eta > 0$ and $\varphi \in \Theta_\eta$ such that for all $\bar{x} \in \Omega$ and for*

all x in the intersection

$$\{x \in \mathbb{R}^m : \operatorname{dist}(x, \Omega) < \varepsilon\} \cap \{x \in \mathbb{R}^m : f(\bar{x}) < f(x) < f(\bar{x}) + \eta\} \tag{2. 1}$$

the following inequality holds

$$\varphi'(f(x) - f(\bar{x})) \operatorname{dist}(0, \partial f(x)) \geq 1. \tag{2. 2}$$

We close this section by presenting two convergence results which will play a determined role in the proof of the results we provide in the next section. The first one was often used in the literature in the context of Fejér monotonicity techniques for proving convergence results of classical algorithms for convex optimization problems or more generally for monotone inclusion problems (see [36]). The second one is probably also known, see for example [60].

Lemma 2.1.2. *Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be real sequences such that $b_n \geq 0$ for all $n \in \mathbb{N}$, $(a_n)_{n\in\mathbb{N}}$ is bounded below and $a_{n+1} + b_n \leq a_n$ for all $n \in \mathbb{N}$. Then $(a_n)_{n\in\mathbb{N}}$ is a monotonically decreasing and convergent sequence and $\sum_{n\in\mathbb{N}} b_n < +\infty$.*

Lemma 2.1.3. *Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be nonnegative real sequences, such that $\sum_{n\in\mathbb{N}} b_n < +\infty$ and $a_{n+1} \leq a \cdot a_n + b \cdot a_{n-1} + b_n$ for all $n \geq 1$, where $a \in \mathbb{R}$, $b \geq 0$ and $a + b < 1$. Then $\sum_{n\in\mathbb{N}} a_n < +\infty$.*

2.1.2 A forward-backward algorithm

In this subsection we present an inertial forward-backward algorithm for a fully nonconvex optimization problem and study its convergence properties. The problem under investigation has the following formulation.

Problem 1. Let $f : \mathbb{R}^m \to (-\infty, +\infty]$ be a proper, lower semicontinuous function which is bounded below and let $g : \mathbb{R}^m \to \mathbb{R}$ be a Fréchet differentiable function with Lipschitz continuous gradient, i.e. there exists $L_{\nabla g} \geq 0$ such that $\|\nabla g(x) - \nabla g(y)\| \leq L_{\nabla g}\|x - y\|$ for all $x, y \in \mathbb{R}^m$. We deal with the optimization problem

$$(P) \inf_{x\in\mathbb{R}^m} [f(x) + g(x)]. \tag{2. 3}$$

In the iterative scheme we propose below, we use also the function $F : \mathbb{R}^m \to \mathbb{R}$, assumed to be $\sigma-$strongly convex, i.e. $F - \frac{\sigma}{2}\|\cdot\|^2$ is convex, Fréchet differentiable and such that ∇F is $L_{\nabla F}$-Lipschitz continuous, where $\sigma, L_{\nabla F} > 0$. The *Bregman distance* to F, denoted by

$D_F : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$, is defined as

$$D_F(x,y) = F(x) - F(y) - \langle \nabla F(y), x-y \rangle \ \forall (x,y) \in \mathbb{R}^m \times \mathbb{R}^m.$$

Notice that the properties of the function F ensure the following inequalities

$$\frac{\sigma}{2}\|x-y\|^2 \leq D_F(x,y) \leq \frac{L_{\nabla F}}{2}\|x-y\|^2 \ \forall x,y \in \mathbb{R}^m. \tag{2. 4}$$

We propose the following iterative scheme.

Algorithm 1. Choose $x_0, x_1 \in \mathbb{R}^m$, $\underline{\alpha}, \overline{\alpha} > 0$, $\beta \geq 0$ and the sequences $(\alpha_n)_{n\geq 1}, (\beta_n)_{n\geq 1}$ fulfilling

$$0 < \underline{\alpha} \leq \alpha_n \leq \overline{\alpha} \ \forall n \geq 1$$

and

$$0 \leq \beta_n \leq \beta \ \forall n \geq 1.$$

Consider the iterative scheme

$$(\forall n \geq 1) \ \ x_{n+1} \in \operatorname*{argmin}_{u \in \mathbb{R}^m} \left\{ D_F(u, x_n) + \alpha_n \langle u, \nabla g(x_n) \rangle + \beta_n \langle u, x_{n-1} - x_n \rangle + \alpha_n f(u) \right\}. \tag{2. 5}$$

Due to the subdifferential sum formula mentioned in the previous section, one can see that any sequence generated by this algorithm satisfies the relation

$$x_{n+1} \in (\nabla F + \alpha_n \partial f)^{-1} (\nabla F(x_n) - \alpha_n \nabla g(x_n) + \beta_n (x_n - x_{n-1})) \ \forall n \geq 1. \tag{2. 6}$$

Further, since f is proper, lower semicontinuous and bounded from below and D_F is coercive in its first argument (that is $\lim_{\|x\| \to +\infty} D_F(x,y) = +\infty$ for all $y \in \mathbb{R}^m$), the iterative scheme is well-defined, meaning that the existence of x_n is guaranteed for each $n \geq 2$, since the objective function in the minimization problem to be solved at each iteration is coercive.

Remark 2.1.1. The condition that f should be bounded below is imposed in order to ensure that in each iteration one can choose at least one x_n (that is the argmin in (2. 5) is nonempty). One can replace this requirement by asking that the objective function in the minimization problem considered in (2. 5) is coercive and the theory presented below still remains valid. This observation is useful when dealing with optimization problems as the ones considered in Subsection 2.1.3.

Before proceeding with the convergence analysis, we discuss the relation of our scheme to other algorithms from the literature. Let us take first $F(x) = \frac{1}{2}\|x\|^2$ for all $x \in \mathbb{R}^m$. In this

case $D_F(x,y) = \frac{1}{2}\|x-y\|^2$ for all $(x,y) \in \mathbb{R}^m \times \mathbb{R}^m$ and $\sigma = L_{\nabla F} = 1$. The iterative scheme becomes

$$(2.\ 7) \qquad (\forall n \geq 1)\ \ x_{n+1} \in \operatorname*{argmin}_{u \in \mathbb{R}^m} \left\{ \frac{\|u - (x_n - \alpha_n \nabla g(x_n) + \beta_n(x_n - x_{n-1}))\|^2}{2\alpha_n} + f(u) \right\}.$$

A similar inertial type algorithm has been analyzed in [177], however in the restrictive case when f is convex. If we take in addition $\beta = 0$, which enforces $\beta_n = 0$ for all $n \geq 1$, then (2. 7) becomes

$$(2.\ 8) \qquad (\forall n \geq 1)\ \ x_{n+1} \in \operatorname*{argmin}_{u \in \mathbb{R}^m} \left\{ \frac{\|u - (x_n - \alpha_n \nabla g(x_n))\|^2}{2\alpha_n} + f(u) \right\},$$

the convergence of which has been investigated in [48] in the full nonconvex setting. Notice that forward-backward algorithms with variable metrics for KL functions have been proposed in [77, 95].

On the other hand, if we take $g(x) = 0$ for all $x \in \mathbb{R}^m$, the iterative scheme in (2. 7) becomes

$$(2.\ 9) \qquad (\forall n \geq 1)\ \ x_{n+1} \in \operatorname*{argmin}_{u \in \mathbb{R}^m} \left\{ \frac{\|u - (x_n + \beta_n(x_n - x_{n-1}))\|^2}{2\alpha_n} + f(u) \right\},$$

which is a proximal point algorithm with inertial/memory effects formulated in the nonconvex setting designed for finding the critical points of f. The iterative scheme without the inertial term, that is when $\beta = 0$ and, so, $\beta_n = 0$ for all $n \geq 1$, has been considered in the context of KL functions in [20].

Let us mention that in the full convex setting, which means that f and g are convex functions, in which case for all $n \geq 2$, x_n is uniquely determined and can be expressed via the *proximal operator* of f, (2. 7) can be derived from the iterative scheme proposed in [163], (2. 8) is the classical forward-backward algorithm (see for example [36] or [78]) and (2. 9) has been analyzed in [9] in the more general context of monotone inclusion problems.

In the convergence analysis of the algorithm the following result will be useful (see for example [164, Lemma 1.2.3]).

Lemma 2.1.4. *Let $g : \mathbb{R}^m \to \mathbb{R}$ be Fréchet differentiable with $L_{\nabla g}$-Lipschitz continuous gradient. Then*

$$g(y) \leq g(x) + \langle \nabla g(x), y - x \rangle + \frac{L_{\nabla g}}{2}\|y - x\|^2, \forall x, y \in \mathbb{R}^m.$$

Let us start now with the investigation of the convergence of the proposed algorithm.

Lemma 2.1.5. *In the setting of Problem 1, let $(x_n)_{n\in\mathbb{N}}$ be a sequence generated by Algorithm 1. Then one has*

$$(f+g)(x_{n+1})+M_1\|x_n-x_{n+1}\|^2 \leq (f+g)(x_n)+M_2\|x_{n-1}-x_n\|^2 \ \forall n\geq 1,$$

where

$$M_1=\frac{\sigma-\overline{\alpha}L_{\nabla g}}{2\overline{\alpha}}-\frac{\beta}{2\underline{\alpha}} \text{ and } M_2=\frac{\beta}{2\underline{\alpha}}. \tag{2. 10}$$

Moreover, for $0<\underline{\alpha}\leq\overline{\alpha}$ and $\beta>0$ satisfying

$$\sigma>\overline{\alpha}L_{\nabla g}+2\beta\frac{\overline{\alpha}}{\underline{\alpha}}, \tag{2. 11}$$

one has $M_1>M_2$.

Proof. Let be $n\geq 1$ fixed. Due to (2. 5) we have

$$\begin{aligned}&D_F(x_{n+1},x_n)+\alpha_n\langle x_{n+1},\nabla g(x_n)\rangle+\beta_n\langle x_{n+1},x_{n-1}-x_n\rangle+\alpha_n f(x_{n+1})\\ \leq\ &D_F(x_n,x_n)+\alpha_n\langle x_n,\nabla g(x_n)\rangle+\beta_n\langle x_n,x_{n-1}-x_n\rangle+\alpha_n f(x_n)\end{aligned}$$

or, equivalently,

$$D_F(x_{n+1},x_n)+\langle x_{n+1}-x_n,\alpha_n\nabla g(x_n)-\beta_n(x_n-x_{n-1})\rangle+\alpha_n f(x_{n+1})\leq\alpha_n f(x_n). \tag{2. 12}$$

On the other hand, by Lemma 2.1.4 we have

$$\langle\nabla g(x_n),x_{n+1}-x_n\rangle\geq g(x_{n+1})-g(x_n)-\frac{L_{\nabla g}}{2}\|x_n-x_{n+1}\|^2.$$

At the same time

$$\langle x_{n+1}-x_n,x_{n-1}-x_n\rangle\geq-\left(\frac{1}{2}\|x_n-x_{n+1}\|^2+\frac{1}{2}\|x_{n-1}-x_n\|^2\right),$$

and from (2. 4) we have

$$\frac{\sigma}{2}\|x_{n+1}-x_n\|^2\leq D_F(x_{n+1},x_n).$$

Hence, (2.12) leads to

$$(f+g)(x_{n+1})+\frac{\sigma-L_{\nabla g}\alpha_n-\beta_n}{2\alpha_n}\|x_{n+1}-x_n\|^2\leq(f+g)(x_n)+\frac{\beta_n}{2\alpha_n}\|x_{n-1}-x_n\|^2. \tag{2. 13}$$

Obviously $M_1 = \frac{\sigma - L_{\nabla g}\overline{\alpha}}{2\overline{\alpha}} - \frac{\beta}{2\underline{\alpha}} \leq \frac{\sigma - L_{\nabla g}\alpha_n - \beta_n}{2\alpha_n}$ and $M_2 = \frac{\beta}{2\underline{\alpha}} \geq \frac{\beta_n}{2\alpha_n}$ thus,

$$(f+g)(x_{n+1}) + M_1\|x_n - x_{n+1}\|^2 \leq (f+g)(x_n) + M_2\|x_{n-1} - x_n\|^2$$

and the first part of the lemma is proved.

Finally, for $0 < \underline{\alpha} \leq \overline{\alpha}$ and $\beta > 0$ satisfying $\sigma > \overline{\alpha}L_{\nabla g} + 2\beta\frac{\overline{\alpha}}{\underline{\alpha}}$, one has that $M_1 > M_2$ and the proof is complete. □

Remark 2.1.2. If $\underline{\alpha}$ and β are positive numbers such that $\sigma > \underline{\alpha}L_{\nabla g} + 2\beta$, then

$$\underline{\alpha} < \frac{\underline{\alpha}\sigma}{\underline{\alpha}L_{\nabla g} + 2\beta}.$$

By choosing

$$\underline{\alpha} \leq \overline{\alpha} < \frac{\underline{\alpha}\sigma}{\underline{\alpha}L_{\nabla g} + 2\beta},$$

relation (2. 11) is satisfied.

On the other hand, if $\overline{\alpha}$ and β are positive numbers such that $\sigma > \overline{\alpha}L_{\nabla g} + 2\beta$, then

$$\frac{2\beta\overline{\alpha}}{\sigma - \overline{\alpha}L_{\nabla g}} < \overline{\alpha}.$$

By choosing

$$\frac{2\beta\overline{\alpha}}{\sigma - \overline{\alpha}L_{\nabla g}} < \underline{\alpha} \leq \overline{\alpha},$$

relation (2. 11) is again satisfied.

Proposition 2.1.1. *In the setting of Problem 1, choose* $\underline{\alpha}, \overline{\alpha}, \beta$ *satisfying* (2. 11) *and* M_1, M_2 *satisfying* (2. 10). *Assume that* $f+g$ *is bounded from below. Then the following statements hold:*

(a) $\sum_{n\geq 1}\|x_n - x_{n-1}\|^2 < +\infty$*;*

(b) the sequence $((f+g)(x_n) + M_2\|x_{n-1} - x_n\|^2)_{n\geq 1}$ *is monotonically decreasing and convergent;*

(c) the sequence $((f+g)(x_n))_{n\in\mathbb{N}}$ *is convergent.*

Proof. For every $n \geq 1$, set $a_n = (f+g)(x_n) + M_2\|x_{n-1} - x_n\|^2$ and $b_n = (M_1 - M_2)\|x_n - x_{n+1}\|^2$. Then obviously from Lemma 2.1.5 one has for every $n \geq 1$

$$a_{n+1} + b_n = (f+g)(x_{n+1}) + M_1\|x_n - x_{n+1}\|^2 \leq (f+g)(x_n) + M_2\|x_{n-1} - x_n\|^2 = a_n.$$

The conclusion follows now from Lemma 2.1.2. □

Lemma 2.1.6. *In the setting of Problem 1, consider the sequences generated by Algorithm 1. For every $n \geq 1$ we have*

$$y_{n+1} \in \partial(f+g)(x_{n+1}), \tag{2. 14}$$

where

$$y_{n+1} = \frac{\nabla F(x_n) - \nabla F(x_{n+1})}{\alpha_n} + \nabla g(x_{n+1}) - \nabla g(x_n) + \frac{\beta_n}{\alpha_n}(x_n - x_{n-1}).$$

Moreover,

$$\|y_{n+1}\| \leq \frac{L_{\nabla F} + \alpha_n L_{\nabla g}}{\alpha_n}\|x_n - x_{n+1}\| + \frac{\beta_n}{\alpha_n}\|x_n - x_{n-1}\| \ \forall n \geq 1 \tag{2. 15}$$

Proof. Let us fix $n \geq 1$. From (2. 6) we have that

$$\frac{\nabla F(x_n) - \nabla F(x_{n+1})}{\alpha_n} - \nabla g(x_n) + \frac{\beta_n}{\alpha_n}(x_n - x_{n-1}) \in \partial f(x_{n+1}),$$

or, equivalently,

$$y_{n+1} - \nabla g(x_{n+1}) \in \partial f(x_{n+1}),$$

which shows that $y_{n+1} \in \partial(f+g)(x_{n+1})$.

The inequality (2. 15) follows now from the definition of y_{n+1} and the triangle inequality. □

Lemma 2.1.7. *In the setting of Problem 1, choose $\underline{\alpha}, \overline{\alpha}, \beta$ satisfying* (2. 11) *and M_1, M_2 satisfying* (2. 10)*. Assume that $f+g$ is coercive, i.e.*

$$\lim_{\|x\| \to +\infty} (f+g)(x) = +\infty.$$

Then any sequence $(x_n)_{n\in\mathbb{N}}$ generated by Algorithm 1 has a subsequence convergent to a critical point of $f+g$. Actually every cluster point of $(x_n)_{n\in\mathbb{N}}$ is a critical point of $f+g$.

Proof. Since $f+g$ is a proper, lower semicontinuous and coercive function, it follows that $\inf_{x\in\mathbb{R}^m}[f(x)+g(x)]$ is finite and the infimum is attained. Hence $f+g$ is bounded from below.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence generated by Algorithm 1. According to Proposition 2.1.1(b), we have

$$(f+g)(x_n) \leq (f+g)(x_n) + M_2\|x_n - x_{n-1}\|^2 \leq (f+g)(x_1) + M_2\|x_1 - x_0\|^2 \ \forall n \geq 1.$$

Since the function $f+g$ is coercive, its lower level sets are bounded, thus the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded.

Let x be a cluster point of $(x_n)_{n\in\mathbb{N}}$. Then there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $x_{n_k}\to x$ as $k\to+\infty$. We show that $(f+g)(x_{n_k})\to(f+g)(x)$ as $k\to+\infty$ and that x is a critical point of $f+g$, that is $0\in\partial(f+g)(x)$.

We show first that $f(x_{n_k})\to f(x)$ as $k\to+\infty$. Since f is lower semicontinuous one has

$$\liminf_{k\to+\infty} f(x_{n_k})\geq f(x).$$

On the other hand, from (2. 5) we have for every $n\geq 1$

$$\begin{aligned}D_F(x_{n+1},x_n)+\alpha_n\langle x_{n+1},\nabla g(x_n)\rangle+\beta_n\langle x_{n+1},x_{n-1}-x_n\rangle+\alpha_n f(x_{n+1})\leq\\ D_F(x,x_n)+\alpha_n\langle x,\nabla g(x_n)\rangle+\beta_n\langle x,x_{n-1}-x_n\rangle+\alpha_n f(x),\end{aligned}$$

which leads to

$$\begin{aligned}&\frac{1}{\alpha_{n_k-1}}\left(D_F(x_{n_k},x_{n_k-1})-D_F(x,x_{n_k-1})\right)+\\ &\frac{1}{\alpha_{n_k-1}}\left(\langle x_{n_k}-x,\alpha_{n_k-1}\nabla g(x_{n_k-1})-\beta_{n_k-1}(x_{n_k-1}-x_{n_k-2})\rangle\right)+\\ &\qquad f(x_{n_k})\leq f(x)\ \forall k\geq 2.\end{aligned}$$

The latter combined with Proposition 2.1.1(a) and (2. 4) shows that $\limsup_{k\to+\infty} f(x_{n_k})\leq f(x)$, hence $\lim_{k\to+\infty} f(x_{n_k})=f(x)$. Since g is continuous, obviously $g(x_{n_k})\to g(x)$ as $k\to+\infty$, thus $(f+g)(x_{n_k})\to(f+g)(x)$ as $k\to+\infty$.

Further, by using the notations from Lemma 2.1.6, we have $y_{n_k}\in\partial(f+g)(x_{n_k})$ for every $k\geq 2$. By Proposition 2.1.1(a) and Lemma 2.1.6 we get $y_{n_k}\to 0$ as $k\to+\infty$.

Concluding, we have:

$$y_{n_k}\in\partial(f+g)(x_{n_k})\ \forall k\geq 2,$$

$$(x_{n_k},y_{n_k})\to(x,0),\ k\to+\infty$$

$$(f+g)(x_{n_k})\to(f+g)(x),\ k\to+\infty.$$

Hence $0\in\partial(f+g)(x)$, that is, x is a critical point of $f+g$. □

Lemma 2.1.8. *In the setting of Problem 1, choose* $\underline{\alpha},\overline{\alpha},\beta$ *satisfying* (2. 11) *and* M_1,M_2

satisfying (2. 10). *Assume that $f+g$ is coercive and consider the function*

$$H : \mathbb{R}^m \times \mathbb{R}^m \to (-\infty, +\infty], H(x,y) = (f+g)(x) + M_2\|x-y\|^2 \ \forall (x,y) \in \mathbb{R}^m \times \mathbb{R}^m.$$

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence generated by Algorithm 1. Then there exist $M, N > 0$ such that the following statements hold:

(H_1) $H(x_{n+1}, x_n) + M\|x_{n+1} - x_n\|^2 \leq H(x_n, x_{n-1})$ for all $n \geq 1$;

(H_2) for all $n \geq 1$, there exists $w_{n+1} \in \partial H(x_{n+1}, x_n)$ such that $\|w_{n+1}\| \leq N(\|x_{n+1} - x_n\| + \|x_n - x_{n-1}\|)$;

(H_3) if $(x_{n_k})_{k\in\mathbb{N}}$ is a subsequence such that $x_{n_k} \to x$ as $k \to +\infty$, then $H(x_{n_k}, x_{n_k-1}) \to H(x,x)$ as $k \to +\infty$ (there exists at least one subsequence with this property).

Proof. For (H_1) just take $M = M_1 - M_2$ and the conclusion follows from Lemma 2.1.5.

Let us prove (H_2). For every $n \geq 1$ we define

$$w_{n+1} = (y_{n+1} + 2M_2(x_{n+1} - x_n), 2M_2(x_n - x_{n+1})),$$

where $(y_n)_{n\geq 2}$ is the sequence introduced in Lemma 2.1.6. The fact that $w_{n+1} \in \partial H(x_{n+1}, x_n)$ follows from Lemma 2.1.6 and the relation

$$\partial H(x,y) = \big(\partial(f+h)(x) + 2M_2(x-y)\big) \times \{2M_2(y-x)\} \ \forall (x,y) \in \mathbb{R}^m \times \mathbb{R}^m. \tag{2. 16}$$

Further, one has (see also Lemma 2.1.6)

$$\|w_{n+1}\| \leq \|y_{n+1} + 2M_2(x_{n+1} - x_n)\| + \|2M_2(x_n - x_{n+1})\| \leq$$

$$\left(\frac{L_{\nabla F}}{\alpha_n} + L_{\nabla g} + 4M_2\right)\|x_{n+1} - x_n\| + \frac{\beta_n}{\alpha_n}\|x_n - x_{n-1}\|.$$

Since $0 < \underline{\alpha} \leq \alpha_n \leq \overline{\alpha}$ and $0 \leq \beta_n \leq \beta$ for all $n \geq 1$, one can choose

$$N = \sup_{n\geq 1}\left\{\frac{L_{\nabla F}}{\alpha_n} + L_{\nabla g} + 4M_2, \frac{\beta_n}{\alpha_n}\right\} < +\infty$$

and the conclusion follows.

For (H_3), consider $(x_{n_k})_{k\in\mathbb{N}}$ a subsequence such that $x_{n_k} \to x$ as $k \to +\infty$. We have shown in the proof of Lemma 2.1.7 that $(f+g)(x_{n_k}) \to (f+g)(x)$ as $k \to +\infty$. From Proposition

2.1.1(a) and the definition of H we easily derive that $H(x_{n_k}, x_{n_k-1}) \to H(x,x) = (f+g)(x)$ as $k \to +\infty$. The existence of such a sequence follows from Lemma 2.1.7. $\square$

In the following we denote by $\omega((x_n)_{n\in\mathbb{N}})$ the set of cluster points of the sequence $(x_n)_{n\in\mathbb{N}}$.

Lemma 2.1.9. *In the setting of Problem 1, choose* $\underline{\alpha}, \overline{\alpha}, \beta$ *satisfying* (2. 11) *and* M_1, M_2 *satisfying* (2. 10). *Assume that* $f+g$ *is coercive and consider the function*

$$H : \mathbb{R}^m \times \mathbb{R}^m \to (-\infty, +\infty], H(x,y) = (f+g)(x) + M_2\|x-y\|^2 \ \forall (x,y) \in \mathbb{R}^m \times \mathbb{R}^m.$$

Let $(x_n)_{n\in\mathbb{N}}$ *be a sequence generated by Algorithm 1. Then the following statements are true:*

(a) $\omega((x_n, x_{n-1})_{n\geq 1}) \subseteq \operatorname{crit}(H) = \{(x,x) \in \mathbb{R}^m \times \mathbb{R}^m : x \in \operatorname{crit}(f+g)\}$*;*

(b) $\lim_{n\to\infty} \operatorname{dist}((x_n, x_{n-1}), \omega((x_n, x_{n-1}))_{n\geq 1}) = 0$*;*

(c) $\omega((x_n, x_{n-1})_{n\geq 1})$ *is nonempty, compact and connected;*

(d) H *is finite and constant on* $\omega((x_n, x_{n-1})_{n\geq 1})$.

Proof. (a) According to Lemma 2.1.7 and Proposition 2.1.1(a) we have $\omega((x_n, x_{n-1})_{n\geq 1}) \subseteq \{(x,x) \in \mathbb{R}^m \times \mathbb{R}^m : x \in \operatorname{crit}(f+g)\}$. The equality $\operatorname{crit}(H) = \{(x,x) \in \mathbb{R}^m \times \mathbb{R}^m : x \in \operatorname{crit}(f+g)\}$ follows from (2. 16).

(b) and (c) can be shown as in [48, Lemma 5], by also taking into consideration [48, Remark 5], where it is noticed that the properties (b) and (c) are generic for sequences satisfying $x_{n+1} - x_n \to 0$ as $n \to +\infty$.

(d) According to Proposition 2.1.1, the sequence $((f+g)(x_n))_{n\in\mathbb{N}}$ is convergent, i.e. $\lim_{n\to+\infty}(f+g)(x_n) = l \in \mathbb{R}$. Take an arbitrary $(x,x) \in \omega((x_n, x_{n-1})_{n\geq 1})$, where $x \in \operatorname{crit}(f+g)$ (we took statement (a) into consideration). From Lemma 2.1.8(H_3) it follows that there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $x_{n_k} \to x$ as $k \to +\infty$ and $H(x_{n_k}, x_{n_k-1}) \to H(x,x)$ as $k \to +\infty$. Moreover, from Proposition 2.1.1 one has $H(x,x) = \lim_{k\to+\infty} H(x_{n_k}, x_{n_k-1}) = \lim_{k\to+\infty}(f+g)(x_{n_k}) + M_2\|x_{n_k} - x_{n_k-1}\|^2 = l$ and the conclusion follows. $\square$

We give now the main result concerning the convergence of the whole sequence $(x_n)_{n\in\mathbb{N}}$.

Theorem 2.1.1. *In the setting of Problem 1, choose* $\underline{\alpha}, \overline{\alpha}, \beta$ *satisfying* (2. 11) *and* M_1, M_2 *satisfying* (2. 10). *Assume that* $f+g$ *is coercive and that*

$$H : \mathbb{R}^m \times \mathbb{R}^m \to (-\infty, +\infty], H(x,y) = (f+g)(x) + M_2\|x-y\|^2 \ \forall (x,y) \in \mathbb{R}^m \times \mathbb{R}^m$$

is a KL function. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence generated by Algorithm 1. Then the following statements are true:

(a) $\sum_{n\in\mathbb{N}} \|x_{n+1} - x_n\| < +\infty$;

(b) there exists $x \in \operatorname{crit}(f+g)$ *such that* $\lim_{n\to+\infty} x_n = x$.

Proof. (a) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence generated by Algorithm 1. According to Lemma 2.1.9 we can consider an element $\bar{x} \in \operatorname{crit}(f+g)$ such that $(\bar{x},\bar{x}) \in \omega((x_n,x_{n-1})_{n\geq 1})$. In analogy to the proof of Lemma 2.1.8 (by taking into account also the decrease property (H1)) one can easily show that $\lim_{n\to+\infty} H(x_n,x_{n-1}) = H(\bar{x},\bar{x})$. We separately treat the following two cases.

I. There exists $\bar{n} \in \mathbb{N}$ such that $H(x_{\bar{n}},x_{\bar{n}-1}) = H(\bar{x},\bar{x})$. The decrease property $(H1)$ in Lemma 2.1.8 implies $H(x_n,x_{n-1}) = H(\bar{x},\bar{x})$ for every $n \geq \bar{n}$. By using again property $(H1)$ in Lemma 2.1.8, one can show inductively that the sequence $(x_n,x_{n-1})_{n\geq\bar{n}}$ is constant. From here the conclusion follows automatically.

II. For all $n \geq 1$ we have $H(x_n,x_{n-1}) > H(\bar{x},\bar{x})$. Take $\Omega := \omega((x_n,x_{n-1})_{n\geq 1})$.

In virtue of Lemma 2.1.9(c) and (d) and Lemma 2.1.1, the KL property of H leads to the existence of positive numbers ε and η and a concave function $\varphi \in \Phi_\eta$ such that for all

$$\begin{aligned}(x,y) \in &\{(u,v) \in \mathbb{R}^m \times \mathbb{R}^m : \operatorname{dist}((u,v),\Omega) < \varepsilon\} \\ &\cap \{(u,v) \in \mathbb{R}^m \times \mathbb{R}^m : H(\bar{x},\bar{x}) < H(u,v) < H(\bar{x},\bar{x}) + \eta\}\end{aligned} \tag{2. 17}$$

one has

$$\varphi'(H(x,y) - H(\bar{x},\bar{x}))\operatorname{dist}((0,0),\partial H(x,y)) \geq 1. \tag{2. 18}$$

Let $n_1 \in \mathbb{N}$ such that $H(x_n,x_{n-1}) < H(\bar{x},\bar{x}) + \eta$ for all $n \geq n_1$. According to Lemma 2.1.9(b), there exists $n_2 \in \mathbb{N}$ such that $\operatorname{dist}((x_n,x_{n-1}),\Omega) < \varepsilon$ for all $n \geq n_2$.

Hence the sequence $(x_n,x_{n-1})_{n\geq\bar{n}}$ where $\bar{n} = \max\{n_1,n_2\}$, belongs to the intersection (2. 17). So we have (see (2. 18))

$$\varphi'(H(x_n,x_{n-1}) - H(\bar{x},\bar{x}))\operatorname{dist}((0,0),\partial H(x_n,x_{n-1})) \geq 1 \ \forall n \geq \bar{n}.$$

Since φ is concave, it holds

$$\begin{aligned}\varphi(H(x_n,x_{n-1})-H(\bar{x},\bar{x}))-\varphi(H(x_{n+1},x_n)-H(\bar{x},\bar{x}))\geq\\ \varphi'(H(x_n,x_{n-1})-H(\bar{x},\bar{x}))\cdot(H(x_n,x_{n-1})-H(x_{n+1},x_n))\geq\\ \frac{H(x_n,x_{n-1})-H(x_{n+1},x_n)}{\operatorname{dist}((0,0),\partial H(x_n,x_{n-1}))}\ \forall n\geq\bar{n}.\end{aligned}$$

Let $M,N>0$ be the real numbers furnished by Lemma 2.1.8. According to Lemma 2.1.8(H_2) there exists $w_n\in\partial H(x_n,x_{n-1})$ such that $\|w_n\|\leq N(\|x_n-x_{n-1}\|+\|x_{n-1}-x_{n-2}\|)$ for all $n\geq 2$. Then obviously $\operatorname{dist}((0,0),\partial H(x_n,x_{n-1}))\leq\|w_n\|$, hence

$$\begin{aligned}\varphi(H(x_n,x_{n-1})-H(\bar{x},\bar{x}))-\varphi(H(x_{n+1},x_n)-H(\bar{x},\bar{x}))\geq\\ \frac{H(x_n,x_{n-1})-H(x_{n+1},x_n)}{\|w_n\|}\geq\\ \frac{H(x_n,x_{n-1})-H(x_{n+1},x_n)}{N(\|x_n-x_{n-1}\|+\|x_{n-1}-x_{n-2}\|)}\ \forall n\geq\bar{n}.\end{aligned}$$

On the other hand, from Lemma 2.1.8(H_1) we obtain that

$$H(x_n,x_{n-1})-H(x_{n+1},x_n)\geq M\|x_{n+1}-x_n\|^2\ \forall n\geq 1.$$

Hence, one has

$$\begin{aligned}\varphi(H(x_n,x_{n-1})-H(\bar{x},\bar{x}))-\varphi(H(x_{n+1},x_n)-H(\bar{x},\bar{x}))\geq\\ \frac{M\|x_{n+1}-x_n\|^2}{N(\|x_n-x_{n-1}\|+\|x_{n-1}-x_{n-2}\|)}\ \forall n\geq\bar{n}.\end{aligned}$$

For all $n\geq 1$, let us denote $\frac{N}{M}(\varphi(H(x_n,x_{n-1})-H(\bar{x},\bar{x}))-\varphi(H(x_{n+1},x_n)-H(\bar{x},\bar{x})))=\varepsilon_n$ and $\|x_n-x_{n-1}\|=a_n$. Then the last inequality becomes

$$\varepsilon_n\geq\frac{a_{n+1}^2}{a_n+a_{n-1}}\ \forall n\geq\bar{n}. \tag{2. 19}$$

Obviously, since $\varphi\geq 0$, for $S\geq 1$ we have

$$\begin{aligned}\sum_{n=1}^{S}\varepsilon_n=&\frac{N}{M}(\varphi(H(x_1,x_0)-H(\bar{x},\bar{x}))-\varphi(H(x_{S+1},x_S)-H(\bar{x},\bar{x})))\\ \leq&\frac{N}{M}(\varphi(H(x_1,x_0)-H(\bar{x},\bar{x}))),\end{aligned}$$

hence $\sum_{n\geq 1}\varepsilon_n < +\infty$.

On the other hand, from (2. 19) we derive

$$a_{n+1} = \sqrt{\varepsilon_n(a_n + a_{n-1})} \leq \frac{1}{4}(a_n + a_{n-1}) + \varepsilon_n \ \forall n \geq \overline{n}.$$

Hence, according to Lemma 2.1.3, $\sum_{n\geq 1} a_n < +\infty$, that is $\sum_{n\in\mathbb{N}} \|x_n - x_{n+1}\| < +\infty$.

(b) It follows from (a) that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, hence it is convergent. Applying Lemma 2.1.7, there exists $x \in \operatorname{crit}(f+g)$ such that $\lim_{n\to+\infty} x_n = x$. □

Remark 2.1.3. As kindly pointed out by an anonymous reviewer, a similar conclusion to the one of Theorem 2.1.1 can be obtained by applying [22, Theorem 2.9] in $\mathbb{R}^m \times \mathbb{R}^m$ endowed with the Euclidean product topology for the function $\widetilde{H} : \mathbb{R}^m \times \mathbb{R}^m \to \overline{\mathbb{R}}, \widetilde{H}(x,y) = (f+g)(x) + \frac{1}{2}(M_1 + M_2)\|x-y\|^2$. Indeed, from Lemma 2.1.5 it yields

$$\widetilde{H}(x_{n+1}, x_n) + \frac{1}{2}(M_1 - M_2)(\|x_{n+1} - x_n\|^2 + \|x_n - x_{n-1}\|^2) \leq \widetilde{H}(x_n, x_{n-1}) \ \forall n \geq 1,$$

which shows that H1 in [22] is fulfilled. The assumptions H2 and H3 in the above-named article are direct consequences of (H_2) and, respectively, (H_3) in Lemma 2.1.8. Under these premises, provided that $\widetilde{H}$ is a KL function, one obtains via [22, Theorem 2.9] the same conclusion as in Theorem 2.1.1.

However, the hypothesis that H is a KL function, as assumed in Theorem 2.1.1, is in our opinion in this context the most natural one, at least in what concerns the way in which it approaches the non-inertial case. Indeed, if β is equal to zero, then M_2 is equal to zero, too, and the conclusion of Theorem 2.1.1 follows by only assuming that $f+g$ is a KL function. On the other hand, in order to apply [22, Theorem 2.9], one would ask that $(x,y) \mapsto (f+g)(x) + \frac{1}{2}M_1\|x-y\|^2$ is a KL function, which is in general a stronger assumption.

Since the class of semi-algebraic functions is closed under addition (see for example [48]) and $(x,y) \mapsto c\|x-y\|^2$ is semi-algebraic for $c > 0$, we obtain also the following direct consequence.

Corollary 2.1.1. *In the setting of Problem 1, choose* $\underline{\alpha}, \overline{\alpha}, \beta$ *satisfying* (2. 11) *and* M_1, M_2 *satisfying* (2. 10). *Assume that* $f+g$ *is coercive and semi-algebraic. Let* $(x_n)_{n\in\mathbb{N}}$ *be a sequence generated by Algorithm 1. Then the following statements are true:*

(a) $\sum_{n\in\mathbb{N}} \|x_{n+1} - x_n\| < +\infty$;

(b) there exists $x \in \operatorname{crit}(f+g)$ *such that* $\lim_{n\to+\infty} x_n = x$.

Remark 2.1.4. As one can notice by taking a closer look at the proof of Lemma 2.1.7, the conclusion of this statement as the ones of Lemma 2.1.8, Lemma 2.1.9, Theorem 2.1.1 and Corollary 2.1.1 remain true, if instead of imposing that $f+g$ is coercive, we assume that $f+g$ is bounded from below and the sequence $(x_n)_{n\in\mathbb{N}}$ generated by Algorithm 1 is bounded. This observation is useful when dealing with optimization problems as the ones considered in Subsection 2.1.3.

2.1.3 Numerical experiments

This section is devoted to the presentation of two numerical experiments which illustrate the applicability of the algorithm proposed in this work. In both numerical experiments we considered $F = \frac{1}{2}\|\cdot\|^2$ and set $\sigma = 1$.

Detecting minimizers of nonconvex optimization problems

As emphasized in [177, Section 5.1] and [41, Exercise 1.3.9] one of the aspects which makes algorithms with inertial/memory effects useful is given by the fact that they are able to detect optimal solutions of minimization problems which cannot be found by their non-inertial variants. In this subsection we show that this phenomenon arises even when solving problems of type (2. 20), where the nonsmooth function f is nonconvex. A similar situation has been addressed in [177], however, by assuming that f is convex.

Consider the optimization problem

$$\inf_{(x_1,x_2)\in\mathbb{R}^2} |x_1| - |x_2| + x_1^2 - \log(1+x_1^2) + x_2^2. \tag{2. 20}$$

The function

$$f : \mathbb{R}^2 \to \mathbb{R}, f(x_1,x_2) = |x_1| - |x_2|,$$

is nonconvex and continuous, the function

$$g : \mathbb{R}^2 \to \mathbb{R}, g(x_1,x_2) = x_1^2 - \log(1+x_1^2) + x_2^2,$$

is continuously differentiable with Lipschitz continuous gradient with Lipschitz constant $L_{\nabla g} = 9/4$ and one can easily prove that $f+g$ is coercive. Furthermore, combining [21, the remarks after Definition 4.1], [46, Remark 5(iii)] and [48, Section 5: Example 4 and Theorem 3], one can easily conclude that H in Theorem 2.1.1 is a KL function. By considering the

first order optimality conditions

$$-\nabla g(x_1,x_2) \in \partial f(x_1,x_2) = \partial(|\cdot|)(x_1) \times \partial(-|\cdot|)(x_2)$$

and by noticing that for all $x \in \mathbb{R}$ we have

$$\partial(|\cdot|)(x) = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \\ \text{[-1,1]}, & \text{if } x = 0 \end{cases}$$

and

$$\partial(-|\cdot|)(x) = \begin{cases} -1, & \text{if } x > 0, \\ 1, & \text{if } x < 0, \\ \{-1,1\}, & \text{if } x = 0, \end{cases}$$

(for the latter, see for example [161]), one can easily determine the two critical points $(0,1/2)$ and $(0,-1/2)$ of (2. 20), which are actually both optimal solutions of this minimization problem. In Figure 2.2 the level sets and the graph of the objective function in (2. 20) are represented.

For $\gamma > 0$ and $x = (x_1,x_2) \in \mathbb{R}^2$ we have (see Remark 2.1.1)

$$\text{prox}_{\gamma f}(x) = \underset{u \in \mathbb{R}^2}{\text{argmin}} \left\{ \frac{\|u-x\|^2}{2\gamma} + f(u) \right\} =$$

$$\text{prox}_{\gamma|\cdot|}(x_1) \times \text{prox}_{\gamma(-|\cdot|)}(x_2),$$

where in the first component one has the well-known shrinkage operator

$$\text{prox}_{\gamma|\cdot|}(x_1) = x_1 - \text{sgn}(x_1) \cdot \min\{|x_1|, \gamma\},$$

while for the proximal operator in the second component the following formula can be proven

$$\text{prox}_{\gamma(-|\cdot|)}(x_2) = \begin{cases} x_2 + \gamma, & \text{if } x_2 > 0 \\ x_2 - \gamma, & \text{if } x_2 < 0 \\ \{-\gamma, \gamma\}, & \text{if } x_2 = 0. \end{cases}$$

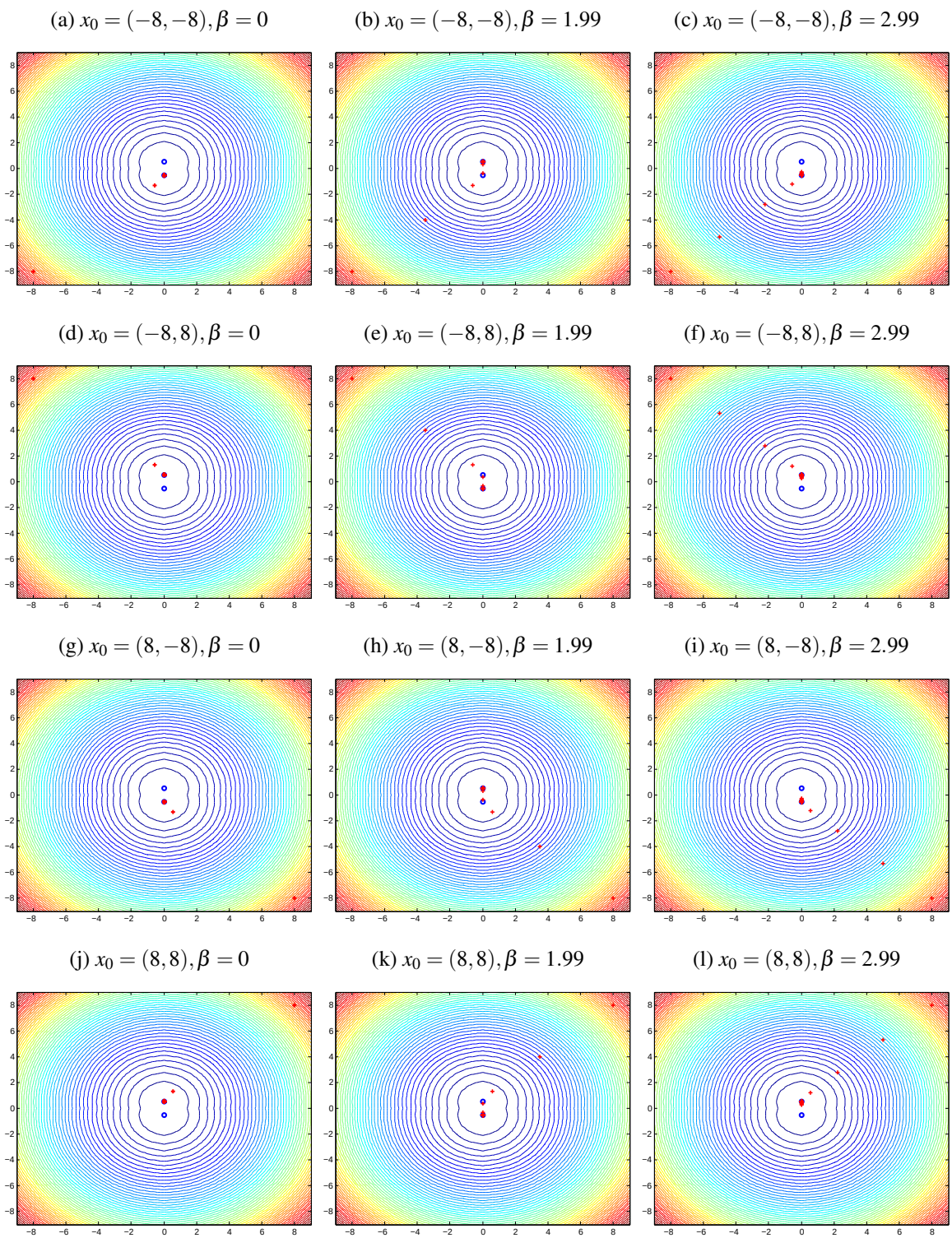

Figure 2.1: Algorithm 1 after 100 iterations and with starting points $(-8,-8), (-8,8), (8,-8)$ and $(8,8)$, respectively: the first column shows the iterates of the non-inertial version ($\beta_n = \beta = 0$ for all $n \geq 1$), the second column the ones of the inertial version with $\beta_n = \beta = 1.99$ for all $n \geq 1$ and the third column the ones of the inertial version with $\beta_n = \beta = 2.99$ for all $n \geq 1$.

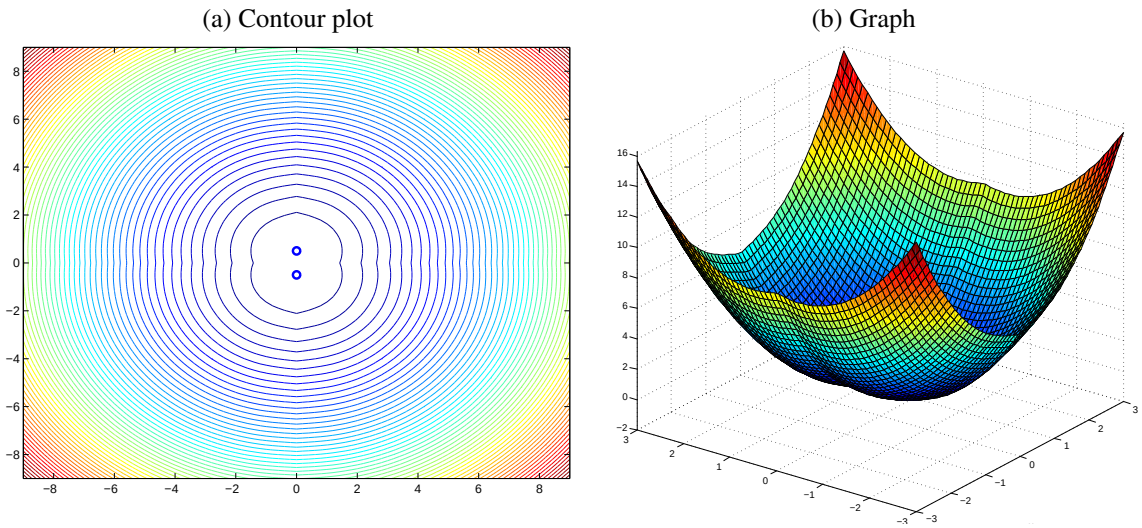

Figure 2.2: Contour plot and graph of the objective function in (2. 20). The two global optimal solutions $(0, 0.5)$ and $(0, -0.5)$ are marked on the first image.

We implemented Algorithm 1 by choosing $\beta_n = \beta = 0$ for all $n \geq 1$ (which corresponds to the non-inertial version), $\beta_n = \beta = 0.199$ for all $n \geq 1$ and $\beta_n = \beta = 0.299$ for all $n \geq 1$, respectively, and by setting $\alpha_n = (0.99999 - 2\beta_n)/L_{\nabla g}$ for all $n \geq 1$. As starting points we considered the corners of the box generated by the points $(\pm 8, \pm 8)$. Figure 2.1 shows that independently of the four starting points we have the following phenomenon: the non-inertial version recovers only one of the two optimal solutions, situation which persists even when changing the value of α_n; on the other hand, the inertial version is capable to find both optimal solutions, namely, one for $\beta = 0.199$ and the other one for $\beta = 0.299$.

Restoration of noisy blurred images

The following numerical experiment concerns the restoration of a noisy blurred image by using a nonconvex misfit functional with nonconvex regularization. For a given matrix $A \in \mathbb{R}^{m \times m}$ describing a blur operator and a given vector $b \in \mathbb{R}^m$ representing the blurred and noisy image, the task is to estimate the unknown original image $\bar{x} \in \mathbb{R}^m$ fulfilling

$$A\bar{x} = b.$$

To this end we solve the following regularized nonconvex minimization problem

$$\inf_{x \in \mathbb{R}^m} \left\{ \sum_{k=1}^{M} \sum_{l=1}^{N} \varphi\big((Ax - b)_{kl}\big) + \lambda \|Wx\|_0 \right\}, \tag{2. 21}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(t) = \log(1+t^2)$, is derived form the Student t distribution, $\lambda > 0$ is a regularization parameter, $W : \mathbb{R}^m \to \mathbb{R}^m$ is a discrete Haar wavelet transform with four levels and $\|y\|_0 = \sum_{i=1}^m |y_i|_0$ ($|\cdot|_0 = |\operatorname{sgn}(\cdot)|$) furnishes the number of nonzero entries of the vector $y = (y_1, ..., y_m) \in \mathbb{R}^m$. In this context, $x \in \mathbb{R}^m$ represents the vectorized image $X \in \mathbb{R}^{M\times N}$, where $m = M \cdot N$ and $x_{i,j}$ denotes the normalized value of the pixel located in the i-th row and the j-th column, for $i = 1, \ldots, M$ and $j = 1, \ldots, N$. Again, by combining [21, the remarks after Definition 4.1], [46, Remark 5(iii)] and [48, Section 5: Example 3, Example 4 and Theorem 3], one can conclude that H in Theorem 2.1.1 is a KL function. It is immediate that (2. 21) can be written in the form (2. 3), by defining $f(x) = \lambda \|Wx\|_0$ and $g(x) = \sum_{k=1}^M \sum_{l=1}^N \varphi\big((Ax-b)_{kl}\big)$ for all $x \in \mathbb{R}^m$. By using that $WW^* = W^*W = I_m$, one can prove the following formula concerning the proximal operator of f

$$\operatorname{prox}_{\gamma f}(x) = W^* \operatorname{prox}_{\lambda\gamma\|\cdot\|_0}(Wx) \ \forall x \in \mathbb{R}^m \ \forall \gamma > 0,$$

where for all $u = (u_1, ..., u_m)$ we have (see [22, Example 5.4(a)])

$$\operatorname{prox}_{\lambda\gamma\|\cdot\|_0}(u) = (\operatorname{prox}_{\lambda\gamma|\cdot|_0}(u_1), ..., \operatorname{prox}_{\lambda\gamma|\cdot|_0}(u_m))$$

and for all $t \in \mathbb{R}$

$$\operatorname{prox}_{\lambda\gamma|\cdot|_0}(t) = \begin{cases} t, & \text{if } |t| > \sqrt{2\lambda\gamma}, \\ \{0,t\}, & \text{if } |t| = \sqrt{2\lambda\gamma}, \\ 0, & \text{otherwise.} \end{cases}$$

For the experiments we used the 256×256 boat test image which we first blurred by using a Gaussian blur operator of size 9×9 and standard deviation 4 and to which we afterward added a zero-mean white Gaussian noise with standard deviation 10^{-6}. In the first row of Figure 2.3 the original boat test image and the blurred and noisy one are represented, while in the second row one has the reconstructed images by means of the non-inertial (for $\beta_n = \beta = 0$ for all $n \geq 1$) and inertial versions (for $\beta_n = \beta = 10^{-7}$ for all $n \geq 1$) of Algorithm 1, respectively. We took as regularization parameter $\lambda = 10^{-5}$ and set $\alpha_n = (0.999999 - 2\beta_n)/L_{\nabla g}$ for all $n \geq 1$, whereby the Lipschitz constant of the gradient of the smooth misfit function is $L_{\nabla g} = 2$.

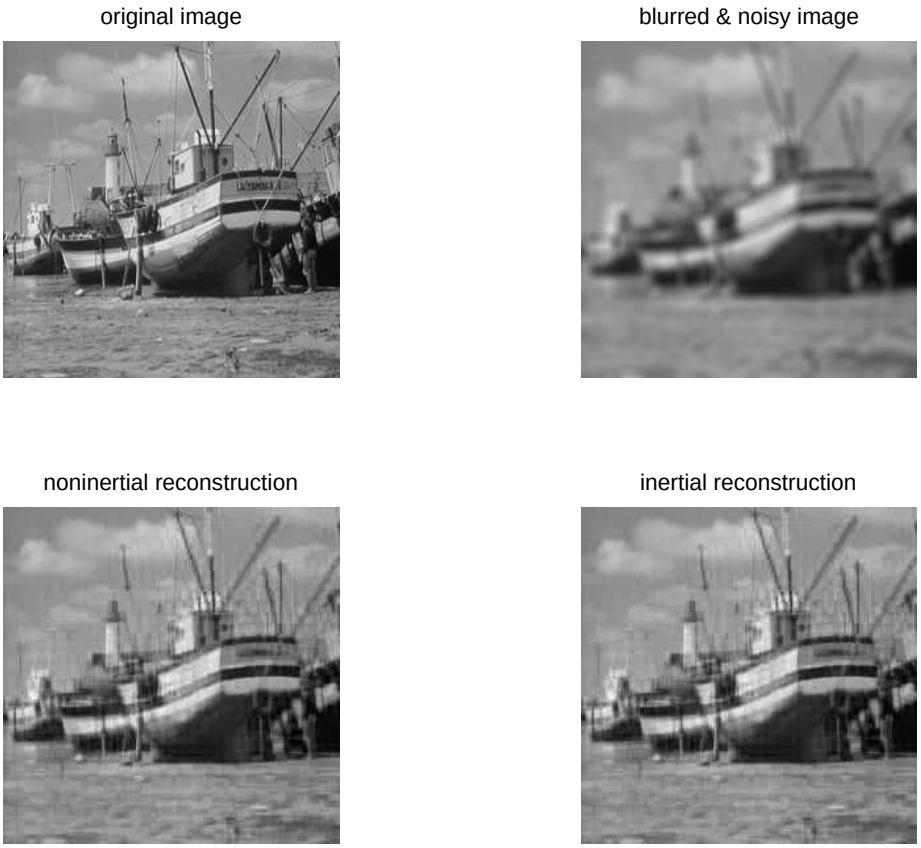

Figure 2.3: The first row shows the original 256×256 boat test image and the blurred and noisy one and the second row the reconstructed images after 300 iterations.

We compared the quality of the recovered images for $\beta_n = \beta$ for all $n \geq 1$ and different values of β by making use of the improvement in signal-to-noise ratio (ISNR), which is defined as

$$\mathrm{ISNR}(n) = 10\log_{10}\left(\frac{\|x-b\|^2}{\|x-x_n\|^2}\right),$$

where x, b and x_n denote the original, observed and estimated image at iteration n, respectively.

In Table 2.1 we list the values of the ISNR-function after 300 iterations, whereby the case $\beta = 0$ corresponds to the non-inertial version of the algorithm. One can notice that for β taking very small values, the inertial version is competitive with the non-inertial one.

Concluding remarks

In this section we proposed a forward-backward proximal-type algorithm with inertial/memory effects for minimizing the sum of a nonsmooth with a smooth function in the nonconvex

β	0.4	0.2	0.01	0.0001	10^{-7}	0
ISNR(300)	2.081946	3.101028	3.492989	3.499428	3.511135	3.511134

Table 2.1: The ISNR values after 300 iterations for different choices of β.

setting. Every sequence of iterates generated by the algorithm is proved to converge to a critical point of the objective function, whenever an appropriate regularization of the latter satisfies the Kurdyka-Łojasiewicz inequality. In this way we extend to the full nonconvex setting the inertial forward-backward type algorithm proposed in [177] for minimizing the sum of a nonsmooth convex with a smooth (not necessarily convex) function.

As it is the case for the particular instances considered in Section 2.1.3, very tight bounds for the parameters used in the iterative scheme are needed. More than that, for these particular instances, there is a minimal difference between the inertial and non-inertial schemes.

In the context of proving convergence for algorithms designed to solve nonsmooth optimization problems with KL functions two approaches can be found in the literature. One of them is the approach proposed in [48], which we also follow in our manuscript, while the second one was used in [22]. As explained in Remark 2.1.3, the two approaches mainly differ in the way the regularization of the objective is constructed. Opting for the approach in [22], one could come to the conclusion by using in a straightforward way the statements of Lemma 2.1.8. However, different to [22], the inertial and non-inertial schemes are treated with our choice of H in an unitary way. Furthermore, in the inertial case, working with H does not assume to have any information about $L_{\nabla g}$, a constant which explicitly appears in the definition of M_1 (see Remark 2.1.3). On the other hand, for the choice of $\underline{\alpha}, \overline{\alpha}$ and β and, consequently, for the defintion of M_2, the Lipschitz constant $L_{\nabla g}$ can be unknown if an upper bound $L > L_{\nabla g}$ is available.

2.2 On the sum of a smooth convex and a nonsmooth convex function

In this section we are concerned with addressing monotone inclusion problem from the perspective of dynamical systems. More precisely, we associate to this constrained variational inequality a second-order dynamical system formulated in terms of the resolvent of the maximal monotone operator A. We emphasize, that in this setting, the minimization of the sum of two convex functions becomes a particular instance.

In the first part of the section we study the existence and uniqueness of (locally) absolutely

continuous trajectories generated by the dynamical system, by appealing to arguments based on the Cauchy-Lipschitz-Picard Theorem (see [108, 198]). In the second part of the section we investigate the convergence of the trajectories to a solution of the constrained variational inequality. We use as tools Lyapunov analysis combined with the continuous version of the Opial Lemma. Under the fulfillment of a condition expressed in terms of the Fitzpatrick function of the cocoercive operator B we are able to show ergodic weak convergence of the orbits. Moreover, if the operator A is strongly monotone, we can prove even strong (non-ergodic) convergence for the generated trajectories.

The results of this section have been published in [63]:[R.I. Boţ, E.R. Csetnek, S. László, *Second order dynamical systems with penalty terms associated to monotone inclusions*].

2.2.1 Minimization via a second order dynamical system

Consider the bilevel optimization problem

$$\inf_{x\in\operatorname{argmin}\psi}\{f(x)+g(x)\}, \tag{2. 22}$$

where $f:\mathcal{H}\longrightarrow\mathbb{R}\cup\{+\infty\}$ is a proper convex and lower semicontinuous function, $g,\psi:\mathcal{H}\longrightarrow\mathbb{R}$ are convex and (Fréchet) differentiable functions both with Lipschitz continuous gradients, and $\operatorname{argmin}\psi$ denotes the set of global minimizers of ψ, assumed to be nonempty.

By making use of the indicator function of $\operatorname{argmin}\psi$, (2. 22) can be rewritten as

$$\inf_{x\in\mathcal{H}}\{f(x)+g(x)+\delta_{\operatorname{argmin}\psi}(x)\}. \tag{2. 23}$$

Obviously, $x\in\operatorname{argmin}\psi$ is an optimal solution of (2.23) if and only if

$$0\in\partial(f+g+\delta_{\operatorname{argmin}\psi})(x),$$

which can be split in

$$0\in\partial f(x)+\nabla g(x)+\partial\delta_{\operatorname{argmin}\psi}(x), \tag{2. 24}$$

provided a suitable qualification condition which guarantees the subdifferential sum rule holds.

Using that $\partial\delta_{\operatorname{argmin}\psi}(x)=N_{\operatorname{argmin}\psi}(x)$ and $\operatorname{argmin}\psi=\operatorname{zer}\nabla\psi$, (2. 24) is nothing else

than

$$0 \in \partial f(x) + \nabla g(x) + N_{\operatorname{zer}\nabla\psi}(x). \tag{2. 25}$$

This motivates us to investigate the following inclusion problem

$$0 \in Ax + Dx + N_C(x), \tag{2. 26}$$

where $A : \mathcal{H} \rightrightarrows \mathcal{H}$ is a maximally monotone operator, $D : \mathcal{H} \longrightarrow \mathcal{H}$ is a L_D^{-1}-cocoercive operator, and $B : \mathcal{H} \longrightarrow \mathcal{H}$ is a L_B^{-1}-cocoercive operator with $L_D, L_B > 0$, $C = \operatorname{zer} B$, and N_C denotes the normal cone operator of the set C. We recall that by the classical Baillon-Haddad Theorem, the gradient of a convex and (Fréchet) differentiable function is L-Lipschitz continuous, for $L > 0$, if and only if it is L^{-1}-cocoercive, see for instance [36, Corollary 18.16]).

In [52], a first order dynamical system has been assigned to the monotone inclusion (2.26), and it has been shown that the generated trajectories converge to a solution of it. In this paper, we assign to (2.26) the following second order dynamical system

$$\begin{cases} \ddot{x}(t) + \gamma(t)\dot{x}(t) + x(t) = J_{\lambda(t)A}\big(x(t) - \lambda(t)D(x(t)) - \lambda(t)\beta(t)B(x(t))\big) \\ x(0) = u_0, \dot{x}(0) = v_0, \end{cases} \tag{2. 27}$$

where $u_0, v_0 \in \mathcal{H}$ and $\gamma, \lambda, \beta : [0, +\infty) \longrightarrow (0, +\infty)$. Dynamical systems governed by resolvents of maximally monotone operators have been considered in [1, 2], and then further developed in [54, 56].

The study of second order dynamical systems is motivated by the fact that the presence of the acceleration term $\ddot{x}(t)$ can lead to better convergence properties of the trajectories. Time discretizations of second order dynamical systems give usually rise to numerical algorithms with inertial terms which have been shown to have improved convergence properties (see [164]). The geometric damping function γ which acts on the velocity can in some situations accelerate the asymptotic properties of the orbits, as emphasized for example in [200].

For $B = 0$ and λ is constant, the differential equation (2. 27) becomes the second order forward-backward dynamical system investigated in [56] in relation to the monotone inclusion problem

$$0 \in Ax + Dx.$$

On the other hand, when particularized to the monotone inclusion system (2. 25), the differ-

ential equation (2. 27) reads

$$\text{(2. 28)} \quad \begin{cases} \ddot{x}(t)+\gamma(t)\dot{x}(t)+x(t)=\operatorname{prox}_{\lambda(t)f}\big(x(t)-\lambda(t)\nabla g(x(t))-\lambda(t)\beta(t)\nabla\psi(x(t))\big) \\ x(0)=u_0, \dot{x}(0)=v_0, \end{cases}$$

where we made used of the fact that the resolvent of the subdifferential of a proper, convex and lower semicontinuous function is the proximal point operator of the latter. In case $f=0$, γ is constant and λ is also constant and identical to 1, (2. 28) leads to the differential equation that has been investigated in [25] and [55].

The first part of the section is devoted to the proof of the existence and uniqueness of (locally) absolutely continuous trajectories generated by the dynamical system (2. 27); an important ingredient for this analysis is the Cauchy-Lipschitz-Picard Theorem (see [108, 198]). The proof of the convergence of the trajectories to a solution of (2. 26) is the main result of the section. Provided that a condition expressed in terms of the Fitzpatrick function of the cocoercive operator B is fulfilled, we prove weak ergodic convergence of the orbits. Furthermore, we show that, if the operator A is strongly monotone, then one obtains even strong (non-ergodic) convergence for the generated trajectories.

In the remaining of this paragraph, we explain the notations we used up to this point and will use throughout the section (see [36, 50, 196]).

The real Hilbert space $\mathcal{H}$ is endowed with *inner product* $\langle\cdot,\cdot\rangle$ and associated *norm* $\|\cdot\|=\sqrt{\langle\cdot,\cdot\rangle}$. The *normal cone* of of a set $S\subseteq\mathcal{H}$ is defined by $N_S(x)=\{u\in\mathcal{H}:\langle y-x,u\rangle\leq 0\ \forall y\in S\}$, if $x\in S$ and $N_S(x)=\emptyset$ for $x\notin S$. The following characterization of the elements of the normal cone of a nonempty set by means of its support function will be used several times in the section: for $x\in S$, $u\in N_S(x)$ if and only if $\sigma_S(u)=\langle x,u\rangle$, where $\sigma_S:\mathcal{H}\to\mathbb{R}\cup\{+\infty\}$ is defined by $\sigma_S(u)=\sup_{y\in S}\langle y,u\rangle$.

Let $A:\mathcal{H}\rightrightarrows\mathcal{H}$ be a set-valued operator. We denote by $\operatorname{Gr}A=\{(x,u)\in\mathcal{H}\times\mathcal{H}:u\in Ax\}$ its *graph*, by $\operatorname{dom}A=\{x\in\mathcal{H}:Ax\neq\emptyset\}$ its *domain* and by $\operatorname{ran}A=\{u\in\mathcal{H}:\exists x\in\mathcal{H}\text{ s.t. }u\in Ax\}$ its *range*. The notation $\operatorname{zer}A=\{x\in\mathcal{H}:0\in Ax\}$ stands for the *set of zeros* of the operator A. We say that A is *monotone* if $\langle x-y,u-v\rangle\geq 0$ for all $(x,u),(y,v)\in\operatorname{Gr}A$. Further, a monotone operator A is said to be *maximally monotone*, if there exists no proper monotone extension of the graph of A on $\mathcal{H}\times\mathcal{H}$. The following characterization of the zeros of a maximally monotone operator will be crucial in the asymptotic analysis of (2. 27): if A is maximally monotone, then

$$z\in\operatorname{zer}A \text{ if and only if } \langle u-z,w\rangle\geq 0 \text{ for all } (u,w)\in\operatorname{Gr}A.$$

The *resolvent* of A, $J_A : \mathcal{H} \rightrightarrows \mathcal{H}$, is defined by $p \in J_A(x)$ if and only if $x \in p + Ap$. Moreover, if A is maximally monotone, then $J_A : \mathcal{H} \to \mathcal{H}$ is single-valued and maximally monotone (cf. [36, Proposition 23.7 and Corollary 23.10]). We will also use the *Yosida approximation* of the operator A, which is defined for $\alpha > 0$ by $A_\alpha = \frac{1}{\alpha}(\mathrm{id} - J_{\alpha A})$, where $\mathrm{id} : \mathcal{H} \to \mathcal{H}, \mathrm{id}(x) = x$ for all $x \in \mathcal{H}$, is the *identity operator* on $\mathcal{H}$.

The notion of *Fitzpatrick function* associated to a monotone operator A will be important in the formulation of the condition under which the convergence of the trajectories is achieved. It is defined as

$$\varphi_A : \mathcal{H} \times \mathcal{H} \to \overline{\mathbb{R}},\ \varphi_A(x,u) = \sup_{(y,v) \in \mathrm{Gr}A} \{\langle x, v\rangle + \langle y, u\rangle - \langle y, v\rangle\},$$

and it is a convex and lower semicontinuous function. Introduced by Fitzpatrick in [93], this notion played in the last years a crucial role in the investigation of maximality of monotone operators by means of convex analysis specific tools (see [36, 37, 50, 64, 136, 138, 142, 196] and the references therein). We notice that, if A is maximally monotone, then φ_A is proper and

$$\varphi_A(x,u) \geq \langle x, u\rangle \ \forall (x,u) \in \mathcal{H} \times \mathcal{H},$$

with equality if and only if $(x,u) \in \mathrm{Gr}A$. We refer the reader to [37] for explicit formulae of Fitzpatrick functions associated to particular classes of monotone operators.

Let $\gamma > 0$ be arbitrary. A single-valued operator $A : \mathcal{H} \to \mathcal{H}$ is said to be *γ-cocoercive*, if $\langle x - y, Ax - Ay\rangle \geq \gamma \|Ax - Ay\|^2$ for all $(x,y) \in \mathcal{H} \times \mathcal{H}$, and *$\gamma$-Lipschitz continuous*, if $\|Ax - Ay\| \leq \gamma \|x - y\|$ for all $(x,y) \in \mathcal{H} \times \mathcal{H}$.

For a proper, convex and lower semicontinuous function $f : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$, its (convex) subdifferential at $x \in \mathcal{H}$ is defined as

$$\partial f(x) = \{u \in \mathcal{H} : f(y) \geq f(x) + \langle u, y - x\rangle \ \forall y \in \mathcal{H}\}.$$

When seen as a set-valued mapping, it is a maximally monotone operator and its resolvent is given by $J_{\gamma \partial f} = \mathrm{prox}_{\gamma f}$ (see [36]), where $\mathrm{prox}_{\gamma f} : \mathcal{H} \to \mathcal{H}$,

$$\mathrm{prox}_{\gamma f}(x) = \underset{y \in \mathcal{H}}{\mathrm{argmin}} \left\{ f(y) + \frac{1}{2\gamma} \|y - x\|^2 \right\}, \tag{2. 29}$$

denotes the proximal point operator of f.

2.2.2 Existence and uniqueness of the trajectory

We start by specifying which type of solutions are we considering in the analysis of the dynamical system (2. 27).

Definition 2.2.1. We say that $x : [0,+\infty) \to \mathcal{H}$ is a strong global solution of (2. 27), if the following properties are satisfied:

(i) $x, \dot{x} : [0,+\infty) \to \mathcal{H}$ are locally absolutely continuous, in other words, absolutely continuous on each interval $[0,b]$ for $0 < b < +\infty$;

(ii) $\ddot{x}(t) + \gamma(t)\dot{x}(t) + \big(x(t) - J_{\lambda(t)A}(x(t) - \lambda(t)D(x(t)) - \lambda(t)\beta(t)B(x(t)))\big) = 0$ for almost every $t \geq 0$;

(iii) $x(0) = u_0$ and $\dot{x}(0) = v_0$.

For proving existence and uniqueness of the strong global solutions of (2. 27), we use the Cauchy-Lipschitz-Picard Theorem for absolutely continues trajectories (see for example [108, Proposition 6.2.1], [198, Theorem 54]). The key argument is that one can rewrite (2. 27) as a particular first order dynamical system in a suitably chosen product space (see also [6]).

To this end we make the following assumption:

$$(H1): \ \gamma, \lambda, \beta : [0,+\infty) \longrightarrow (0,+\infty) \text{ are continuous on each interval } [0,b], \text{ for } 0 < b < +\infty,$$

which also describes the framework in which we will carry out the convergence analysis in the forthcoming subsections.

Theorem 2.2.1. *Suppose that γ, λ and β satisfy $(H1)$. Then for every $u_0, v_0 \in \mathcal{H}$ there exists a unique strong global solution of* (2.27).

Proof. Define $X : [0,+\infty) \longrightarrow \mathcal{H} \times \mathcal{H}$ as $X(t) = (x(t), \dot{x}(t))$. Then (2. 27) is equivalent to

$$\begin{cases} \dot{X}(t) = F(t, X(t)) \\ X(0) = (u_0, v_0), \end{cases} \tag{2. 30}$$

where $F(t,u,v) = \big(v, -\gamma(t)v - u + J_{\lambda(t)A}(u - \lambda(t)D(u) - \lambda(t)\beta(t)B(u))\big)$.

First we show that $F(t,\cdot,\cdot)$ is Lipschitz continuous with a Lipschitz constant $L(t) \in L^1_{loc}([0,+\infty))$, for every $t \geq 0$. Indeed,

$$\|F(t,u,v) - F(t,\bar{u},\bar{v})\| = \sqrt{\|v-\bar{v}\|^2 + \|\gamma(t)(\bar{v}-v) + (\bar{u}-u) + (J_{\lambda(t)A}(s) - J_{\lambda(t)A}(\bar{s}))\|^2}$$

$$\leq \sqrt{\|v-\bar{v}\|^2 + 2\|\gamma(t)(\bar{v}-v) + (\bar{u}-u)\|^2 + 2\|J_{\lambda(t)A}(s) - J_{\lambda(t)A}(\bar{s})\|^2}$$

$$\leq \sqrt{(1+4\gamma^2(t))\|v-\bar{v}\|^2+4\|(\bar{u}-u)\|^2+2\|J_{\lambda(t)A}(s)-J_{\lambda(t)A}(\bar{s})\|^2},$$

where $s = u-\lambda(t)D(u)-\lambda(t)\beta(t)B(u)$ and $\bar{s} = \bar{u}-\lambda(t)D(\bar{u})-\lambda(t)\beta(t)B(\bar{u})$.

By using the nonexpansivity of $J_{\lambda(t)A}$ we get

$$\|J_{\lambda(t)A}(s)-J_{\lambda(t)A}(\bar{s})\| \leq \|(u-\bar{u})+\lambda(t)(D(\bar{u})-D(u))+\lambda(t)\beta(t)(B(\bar{u})-B(u)\|$$

$$\leq (1+\lambda(t)L_D+\lambda(t)\beta(t)L_B)\|u-\bar{u}\|.$$

Hence,

$$\|F(t,u,v)-F(t,\bar{u},\bar{v})\| \leq$$

$$\leq \sqrt{(1+4\gamma^2(t))\|v-\bar{v}\|^2+(4+2(1+\lambda(t)L_D+\lambda(t)\beta(t)L_B)^2)\|u-\bar{u}\|^2}$$

$$\leq \sqrt{5+4\gamma^2(t)+2(1+\lambda(t)L_D+\lambda(t)\beta(t)L_B)^2}\sqrt{\|u-\bar{u}\|^2+\|v-\bar{v}\|^2}$$

$$\leq (\sqrt{5}+2\gamma(t)+\sqrt{2}(1+\lambda(t)L_D+\lambda(t)\beta(t)L_B))\|(u,v)-(\bar{u},\bar{v})\|.$$

Since $\gamma,\lambda,\beta \in L^1_{loc}([0,+\infty))$, it follows that

$$L(t) := \sqrt{5}+2\gamma(t)+\sqrt{2}(1+\lambda(t)L_D+\lambda(t)\beta(t)L_B)$$

is also locally integrable on $[0,+\infty)$.

Next we show that $F(\cdot,u,v) \in L^1_{loc}([0,+\infty),\mathcal{H}\times\mathcal{H})$ for all $u,v \in \mathcal{H}$. We fix $u,v \in \mathcal{H}$ and $b > 0$, and notice that

$$\int_0^b \|F(t,u,v)\|dt = \int_0^b \sqrt{\|v\|^2+\|\gamma(t)v+u-J_{\lambda(t)A}(u-\lambda(t)D(u)-\lambda(t)\beta(t)B(u))\|^2}dt$$

$$\leq \int_0^b \sqrt{(1+2\gamma^2(t))\|v\|^2+4\|u\|^2+4\|J_{\lambda(t)A}(u-\lambda(t)D(u)-\lambda(t)\beta(t)B(u))\|^2}dt.$$

According to $(H1)$, there exist positive numbers $\underline{\lambda}$ and $\underline{\beta}$ such that $0 < \underline{\lambda} \leq \lambda(t)$ and $0 < \underline{\beta} \leq \beta(t)$ for all $t \in [0,b]$. Hence

$$\|J_{\lambda(t)A}(u-\lambda(t)D(u)-\lambda(t)\beta(t)B(u))\| =$$

$$\|J_{\lambda(t)A}(u-\lambda(t)D(u)-\lambda(t)\beta(t)B(u))-J_{\lambda(t)A}(u-\underline{\lambda}D(u)-\underline{\lambda}\underline{\beta}B(u))+$$

$$+J_{\lambda(t)A}(u-\underline{\lambda}D(u)-\underline{\lambda}\underline{\beta}B(u))\|$$

$$\leq (\lambda(t)-\underline{\lambda})\|D(u)\|+(\lambda(t)\beta(t)-\underline{\lambda}\underline{\beta})\|B(u)\|+\|J_{\lambda(t)A}(u-\underline{\lambda}D(u)-\underline{\lambda}\underline{\beta}B(u))\|.$$

In addition,

$$\|J_{\lambda(t)A}(u-\underline{\lambda}D(u)-\underline{\lambda}\underline{\beta}B(u))\| =$$

$$\|J_{\lambda(t)A}(u-\underline{\lambda}D(u)-\underline{\lambda}\underline{\beta}B(u))-J_{\underline{\lambda}A}(u-\underline{\lambda}D(u)-\underline{\lambda}\underline{\beta}B(u))+J_{\underline{\lambda}A}(u-\underline{\lambda}D(u)-\underline{\lambda}\underline{\beta}B(u))\|$$

$$\leq (\lambda(t)-\underline{\lambda})\|A_{\underline{\lambda}}(u-\underline{\lambda}D(u)-\underline{\lambda}\underline{\beta}B(u))\|+\|J_{\underline{\lambda}A}(u-\underline{\lambda}D(u)-\underline{\lambda}\underline{\beta}B(u))\|,$$

where the last inequality follows from the Lipschitz property of the resolvent operator as a function of the step size, which basically follows by combining [66, Proposition 2.6] and [36, Proposition 23.28] (see also [2, Proposition 3.1]). Hence,

$$\int_0^b \|F(t,u,v)\|dt \leq$$

$$\int_0^b \Big((1+\sqrt{2}\gamma(t))\|v\|+2\|u\|+2(\lambda(t)-\underline{\lambda})\|D(u)\|+2(\lambda(t)\beta(t)-\underline{\lambda}\underline{\beta})\|B(u)\|\Big)\,dt$$

$$+\int_0^b \Big(2(\lambda(t)-\underline{\lambda})\|A_{\underline{\lambda}}(u-\underline{\lambda}D(u)-\underline{\lambda}\underline{\beta}B(u))\|+2\|J_{\underline{\lambda}A}(u-\underline{\lambda}D(u)-\underline{\lambda}\underline{\beta}B(u))\|\Big)\,dt.$$

Hence, $F(\cdot,u,v)\in L^1_{loc}([0,+\infty),\mathcal{H}\times\mathcal{H})$ for all $u,v\in\mathcal{H}$. The conclusion of the theorem follows by applying the Cauchy-Lipschitz-Picard theorem to the first order dynamical system (2. 30).

□

2.2.3 Some preparatory lemmas

In this section we provide some preparatory lemmas which will be used when proving the convergence of the trajectories generated by the dynamical system (2. 27). We start by recalling two central results; see for example [2, Lemma 5.1] and [2, Lemma 5.2], respectively.

Lemma 2.2.1. *Suppose that $F:[0,+\infty)\to\mathbb{R}$ is locally absolutely continuous and bounded below and that there exists $G\in L^1([0,+\infty))$ such that for almost every $t\in[0,+\infty)$*

$$\frac{d}{dt}F(t)\leq G(t).$$

Then there exists $\lim_{t\to\infty}F(t)\in\mathbb{R}$.

Lemma 2.2.2. *If $1\leq p<\infty$, $1\leq r\leq\infty$, $F:[0,+\infty)\to[0,+\infty)$ is locally absolutely continuous, $F\in L^p([0,+\infty))$, $G:[0,+\infty)\to\mathbb{R}$, $G\in L^r([0,+\infty))$ and for almost every $t\in[0,+\infty)$*

$$\frac{d}{dt}F(t)\leq G(t),$$

then $\lim_{t\to+\infty} F(t) = 0$.

Lemma 2.2.3. *Suppose that* $(H1)$ *holds and let* x *be the unique strong global solution of* (2. 27). *Take* $(x^*, w) \in \mathrm{Gr}(A + D + N_C)$ *such that* $w = v + Dx^* + p$, *where* $v \in Ax^*$ *and* $p \in N_C(x^*)$. *For every* $t \geq 0$ *consider the function* $h(t) = \frac{1}{2}\|x(t) - x^*\|^2$. *Then the following inequality holds for almost every* $t \geq 0$:

$$\ddot{h}(t) + \gamma(t)\dot{h}(t) + \lambda(t)\left(\frac{1}{L_D} - \lambda(t)\right)\|D(x(t)) - Dx^*\|^2 - \|\dot{x}(t)\|^2 \leq \tag{2. 31}$$

$$\lambda(t)\beta(t)\left(\sup_{u\in C}\varphi_B\left(u, \frac{p}{\beta(t)}\right) - \sigma_C\left(\frac{p}{\beta(t)}\right)\right) +$$

$$\lambda^2(t)\|Dx^* + v\|^2 + \lambda(t)\langle w, x^* - x(t)\rangle + \frac{\lambda^2(t)\beta^2(t)}{2}\|B(x(t))\|^2.$$

Proof. We have $\dot{h}(t) = \langle \dot{x}(t), x(t) - x^*\rangle$ and $\ddot{h}(t) = \langle \ddot{x}(t), x(t) - x^*\rangle + \|\dot{x}(t)\|^2$ for every $t \geq 0$. By using the definition of the resolvent, the differential equation in (2. 27) can be written for almost every $t \geq 0$ as

$$x(t) - \lambda(t)D(x(t)) - \lambda(t)\beta(t)B(x(t)) \in \ddot{x}(t) + \gamma(t)\dot{x}(t) + x(t) + \lambda(t)A(\ddot{x}(t) + \gamma(t)\dot{x}(t) + x(t))$$

or, equivalently,

$$-\frac{1}{\lambda(t)}\ddot{x}(t) - \frac{\gamma(t)}{\lambda(t)}\dot{x}(t) - D(x(t)) - \beta(t)B(x(t)) \in A(\ddot{x}(t) + \gamma(t)\dot{x}(t) + x(t)). \tag{2. 32}$$

Since $v \in Ax^*$ and A is monotone, we get for almost every $t \geq 0$

$$\left\langle v + \frac{1}{\lambda(t)}\ddot{x}(t) + \frac{\gamma(t)}{\lambda(t)}\dot{x}(t) + D(x(t)) + \beta(t)B(x(t)), x^* - \ddot{x}(t) - \gamma(t)\dot{x}(t) - x(t)\right\rangle \geq 0.$$

It follows that

$$\lambda(t)\langle D(x(t)) + \beta(t)B(x(t)) + v, x^* - \ddot{x}(t) - \gamma(t)\dot{x}(t) - x(t)\rangle \geq \tag{2. 33}$$

$$\langle \ddot{x}(t) + \gamma(t)\dot{x}(t), -x^* + \ddot{x}(t) + \gamma(t)\dot{x}(t) + x(t)\rangle = \ddot{h}(t) + \gamma(t)\dot{h}(t) + \|\ddot{x}(t) + \gamma(t)\dot{x}(t)\|^2 - \|\dot{x}(t)\|^2$$

for almost every $t \geq 0$. Hence, for almost every $t \geq 0$

$$\ddot{h}(t) + \gamma(t)\dot{h}(t) - \|\dot{x}(t)\|^2 \leq \lambda(t)\langle D(x(t)) + \beta(t)B(x(t)) + v, x^* - x(t)\rangle$$
$$+\lambda(t)\langle D(x(t)) + \beta(t)B(x(t)) + v, -\ddot{x}(t) - \gamma(t)\dot{x}(t)\rangle - \|\ddot{x}(t) + \gamma(t)\dot{x}(t)\|^2 \leq$$
$$\lambda(t)\langle D(x(t)) + \beta(t)B(x(t)) + v, x^* - x(t)\rangle + \frac{\lambda^2(t)}{4}\|D(x(t)) + \beta(t)B(x(t)) + v\|^2,$$

and from here, by using mean inequalities,

$$\ddot{h}(t) + \gamma(t)\dot{h}(t) - \|\dot{x}(t)\|^2 \leq \lambda(t)\langle D(x(t)) + \beta(t)B(x(t)) + v, x^* - x(t)\rangle +$$
$$\frac{\lambda^2(t)\beta^2(t)}{2}\|B(x(t))\|^2 + \lambda^2(t)\|Dx^* + v\|^2 + \lambda^2(t)\|D(x(t)) - Dx^*\|^2.$$

Since $v = w - Dx^* - p$, we obtain for the first summand of the term on the right-hand side of the above inequality for every $t \geq 0$ the following estimate

$$\lambda(t)\langle D(x(t)) + \beta(t)B(x(t)) + v, x^* - x(t)\rangle =$$
$$\lambda(t)\langle D(x(t)) + \beta(t)B(x(t)) + w - Dx^* - p, x^* - x(t)\rangle =$$
$$\lambda(t)\langle D(x(t)) - Dx^*, x^* - x(t)\rangle + \lambda(t)\langle w, x^* - x(t)\rangle +$$
$$\lambda(t)\beta(t)\left[\langle B(x(t)), x^*\rangle + \left\langle \frac{p}{\beta(t)}, x(t)\right\rangle - \langle B(x(t)), x(t)\rangle - \left\langle \frac{p}{\beta(t)}, x^*\right\rangle\right] \leq$$
$$-\frac{\lambda(t)}{L_D}\|D(x(t)) - Dx^*\|^2 + \lambda(t)\langle w, x^* - x(t)\rangle + \lambda(t)\beta(t)\left(\sup_{u\in C}\varphi_B\left(u, \frac{p}{\beta(t)}\right) - \sigma_C\left(\frac{p}{\beta(t)}\right)\right).$$

Hence, for almost every $t \geq 0$, we have

$$\ddot{h}(t) + \gamma(t)\dot{h}(t) - \|\dot{x}(t)\|^2 \leq$$
$$\frac{\lambda^2(t)\beta^2(t)}{2}\|B(x(t))\|^2 + \lambda^2(t)\|Dx^* + v\|^2 + \lambda^2(t)\|D(x(t)) - Dx^*\|^2$$
$$-\frac{\lambda(t)}{L_D}\|D(x(t)) - Dx^*\|^2 + \lambda(t)\langle w, x^* - x(t)\rangle$$
$$+\lambda(t)\beta(t)\left(\sup_{u\in C}\varphi_B\left(u, \frac{p}{\beta(t)}\right) - \sigma_C\left(\frac{p}{\beta(t)}\right)\right),$$

which is nothing else than the desired conclusion. □

Lemma 2.2.4. *Suppose that* $(H1)$ *holds and let* x *be the unique strong global solution of*

(2. 27). *Take* $x^* \in C \cap \mathrm{dom} A$ *and* $v \in Ax^*$. *For every* $t \geq 0$ *consider the function* $h(t) = \frac{1}{2}\|x(t) - x^*\|^2$. *Then for every* $\varepsilon > 0$ *the following inequality holds for almost every* $t \geq 0$*:*
(2. 34)

$$\ddot{h}(t) + \gamma(t)\dot{h}(t) + \frac{1+2\varepsilon}{2+2\varepsilon}\|\ddot{x}(t) + \gamma(t)\dot{x}(t)\|^2 - \|\dot{x}(t)\|^2 + \frac{\varepsilon\lambda(t)\beta(t)}{1+\varepsilon}\langle B(x(t)), x(t) - x^*\rangle \leq$$

$$\lambda(t)\beta(t)\left(\frac{1+\varepsilon}{2}\lambda(t)\beta(t) - \frac{1}{(1+\varepsilon)L_B}\right)\|B(x(t))\|^2 +$$

$$\lambda(t)\langle D(x(t)) + v, x^* - \ddot{x}(t) - \gamma(t)\dot{x}(t) - x(t)\rangle.$$

Proof. Let be $\varepsilon > 0$ fixed. According to (2. 33) in the proof of the above lemma, we have for almost every $t \geq 0$

$$\ddot{h}(t) + \gamma(t)\dot{h}(t) + \|\ddot{x}(t) + \gamma(t)\dot{x}(t)\|^2 - \|\dot{x}(t)\|^2 \leq \lambda(t)\beta(t)\langle B(x(t)), x^* - x(t)\rangle +$$

$$\lambda(t)\beta(t)\langle B(x(t)), -\ddot{x}(t) - \gamma(t)\dot{x}(t)\rangle + \lambda(t)\langle D(x(t)) + v, x^* - \ddot{x}(t) - \gamma(t)\dot{x}(t) - x(t)\rangle.$$

Since B is $\frac{1}{L_B}$-cocoercive and $Bx^* = 0$ we have $\langle B(x(t)), x^* - x(t)\rangle \leq -\frac{1}{L_B}\|B(x(t))\|^2$, hence

$$\lambda(t)\beta(t)\langle B(x(t)), x^* - x(t)\rangle \leq -\frac{\lambda(t)\beta(t)}{(1+\varepsilon)L_B}\|B(x(t))\|^2 + \frac{\varepsilon}{1+\varepsilon}\lambda(t)\beta(t)\langle B(x(t)), x^* - x(t)\rangle,$$

for every $t \geq 0$. Consequently,

$$\ddot{h}(t) + \gamma(t)\dot{h}(t) + \|\ddot{x}(t) + \gamma(t)\dot{x}(t)\|^2 - \|\dot{x}(t)\|^2 + \frac{\varepsilon\lambda(t)\beta(t)}{1+\varepsilon}\langle B(x(t)), x(t) - x^*\rangle \leq$$

$$-\frac{\lambda(t)\beta(t)}{(1+\varepsilon)L_B}\|B(x(t))\|^2 + \lambda(t)\beta(t)\langle B(x(t)), -\ddot{x}(t) - \gamma(t)\dot{x}(t)\rangle +$$

$$\lambda(t)\langle D(x(t)) + v, x^* - \ddot{x}(t) - \gamma(t)\dot{x}(t) - x(t)\rangle,$$

which, combined with

$$\lambda(t)\beta(t)\langle B(x(t)), -\ddot{x}(t) - \gamma(t)\dot{x}(t)\rangle \leq$$

$$\frac{1}{2(1+\varepsilon)}\|\ddot{x}(t) + \gamma(t)\dot{x}(t)\|^2 + \frac{(1+\varepsilon)\lambda^2(t)\beta^2(t)}{2}\|B(x(t))\|^2,$$

implies for almost every $t \geq 0$ relation (2. 34). □

Lemma 2.2.5. *Suppose that* $(H1)$ *holds and let* x *be the unique strong global solution of* (2. 27). *Furthermore, suppose that* $\limsup_{t \longrightarrow +\infty} \lambda(t)\beta(t) < \frac{1}{L_B}$. *Take* $x^* \in C \cap \mathrm{dom} A$ *and*

$v \in Ax^*$. For every $t \geq 0$ consider the function $h(t) = \frac{1}{2}\|x(t) - x^*\|^2$. Then there exist $a, b, c > 0$ and $t_0 > 0$ such that for almost every $t \geq t_0$ the following inequality holds:

(2. 35)

$$\ddot{h}(t) + \gamma(t)\dot{h}(t) + c\|\ddot{x}(t) + \gamma(t)\dot{x}(t)\|^2 + a\lambda(t)\beta(t)\Big(\langle B(x(t)), x(t) - x^*\rangle + \|B(x(t))\|^2\Big) \leq$$

$$\left(b\lambda^2(t) - \frac{\lambda(t)}{L_D}\right)\|D(x(t)) - Dx^*\|^2 + \lambda(t)\langle Dx^* + v, x^* - x(t)\rangle +$$

$$b\lambda^2(t)\|Dx^* + v\|^2 + \|\dot{x}(t)\|^2.$$

Proof. Let be $\varepsilon > 0$. According to the previous lemma, (2. 34) holds for almost every $t \geq 0$. We estimate the last summand in the right-hand side of (2. 34) by using the mean inequality and the cocoercieveness of D. For every $t \geq 0$ we obtain

$$\lambda(t)\langle D(x(t)) + v, x^* - \ddot{x}(t) - \gamma(t)\dot{x}(t) - x(t)\rangle \leq$$

$$\frac{\varepsilon}{4(1+\varepsilon)}\|\ddot{x}(t) + \gamma(t)\dot{x}(t)\|^2 + \frac{\lambda^2(t)(1+\varepsilon)}{\varepsilon}\|D(x(t)) + v\|^2 + \lambda(t)\langle D(x(t)) + v, x^* - x(t)\rangle =$$

$$\frac{\varepsilon}{4(1+\varepsilon)}\|\ddot{x}(t) + \gamma(t)\dot{x}(t)\|^2 + \frac{\lambda^2(t)(1+\varepsilon)}{\varepsilon}\|D(x(t)) + v\|^2 +$$

$$\lambda(t)\langle D(x(t)) - Dx^*, x^* - x(t)\rangle + \lambda(t)\langle Dx^* + v, x^* - x(t)\rangle \leq$$

$$\frac{\varepsilon}{4(1+\varepsilon)}\|\ddot{x}(t) + \gamma(t)\dot{x}(t)\|^2 + \frac{\lambda^2(t)(1+\varepsilon)}{\varepsilon}\|D(x(t)) + v\|^2 +$$

$$-\frac{\lambda(t)}{L_D}\|D(x(t)) - Dx^*\|^2 + \lambda(t)\langle Dx^* + v, x^* - x(t)\rangle,$$

which, combined with $\|D(x(t)) + v\|^2 \leq 2\|D(x(t)) - Dx^*\|^2 + 2\|Dx^* + v\|^2$, implies

$$\lambda(t)\langle D(x(t)) + v, x^* - \ddot{x}(t) - \gamma(t)\dot{x}(t) - x(t)\rangle \leq$$

$$\frac{\varepsilon}{4(1+\varepsilon)}\|\ddot{x}(t) + \gamma(t)\dot{x}(t)\|^2 + \left(\frac{2\lambda^2(t)(1+\varepsilon)}{\varepsilon} - \frac{\lambda(t)}{L_D}\right)\|D(x(t)) - Dx^*\|^2 +$$

$$\frac{2\lambda^2(t)(1+\varepsilon)}{\varepsilon}\|Dx^* + v\|^2 + \lambda(t)\langle Dx^* + v, x^* - x(t)\rangle.$$

Using the above estimate in (2. 34), we obtain for almost every $t \geq 0$

$$\ddot{h}(t) + \gamma(t)\dot{h}(t) + \frac{1+2\varepsilon}{2+2\varepsilon}\|\ddot{x}(t) + \gamma(t)\dot{x}(t)\|^2 - \|\dot{x}(t)\|^2 + \frac{\varepsilon\lambda(t)\beta(t)}{1+\varepsilon}\langle B(x(t)), x(t) - x^*\rangle \leq$$

$$\lambda(t)\beta(t)\left(\frac{1+\varepsilon}{2}\lambda(t)\beta(t)-\frac{1}{(1+\varepsilon)L_B}\right)\|B(x(t))\|^2+\frac{\varepsilon}{4(1+\varepsilon)}\|\ddot{x}(t)+\gamma(t)\dot{x}(t)\|^2+$$
$$\left(\frac{2\lambda^2(t)(1+\varepsilon)}{\varepsilon}-\frac{\lambda(t)}{L_D}\right)\|D(x(t))-Dx^*\|^2+\frac{2\lambda^2(t)(1+\varepsilon)}{\varepsilon}\|Dx^*+v\|^2+$$
$$\lambda(t)\langle Dx^*+v,x^*-x(t)\rangle$$

or, equivalently

$$\ddot{h}(t)+\gamma(t)\dot{h}(t)+\frac{2+3\varepsilon}{4(1+\varepsilon)}\|\ddot{x}(t)+\gamma(t)\dot{x}(t)\|^2-\|\dot{x}(t)\|^2+$$
$$\frac{\varepsilon\lambda(t)\beta(t)}{1+\varepsilon}\left(\langle B(x(t)),x(t)-x^*\rangle+\|B(x(t))\|^2\right)\leq$$
$$\lambda(t)\beta(t)\left(\frac{1+\varepsilon}{2}\lambda(t)\beta(t)-\frac{1}{(1+\varepsilon)L_B}+\frac{\varepsilon}{1+\varepsilon}\right)\|B(x(t))\|^2+$$
$$\left(\frac{2\lambda^2(t)(1+\varepsilon)}{\varepsilon}-\frac{\lambda(t)}{L_D}\right)\|D(x(t))-Dx^*\|^2+$$
$$\frac{2\lambda^2(t)(1+\varepsilon)}{\varepsilon}\|Dx^*+v\|^2+\lambda(t)\langle Dx^*+v,x^*-x(t)\rangle.$$

Since $\limsup_{t\longrightarrow+\infty}\lambda(t)\beta(t)<\frac{1}{L_B}$, there exists $t_0>0$ such that

$$\frac{1+\varepsilon}{2}\lambda(t)\beta(t)-\frac{1}{(1+\varepsilon)L_B}+\frac{\varepsilon}{1+\varepsilon}<\frac{1+\varepsilon}{2L_B}-\frac{1}{(1+\varepsilon)L_B}+\frac{\varepsilon}{1+\varepsilon}$$

for every $t\geq t_0$. Further, we notice that

$$\frac{1+\varepsilon}{2L_B}-\frac{1}{(1+\varepsilon)L_B}+\frac{\varepsilon}{1+\varepsilon}\leq 0$$

for every $\varepsilon\in\left(0,\sqrt{(1+L_B)^2+1}-(1+L_B)\right]$. By chosing ε_0 from this interval and defining

$$a:=\frac{\varepsilon_0}{1+\varepsilon_0},b:=\frac{2(1+\varepsilon_0)}{\varepsilon_0}\text{ and }c:=\frac{2+3\varepsilon_0}{4(1+\varepsilon_0)},$$

the conclusion follows. □

Remark 2.2.1. In the proof of the above theorem, the choice $\varepsilon_0\leq\sqrt{(1+L_B)^2+1}-(1+L_B)<\sqrt{2}-1$, implies that $a<1-\frac{1}{\sqrt{2}}$ and $\frac{1}{2}<c<\frac{3}{4}-\frac{\sqrt{2}}{8}$.

Lemma 2.2.6. *Suppose that $(H1)$ holds and let x be the unique strong global solution of*

(2. 27). *Furthermore, suppose that* $\limsup_{t\longrightarrow+\infty}\lambda(t)\beta(t)<\frac{1}{L_B}$ *and* $\lim_{t\longrightarrow+\infty}\lambda(t)=0$. *Take* $(x^*,w)\in \mathrm{Gr}(A+D+N_C)$ *such that* $w=v+Dx^*+p$, *where* $v\in Ax^*$ *and* $p\in N_C(x^*)$. *For every* $t\geq 0$ *consider the function* $h(t)=\frac{1}{2}\|x(t)-x^*\|^2$. *Then there exist* $a,b,c>0$ *and* $t_1>0$ *such that for almost every* $t\geq t_1$ *the following inequality holds:*

(2. 36)

$$\ddot{h}(t)+\gamma(t)\dot{h}(t)+c\|\ddot{x}(t)+\gamma(t)\dot{x}(t)\|^2+a\lambda(t)\beta(t)\left(\frac{1}{2}\langle B(x(t)),x(t)-x^*\rangle+\|B(x(t))\|^2\right)\leq$$

$$\frac{a\lambda(t)\beta(t)}{2}\left(\sup_{u\in C}\varphi_B\left(u,\frac{2p}{a\beta(t)}\right)-\sigma_C\left(\frac{2p}{a\beta(t)}\right)\right)+b\lambda^2(t)\|Dx^*+v\|^2+$$
$$\lambda(t)\langle w,x^*-x(t)\rangle+\|\dot{x}(t)\|^2.$$

Proof. According to Lemma 2.2.5, there exist $a,b,c>0$ and $t_0>0$ such that for almost every $t\geq t_0$ the inequality (2. 35) holds. Since $\lim_{t\longrightarrow+\infty}\lambda(t)=0$, there exists $t_1\geq t_0$ such that $\lambda(t)\leq\frac{1}{bL_D}$, hence $b\lambda^2(t)-\frac{\lambda(t)}{L_D}\leq 0$ for every $t\geq t_1$. Consequently, we can omit for every $t\geq t_1$ the term $\left(b\lambda^2(t)-\frac{\lambda(t)}{L_D}\right)\|D(x(t))-Dx^*\|^2$ in (2. 35) and obtain that the inequality

(2. 37)

$$\ddot{h}(t)+\gamma(t)\dot{h}(t)+c\|\ddot{x}(t)+\gamma(t)\dot{x}(t)\|^2+a\lambda(t)\beta(t)\Big(\langle B(x(t)),x(t)-x^*\rangle+\|B(x(t))\|^2\Big)\leq$$

$$\lambda(t)\langle Dx^*+v,x^*-x(t)\rangle+b\lambda^2(t)\|Dx^*+v\|^2+\|\dot{x}(t)\|^2$$

holds for almost every $t\geq t_1$.

Since $Dx^*+v=w-p$, we have for every $t\geq 0$

$$\frac{a\lambda(t)\beta(t)}{2}\langle B(x(t)),x^*-x(t)\rangle+\lambda(t)\langle Dx^*+v,x^*-x(t)\rangle=$$

$$\frac{a\lambda(t)\beta(t)}{2}\left(\langle B(x(t)),x^*\rangle+\left\langle\frac{2p}{a\beta(t)},x(t)\right\rangle-\langle B(x(t)),x(t)\rangle-\left\langle\frac{2p}{a\beta(t)},x^*\right\rangle\right)+$$
$$\lambda(t)\langle w,x^*-x(t)\rangle\leq$$
$$\frac{a\lambda(t)\beta(t)}{2}\left(\sup_{u\in C}\varphi_B\left(u,\frac{2p}{a\beta(t)}\right)-\sigma_C\left(\frac{2p}{a\beta(t)}\right)\right)+\lambda(t)\langle w,x^*-x(t)\rangle.$$

On the other hand, (2. 37) can be equivalently written for almost every $t\geq t_1$ as

$$\ddot{h}(t)+\gamma(t)\dot{h}(t)+c\|\ddot{x}(t)+\gamma(t)\dot{x}(t)\|^2+a\lambda(t)\beta(t)\Big(\frac{1}{2}\langle B(x(t)),x(t)-x^*\rangle+\|B(x(t))\|^2\Big)\leq$$

$$\frac{a\lambda(t)\beta(t)}{2}\langle B(x(t)),x^*-x(t)\rangle+\lambda(t)\langle Dx^*+v,x^*-x(t)\rangle+b\lambda^2(t)\|Dx^*+v\|^2+\|\dot{x}(t)\|^2,$$

hence

$$\ddot{h}(t)+\gamma(t)\dot{h}(t)+c\|\ddot{x}(t)+\gamma(t)\dot{x}(t)\|^2+a\lambda(t)\beta(t)\left(\frac{1}{2}\langle B(x(t)),x(t)-x^*\rangle+\|B(x(t))\|^2\right)\leq$$

$$\frac{a\lambda(t)\beta(t)}{2}\left(\sup_{u\in C}\varphi_B\left(u,\frac{2p}{a\beta(t)}\right)-\sigma_C\left(\frac{2p}{a\beta(t)}\right)\right)+b\lambda^2(t)\|Dx^*+v\|^2+$$
$$\lambda(t)\langle w,x^*-x(t)\rangle+\|\dot{x}(t)\|^2.$$

□

2.2.4 The convergence of the trajectories

For the proof of the convergence of the trajectories generated by the dynamical system (2. 27) we will utilize the following assumptions:

$(H2): A+N_C$ is maximally monotone and $\operatorname{zer}(A+D+N_C)\neq\emptyset$;

$(H3): \lambda\in L^2([0,+\infty))\setminus L^1([0,+\infty))$ and $\lim_{t\longrightarrow+\infty}\lambda(t)=0$;

(H_{fitz}): For every $p\in\operatorname{ran}N_C$, $\int_0^{+\infty}\lambda(t)\beta(t)\left(\sup_{u\in C}\varphi_B\left(u,\frac{p}{\beta(t)}\right)-\sigma_C\left(\frac{p}{\beta(t)}\right)\right)dt<+\infty$.

Remark 2.2.2. (i) The first assumption in $(H2)$ is fulfilled when a regularity condition which ensures the maximality of the sum of two maximally monotone operators holds. This is a widely studied topic in the literature; we refer the reader to [36, 49, 50, 196] for such conditions, including the classical Rockafellar's condition expressed in terms of the domains of the involved operators.

(ii) With respect to (H_{fitz}), we would like to remind that a similar condition formulated in terms of the Fitzpatrick function has been considered for the first time in [53] in the discrete setting. Its continuous version has been introduced in [52] and further used also in [23].

This class of conditions, widely used in the context of penalization approaches, has its origin in [24]. Here, in the particular case $C=\operatorname{argmin}\psi$, where $\psi:\mathcal{H}\to\mathbb{R}$ is a convex and differentiable function with Lipschitz continuous gradient and such that $\min\psi=0$, the condition

(H): For every $p\in\operatorname{ran}N_C$, $\int_0^{+\infty}\lambda(t)\beta(t)\left[\psi^*\left(\frac{p}{\beta(t)}\right)-\sigma_C\left(\frac{p}{\beta(t)}\right)\right]dt<+\infty$.

has been used in the asymptotic analysis of a coupled dynamical system with multiscale aspects. The function $\psi^*:\mathcal{H}\to\mathbb{R}\cup\{+\infty\}$, $\psi^*(u)=\sup_{x\in\mathcal{H}}\{\langle u,x\rangle-\psi(x)\}$, denotes the Fenchel conjugate of ψ.

According to [37], it holds

$$\varphi_{\nabla\psi}(x,u)\leq\psi(x)+\psi^*(u)\ \forall(x,u)\in\mathcal{H}\times\mathcal{H}. \tag{2. 38}$$

Since $\psi(x) = 0$ for $x \in C$, condition (H_{fitz}) applied to $B = \nabla\psi$ is fulfilled, provided that (H) is fulfilled. For several particular situations where (H_{fitz}) is verified (in its continuous or discrete version) we refer the reader to [24, 26, 27, 35, 176, 183].

For proving the convergence of the trajectories generated by the dynamical system (2. 27) we will also make use of the following ergodic version of the continuous Opial Lemma (see [24, Lemma 2.3]).

Lemma 2.2.7. *Let $S \subseteq \mathcal{H}$ be a nonempty set, $x : [0,+\infty) \to \mathcal{H}$ a given map and $\lambda : [0,+\infty) \to (0,+\infty)$ such that $\int_0^{+\infty} \lambda(t) = +\infty$. Define $\tilde{x} : [0,+\infty) \to \mathcal{H}$ by*

$$\tilde{x}(t) = \frac{1}{\int_0^t \lambda(s)ds} \int_0^t \lambda(s)x(s)ds.$$

Assume that

(i) for every $z \in S$, $\lim_{t\to+\infty} \|x(t) - z\|$ exists;

(ii) every weak sequential cluster point of the map $\tilde{x}$ belongs to S.

Then there exists $x_\infty \in S$ such that $w - \lim_{t\to+\infty} \tilde{x}(t) = x_\infty$.

We can state now the main result of this paper.

Theorem 2.2.2. *Suppose that $(H1) - (H3)$ and (H_{fitz}) hold, and let x be the unique strong global solution of* (2. 27). *Furthermore, suppose that $\limsup_{t\longrightarrow+\infty} \lambda(t)\beta(t) < \frac{1}{L_B}$, γ is locally absolutely continuous and for almost every $t \geq 0$ it holds $\gamma(t) \geq \sqrt{2}$ and $\dot{\gamma}(t) \leq 0$. Let $\tilde{x} : [0,+\infty) \longrightarrow \mathcal{H}$ be defined by*

$$\tilde{x}(t) = \frac{1}{\int_0^t \lambda(s)ds} \int_0^t \lambda(s)x(s)ds.$$

Then the following statements hold:

(i) for every $x^* \in \mathrm{zer}(A + D + N_C)$, $\|x(t) - x^*\|$ converges as $t \longrightarrow +\infty$; in addition, $\dot{x}, \ddot{x} \in L^2([0,+\infty), \mathcal{H})$, $\lambda(\cdot)\beta(\cdot)\|B(x(\cdot))\|^2 \in L^1([0,+\infty))$, and $\int_0^{+\infty} \lambda(t)\beta(t)\langle B(x(t)), x(t) - x^*\rangle dt < +\infty$.
Moreover, $\lim_{t\to+\infty} \dot{x}(t) = \lim_{t\to+\infty} \dot{h}(t) = 0$, where $h(t) = \frac{1}{2}\|x(t) - x^*\|^2$;

(ii) $\tilde{x}(t)$ converges weakly as $t \longrightarrow +\infty$ to an element in $\mathrm{zer}(A + D + N_C)$;

(iii) if, additionally, A is strongly monotone, then $x(t)$ converges strongly as $t \longrightarrow +\infty$ to the unique element of $\mathrm{zer}(A + D + N_C)$.

Proof. (i) Let be $x^* \in \operatorname{zer}(A+D+N_C)$, thus $(x^*,0) \in \operatorname{Gr}(A+D+N_C)$ and $0 = v + Dx^* + p$ for $v \in Ax^*$ and $p \in N_C(x^*)$. According to Lemma 2.2.6 and Remark 2.2.1, there exist $a,b,c > 0$, with $a < 1 - \frac{1}{\sqrt{2}}$ and $\frac{1}{2} < c < \frac{3}{4} - \frac{\sqrt{2}}{8}$, and $t_1 > 0$ such that for almost every $t \geq t_1$ it holds

$$\ddot{h}(t) + \gamma(t)\dot{h}(t) + c\|\ddot{x}(t) + \gamma(t)\dot{x}(t)\|^2 + a\lambda(t)\beta(t)\left(\frac{1}{2}\langle B(x(t)), x(t) - x^*\rangle + \|B(x(t))\|^2\right) \leq$$

$$\frac{a\lambda(t)\beta(t)}{2}\left(\sup_{u\in C}\varphi_B\left(u, \frac{2p}{a\beta(t)}\right) - \sigma_C\left(\frac{2p}{a\beta(t)}\right)\right) + b\lambda^2(t)\|Dx^* + v\|^2 + \|\dot{x}(t)\|^2.$$

On the other hand, since

$$\gamma(t)\dot{h}(t) = \frac{d}{dt}(\gamma(t)h(t)) - \dot{\gamma}(t)h(t) \geq \frac{d}{dt}(\gamma(t)h(t)),$$

it holds

$$\ddot{h}(t) + \gamma(t)\dot{h}(t) + c\|\ddot{x}(t) + \gamma(t)\dot{x}(t)\|^2 - \|\dot{x}(t)\|^2 \geq$$

$$\frac{d}{dt}\left(\dot{h}(t) + \gamma(t)h(t) + c\gamma(t)\|\dot{x}(t)\|^2\right) + (c\gamma^2(t) - c\dot{\gamma}(t) - 1)\|\dot{x}(t)\|^2 + c\|\ddot{x}(t)\|^2$$

for every $t \geq 0$.

By combining these two inequalities, we obtain for almost every $t \geq t_1$

$$\frac{d}{dt}\left(\dot{h}(t) + \gamma(t)h(t) + c\gamma(t)\|\dot{x}(t)\|^2\right) + (c\gamma^2(t) - c\dot{\gamma}(t) - 1)\|\dot{x}(t)\|^2 + c\|\ddot{x}(t)\|^2 + \tag{2. 39}$$

$$a\frac{\lambda(t)\beta(t)}{2}\langle B(x(t)), x(t) - x^*\rangle + a\lambda(t)\beta(t)\|B(x(t))\|^2 \leq$$

$$\frac{a\lambda(t)\beta(t)}{2}\left(\sup_{u\in C}\varphi_B\left(u, \frac{2p}{a\beta(t)}\right) - \sigma_C\left(\frac{2p}{a\beta(t)}\right)\right) + b\lambda^2(t)\|Dx^* + v\|^2.$$

Since $\gamma(t) \geq \sqrt{2}$ and $c > \frac{1}{2}$ one has $c\gamma^2(t) - c\dot{\gamma}(t) - 1 \geq 2c - 1 > 0$ for almost every $t \geq 0$. By using that $\langle B(x(t)), x(t) - x^*\rangle \geq \frac{1}{L_B}\|B(x(t))\|^2$ for every $t \geq 0$ and by neglecting the nonnegative terms on the left-hand side of (2. 39), we get for almost every $t \geq t_1$

$$\frac{d}{dt}\left(\dot{h}(t) + \gamma(t)h(t) + c\gamma(t)\|\dot{x}(t)\|^2\right) \leq$$

$$\frac{a\lambda(t)\beta(t)}{2}\left(\sup_{u\in C}\varphi_B\left(u, \frac{2p}{a\beta(t)}\right) - \sigma_C\left(\frac{2p}{a\beta(t)}\right)\right) + b\lambda^2(t)\|Dx^* + v\|^2.$$

Further, by integration, we easily derive that there exists $M > 0$ such that for every $t \geq 0$

$$\dot{h}(t) + \gamma(t)h(t) + c\gamma(t)\|\dot{x}(t)\|^2 \leq M. \tag{2. 40}$$

Hence, $\dot{h}(t) + \gamma(t)h(t) \leq M$, which leads to $\dot{h}(t) + \sqrt{2}h(t) \leq M$ for all $t \geq 0$. Consequently, $\frac{d}{dt}(h(t)e^{\sqrt{2}t}) \leq Me^{\sqrt{2}t}$; therefore, by integrating this inequality from 0 to $T > 0$, one obtains

$$h(T) \leq \frac{M}{\sqrt{2}} - \frac{M}{\sqrt{2}}e^{\sqrt{2}(-T)} + h(0)e^{\sqrt{2}(-T)},$$

which shows that h is bounded, hence x is bounded. Combining this with

$$\langle \dot{x}(t), x(t) - x^* \rangle + c\sqrt{2}\|\dot{x}(t)\|^2 \leq M \ \forall t \geq 0,$$

which is a consequence of (2. 40), we derive that $\dot{x}$ is bounded, too. In conclusion, the map $t \mapsto \dot{h}(t) + \gamma(t)h(t) + c\gamma(t)\|\dot{x}(t)\|^2$ is bounded from below. By taking into account relation (2. 39) and applying Lemma 2.2.1, we obtain

$$\lim_{t \longrightarrow +\infty} \left(\dot{h}(t) + \gamma(t)h(t) + c\gamma(t)\|\dot{x}(t)\|^2\right) \in \mathbb{R} \tag{2. 41}$$

and

$$\int_0^{+\infty} \|\dot{x}(t)\|^2 dt, \int_0^{+\infty} \|\ddot{x}(t)\|^2 dt, \int_0^{+\infty} \lambda(t)\beta(t)\langle B(x(t)), x(t) - x^* \rangle dt,$$

$$\int_0^{+\infty} \lambda(t)\beta(t)\|B(x(t))\|^2 \in \mathbb{R}.$$

Since

$$\frac{d}{dt}\left(\frac{1}{2}\|\dot{x}(t)\|^2\right) = \langle \ddot{x}(t), \dot{x}(t) \rangle \leq \frac{1}{2}\|\ddot{x}(t)\|^2 + \frac{1}{2}\|\dot{x}(t)\|^2$$

for every $t \geq 0$ and the function on the right-hand side of the above inequality belongs to $L^1([0, +\infty))$, according to Lemma 2.2.2 one has $\lim_{t \to +\infty} \dot{x}(t) = 0$. Further, the equality $\dot{h}(t) = \langle \dot{x}(t), x(t) - x^* \rangle$ leads to $-\|\dot{x}(t)\|\|x(t) - x^*\| \leq \dot{h}(t) \leq \|\dot{x}(t)\|\|x(t) - x^*\|$ for every $t \geq 0$. Since $\lim_{t \to +\infty} \dot{x}(t) = 0$ and $\|x(\cdot) - x^*\|$ is bounded, one obtains $\lim_{t \to +\infty} \dot{h}(t) = 0$.

From

$$\lim_{t \longrightarrow +\infty} (\dot{h}(t) + \gamma(t)h(t) + c\gamma(t)\|\dot{x}(t)\|^2) \in \mathbb{R}$$

,

$$\lim_{t \longrightarrow +\infty} \dot{h}(t) = 0$$

and

$$\lim_{t\longrightarrow+\infty} c\gamma(t)\|\dot{x}(t)\|^2 = 0,$$

one obtains that the limit $\lim_{t\longrightarrow+\infty}\gamma(t)h(t)$ exists and it is a finite number. On the other hand, since the limit $\lim_{t\longrightarrow+\infty}\gamma(t)\geq\sqrt{2}$ exists and it is a positive number, one can conclude that $\lim_{t\longrightarrow+\infty}h(t)$ exists and it is finite. Consequently, $\|x(t)-x^*\|$ converges as $t\longrightarrow+\infty$.

(ii) We show that every weak sequential limit point of $\tilde{x}$ belongs to $\mathrm{zer}(A+D+N_C)$. Indeed, let x_0 be a weak sequential limit point of $\tilde{x}$; thus, there exists a sequence $(s_n)_{n\geq 0}$ with $s_n\longrightarrow+\infty$ and $\tilde{x}(s_n)\longrightarrow x_0$ as $n\longrightarrow+\infty$.

Take an arbitrary $(x^*,w)\in\mathrm{Gr}(A+D+N_C)$ with $w=v+Dx^*+p$, $v\in Ax^*$ and $p\in N_C(x^*)$. Since $\lim_{t\longrightarrow+\infty}\lambda(t)=0$, there exists $t_2>0$ such that for every $t\geq t_2$ one has $\lambda(t)\left(\frac{1}{L_D}-\lambda(t)\right)\geq 0$. From (2. 31) we obtain

$$\ddot{h}(t)+\gamma(t)\dot{h}(t)\leq$$

$$\lambda(t)\beta(t)\left(\sup_{u\in C}\varphi_B\left(u,\frac{p}{\beta(t)}\right)-\sigma_C\left(\frac{p}{\beta(t)}\right)\right)+\lambda^2(t)\|Dx^*+v\|^2+$$

$$\frac{\lambda^2(t)\beta^2(t)}{2}\|B(x(t))\|^2+\|\dot{x}(t)\|^2+\lambda(t)\langle w,x^*-x(t)\rangle.$$

for every $t\geq t_2$. By integrating from t_2 to $T>t_2$, we get from here

$$\int_{t_2}^{T}\ddot{h}(t)+\gamma(t)\dot{h}(t)dt\leq L+\left\langle w,\left(\int_{t_2}^{T}\lambda(t)dt\right)x^*-\int_{t_2}^{T}\lambda(t)x(t)dt\right\rangle,$$

where

$$L:=\int_{t_2}^{T}\lambda(t)\beta(t)\left(\sup_{u\in C}\varphi_B\left(u,\frac{p}{\beta(t)}\right)-\sigma_C\left(\frac{p}{\beta(t)}\right)\right)dt+$$

$$\int_{t_2}^{T}\left(\lambda^2(t)\|Dx^*+v\|^2+\frac{\lambda^2(t)\beta^2(t)}{2}\|B(x(t))\|^2+\|\dot{x}(t)\|^2\right)dt.$$

Since $\gamma(t)\dot{h}(t)\geq\frac{d}{dt}(\gamma(t)h(t))$ and $\gamma(T)h(T)\geq 0$, we obtain

$$-\gamma(t_2)h(t_2)\leq L+\left\langle w,\left(\int_{t_2}^{T}\lambda(t)dt\right)x^*-\int_{t_2}^{T}\lambda(t)x(t)dt\right\rangle-\dot{h}(T)+\dot{h}(t_2),$$

hence

$$\frac{-\gamma(t_2)h(t_2)}{\int_0^T\lambda(t)dt}\leq\frac{L_1-\dot{h}(T)}{\int_0^T\lambda(t)dt}+\left\langle w,x^*-\frac{\int_0^T\lambda(t)x(t)dt}{\int_0^T\lambda(t)dt}\right\rangle,$$

where $L_1:=L+\dot{h}(t_2)+\left\langle w,\int_0^{t_2}\lambda(t)x(t)dt-\left(\int_0^{t_2}\lambda(t)dt\right)x^*\right\rangle\in\mathbb{R}$.

We choose in the above inequality $T = s_n$ for those n for which $s_n > t_2$, let n converge to $+\infty$ and so, by taking into account that $\int_0^{+\infty} \lambda(t)dt = +\infty$ and $\dot{h}$ is bounded, we obtain

$$\langle w, x^* - x_0 \rangle \geq 0.$$

Since $(x^*, w) \in \mathrm{Gr}(A + D + N_C)$ was arbitrary chosen, it follows that $x_0 \in \mathrm{zer}(A + D + N_C)$. Hence, by Lemma 2.2.7, $\tilde{x}(t)$ converges weakly as $t \longrightarrow +\infty$ to an element in $\mathrm{zer}(A+D+N_C)$.

(iii) Assume now that A is strongly monotone, i.e. there exists $\eta > 0$ such that

$$\langle u^* - v^*, x - y \rangle \geq \eta \|x - y\|^2, \text{ for all } (x, u^*), (y, v^*) \in \mathrm{Gr}(A).$$

Let x^* be the unique element of $\mathrm{zer}(A + D + N_C)$. Then $0 = v + Dx^* + p$, for $v \in Ax^*$ and $p \in N_C(x^*)$.

Since $v \in Ax^*$ and A is $\eta-$strongly monotone, from (2. 32) we obtain for almost every $t \geq 0$ one has:

$$\left\langle v + \frac{1}{\lambda(t)}\ddot{x}(t) + \frac{\gamma(t)}{\lambda(t)}\dot{x}(t) + D(x(t)) + \beta(t)B(x(t)), x^* - \ddot{x}(t) - \gamma(t)\dot{x}(t) - x(t) \right\rangle \geq$$

$$\eta \|x^* - \ddot{x}(t) - \gamma(t)\dot{x}(t) - x(t)\|^2.$$

By repeating the arguments in the proof of Lemma 2.2.3, we easily derive for almost every $t \geq 0$ it holds

$$\ddot{h}(t) + \gamma(t)\dot{h}(t) - \|\dot{x}(t)\|^2 + \eta\lambda(t)\|x^* - \ddot{x}(t) - \gamma(t)\dot{x}(t) - x(t)\|^2 \leq$$

$$\left(\lambda^2(t) - \frac{\lambda(t)}{L_D}\right)\|D(x(t)) - Dx^*\|^2 + \lambda(t)\beta(t)\left(\sup_{u \in C} \varphi_B\left(u, \frac{p}{\beta(t)}\right) - \sigma_C\left(\frac{p}{\beta(t)}\right)\right) +$$

$$\frac{\lambda^2(t)\beta^2(t)}{2}\|B(x(t))\|^2 + \lambda^2(t)\|v + Dx^*\|^2.$$

Since $\lim_{t \longrightarrow +\infty} \lambda(t) = 0$, there exists $t_3 > 0$ such that $\lambda^2(t) - \frac{\lambda(t)}{L_D} \leq 0$ for every $t \geq t_3$. Thus for almost every $t \geq t_3$

$$\ddot{h}(t) + \gamma(t)\dot{h}(t) + \eta\lambda(t)\|x^* - \ddot{x}(t) - \gamma(t)\dot{x}(t) - x(t)\|^2 \leq$$

$$\lambda(t)\beta(t)\left(\sup_{u \in C} \varphi_B\left(u, \frac{p}{\beta(t)}\right) - \sigma_C\left(\frac{p}{\beta(t)}\right)\right) + \frac{\lambda^2(t)\beta^2(t)}{2}\|B(x(t))\|^2 +$$

$$\lambda^2(t)\|v + Dx^*\|^2 + \|\dot{x}(t)\|^2.$$

Combining this inequality with

$$\|x^* - x(t)\|^2 \leq 2\|\ddot{x}(t) + \gamma(t)\dot{x}(t)\|^2 + 2\|x^* - x(t) - \ddot{x}(t) - \gamma(t)\dot{x}(t)\|^2,$$

we derive for almost every $t \geq t_3$

$$\ddot{h}(t) + \gamma(t)\dot{h}(t) + \frac{\eta\lambda(t)}{2}\|x^* - x(t)\|^2 \leq$$

$$\eta\lambda(t)\|\ddot{x}(t) + \gamma(t)\dot{x}(t)\|^2 + \lambda(t)\beta(t)\left(\sup_{u\in C}\varphi_B\left(u, \frac{p}{\beta(t)}\right) - \sigma_C\left(\frac{p}{\beta(t)}\right)\right) +$$

$$\frac{\lambda^2(t)\beta^2(t)}{2}\|B(x(t))\|^2 + \lambda^2(t)\|v + Dx^*\|^2 + \|\dot{x}(t)\|^2.$$

By using that $\gamma(t)\dot{h}(t) \geq \frac{d}{dt}(\gamma(t)h(t))$ and $\|\ddot{x}(t) + \gamma(t)\dot{x}(t)\|^2 \leq 2\|\ddot{x}(t)\|^2 + 2\gamma^2(t)\|\dot{x}(t)\|^2$ for every $t \geq 0$, by integrating from t_3 to $T > t_3$, by letting T converge to $+\infty$, and by using (i), we obtain

$$\int_0^{+\infty} \lambda(t)\|x^* - x(t)\|^2 dt < +\infty.$$

Since $\lambda \in L^2([0,+\infty)) \setminus L^1([0,+\infty))$, it follows that $\|x(t) - x^*\| \longrightarrow 0$ as $t \longrightarrow +\infty$. □

Remark 2.2.3. We want to emphasize that unlike other papers addressing the asymptotic analysis of second order dynamical systems, and where the damping function $\gamma(t)$ was assumed to be strictly greater than $\sqrt{2}$ for all $t \geq 0$ (see [19, 56]), in Theorem 2.2.2 we allow for it take this value, too.

We close the section by formulating Theorem 2.2.2 in the context of the optimization problem (2. 22), where we also use the relation between the assumptions (H) and (H_{fitz}) (see Remark 2.2.2).

Corollary 2.2.1. *Consider the optimization problem* (2. 22). *Suppose that its system of optimality conditions*

$$0 \in \partial f(x) + \nabla g(x) + N_{\mathrm{zer}\,\nabla\psi}(x)$$

is solvable, that $(H1) - (H3)$ *and* (H) *hold, and let* x *be the unique strong global solution of* (2. 28). *Furthermore, suppose that* $\limsup_{t\longrightarrow+\infty}\lambda(t)\beta(t) < \frac{1}{L_B}$, γ *is locally absolutely continuous and for almost every* $t \geq 0$ *it holds* $\gamma(t) \geq \sqrt{2}$ *and* $\dot{\gamma}(t) \leq 0$. *Let* $\tilde{x} : [0,+\infty) \longrightarrow \mathcal{H}$ *be defined by*

$$\tilde{x}(t) = \frac{1}{\int_0^t \lambda(s)ds}\int_0^t \lambda(s)x(s)ds.$$

Then the following statements hold:

(i) *for every* $x^* \in \operatorname{zer}(\partial f + \nabla g + N_{\operatorname{zer}\nabla\psi})$, $\|x(t) - x^*\|$ *converges as* $t \longrightarrow +\infty$*; in addition,* $\dot{x}, \ddot{x} \in L^2([0,+\infty), \mathcal{H})$, $\lambda(\cdot)\beta(\cdot)\|\nabla\psi(x(\cdot))\|^2 \in L^1([0,+\infty))$, $\int_0^{+\infty} \lambda(t)\beta(t)\langle \nabla\psi(x(t)), x(t) - x^*\rangle dt < +\infty$*, and* $\lim_{t\to+\infty} \dot{x}(t) = \lim_{t\to+\infty} \dot{h}(t) = 0$, *where* $h(t) = \frac{1}{2}\|x(t) - x^*\|^2$;

(ii) $\tilde{x}(t)$ *converges weakly as* $t \longrightarrow +\infty$ *to an element in* $\operatorname{zer}(\partial f + \nabla g + N_{\operatorname{zer}\nabla\psi})$*, which is also an optimal solution of* (2. 22)*;*

(iii) *if, additionally,* f *is strongly convex, then* $x(t)$ *converges strongly as* $t \longrightarrow +\infty$ *to the unique element of* $\operatorname{zer}(\partial f + \nabla g + N_{\operatorname{zer}\nabla\psi})$*, which is the unique optimal solution of* (2. 22).

3

Variational inequalities and coincidence points

3.1 A coincidence point result via variational inequalities

General variational inequalities can be used to study a wide class of problems including unilateral, moving boundary, obstacle, free boundary and equilibrium problems arising in various areas of pure and applied sciences (see [168]).

In [132]:[S. László, *Some existence results of solutions for general variational inequalities*, J. Optim. Theory Appl., **150**, 425-443 (2011)] has been studied the so-called general variational inequality of Stampacchia type and was established several existence result of the solution for this problem. Since general variational inequalities provide us with a unified, simple, and natural framework to study a wide class of problems including unilateral, moving boundary, obstacle, free boundary and equilibrium problems arising in various areas of pure and applied sciences (see [168]), the existence results of the solution of this problem are helpful in the sense that one would like to know if a solution of a general variational inequality exists before one actually devises some plausible algorithms for solving the problem. While existence results of the solution for Stampacchia variational inequalities were abundant in the last years (see, for instance, [155]), this is not the case of general variational inequality of Stampacchia type.

In this section, by making use of a new class of operators, we establish some existence results of the solution for an extended general variational inequality already considered in the literature. As application we obtain a new coincidence point theorem in a Hilbert space setting. Let us mention that the results from this section were partially published in [10]:[A.

Amini-Harandi, S. László, *A coincidence point result via variational inequalities*, Fixed Point Theory, **15**, 87-98 (2014).]

3.1.1 General variational inequalities

In what follows, unless is otherwise specified, we assume that X is a real Banach space and X^* is the topological dual of X. We denote by $\langle x^*, x\rangle$ the value of the linear, continuous functional $x^* \in X^*$ in $x \in X$. Consider the set $K \subseteq X$ and let $A : K \longrightarrow X^*$ be a given operator.

Recall, that Stampacchia variational inequality, $VI_S(A,K)$, consists in finding an element $x \in K$, such that

$$\langle A(x), y-x\rangle \geq 0 \text{ for all } y \in K,$$

where the set K is convex and closed (see, for instance, [91, 92, 94, 126, 155, 199]).

In recent years, many generalizations of this problem have been considered, studied and applied in various directions (see, for instance, [40, 91, 126, 132, 168]). General variational inequalities were introduced in [168]. It was realized that the general variational inequality can be used to study both the odd- and even-order free and moving boundary value problems. It has been shown that general variational inequalities provide us with a unified, simple, and natural framework to study a wide class of problems including unilateral, moving boundary, obstacle, free boundary and equilibrium problems arising in various areas of pure and applied sciences.

Recently László (see [132]), studied the so-called general variational inequality of Stampacchia type, $VI_S(A,a,K)$, which consists in finding an element $x \in K$ such that

$$\langle A(x), a(y)-a(x)\rangle \geq 0, \text{ for all } y \in K,$$

where $a : K \longrightarrow X$ is another given operator (see also [168]).

The problem that we shall study in this section is the so-called extended general variational inequality, $VI(A,\psi,K)$, which was introduced in [83] and consists in finding an element $x \in K$ such that

$$\langle A(x), \psi(x,y)\rangle \geq 0, \text{ for all } y \in K,$$

where $\psi : K \times K \longrightarrow X$ is a given operator. Let us mention that this problem is a particular case of some more general problems already studied in the literature (see [115] and [127]). From another point of view, the problem studied in this section is an extension of the general variational inequality of Stampacchia type.

In [132] some sufficient conditions that ensure the existence of the solutions for the gen-

eral variational inequalities of Stampacchia type were provided. It was introduced a new class of operators, the class of operators of type ql, and it was shown that the existence results mentioned above fail outside of this class. It was also shown that the concept of operator of type ql, on the one hand, may be viewed as a generalization of the monotonicity of a real valued function of one real variable and, on the other hand, may be viewed as a generalization of a linear operator.

In what follows we introduce a new class of operators, the class of operators of type g-ql, that contains in particular the set of operators of type ql and we extend some results already established in [132] for general variational inequalities of Stampacchia type involving operators of type ql.

3.1.2 Operators of type g-ql

In this paragraph we generalize the concept of operator of type ql which was introduced in [132]. In what follows such an operator will be called operator of type g-ql. We also show that this concept is more general than the concept of operator of type ql, providing an example of operator of type g-ql which is not of type ql. Let us recall some definitions and results that we will need in what follows.

Let X be a real linear space. For $x, y \in X$, we denote by $[x,y] := \{(1-t)x + ty : t \in [0,1]\}$ the closed line segment with the endpoints x and y. Let Y be another real linear space. An operator $B : D \subseteq X \longrightarrow Y$ is said to be of type *ql* (see [132]), if

$$B([x,y] \cap D) \subseteq [B(x), B(y)], \text{ for every } x, y \in D.$$

This concept can be seen as an extension of the notion of monotonicity. On the other hand, from another point of view the concept of an operator of type ql extends the concept of an affine operator.

Theorem 3.1.1. *(Proposition 3.2, [132]) Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a function. Then f is of type ql if and only if f is monotone.*

Theorem 3.1.2. *(Proposition 3.3, [132]) Let $B : X \longrightarrow Y$ be a linear operator. Then B is of type ql.*

Theorem 3.1.3. *(Theorem 3.2, [132]) Let $D \subseteq X$ be a convex set, and let $B : D \longrightarrow Y$ be an operator of type ql. Then for every $n \in \mathbb{N}$ and every $x_1, \ldots, x_n \in D$, we have*

$$B(\mathrm{co}\{x_1, \ldots, x_n\}) \subseteq \mathrm{co}\{B(x_1), \ldots, B(x_n)\},$$

where $\mathrm{co}(E)$ *denotes the convex hull of the set* $E \subseteq Y$.

In what follows we introduce the concept of an operator of type g-ql.

Definition 3.1.1. Let X and Y be two real linear spaces and let $D \subseteq X$ be a convex set. An operator $B : D \longrightarrow Y$ is said to be of type g-ql if, for every $n \in \mathbb{N}$ and every $x_1, \ldots, x_n \in D$, there exist $y_1, \ldots, y_n \in D$, not necessarily all different, such that for any subset $\{y_{i_1}, \ldots, y_{i_k}\} \subseteq \{y_1, \ldots, y_n\}$, $1 \leq k \leq n$, we have

$$B(\mathrm{co}\{y_{i_1}, \ldots, y_{i_k}\}) \subseteq \mathrm{co}\{B(x_{i_1}), \ldots, B(x_{i_k})\}.$$

From Theorem 3.1.3 we obtain immediately, that every operator of type ql is of type g-ql. The following theorem provides us other examples.

Theorem 3.1.4. *Let* $I \subseteq \mathbb{R}$ *and let* $f : I \longrightarrow \mathbb{R}$ *be a function. If there exists an interval* $\widetilde{I} \subseteq I$ *such that* $f(\widetilde{I}) = f(I)$ *and the restriction of* f *to* $\widetilde{I}$, $f_{|_{\widetilde{I}}}$, *is monotone then* f *is of type g-ql.*

Proof. Let $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in I$. Since $f(\widetilde{I}) = f(I)$ then there exist $y_1, \ldots, y_n \in \widetilde{I}$ such that $f(x_i) = f(y_i)$ for each $i \in \{1, \ldots, n\}$. Since $f_{|_{\widetilde{I}}}$ is monotone then by Theorem 3.1.1 $f_{|_{\widetilde{I}}}$ is of type ql. Then by Theorem 3.1.3, we have

$$f(\mathrm{co}\{y_{i_1}, \ldots, y_{i_k}\}) \subseteq \mathrm{co}\{f(y_{i_1}), \ldots, f(y_{i_k})\} = \mathrm{co}\{f(x_{i_1}), \ldots, f(x_{i_k})\},$$

for any subset $\{y_{i_1}, \ldots, y_{i_k}\} \subseteq \{y_1, \ldots, y_n\}$, $1 \leq k \leq n$. Hence f is of type g-ql. □

Next we provide some examples of operators of type g-ql which are not of type ql.

Example 3.1.1. Let $X = \mathbb{R}$ and consider the functions $f, g : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^2$ and $g(x) = \begin{cases} 0, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$ Since f and g are not monotone according to Theorem 3.1.1 f and g are not of the type ql. On the other hand f and g are of type g-ql (let $\widetilde{I} = [0, \infty)$ and apply Theorem 3.1.4).

It can be easily observed, that Theorem 3.1.4 can be extended for the general case as well. More precisely we have the following result.

Theorem 3.1.5. *Let* X *and* Y *be two real linear spaces and let* $D \subseteq X$ *be a convex set. Let* $B : D \longrightarrow Y$ *be an operator, and assume that there exists a convex subset* $D_1 \subseteq D$, *such that the restriction of* B *on* D_1, $B|_{D_1} : D_1 \longrightarrow Y$ *is of type ql, and* $B(D_1) = B(D)$. *Then* B *is of type g-ql.*

The proof is similar to the proof of Theorem 3.1.4 therefore we omit it. In what follows we extend the concept of operator of type g-ql for operators defined on subsets of a product space.

Definition 3.1.2. Let X and Y be two real linear spaces, and let $D \subseteq X$ be a convex set. An operator $\psi : D \times D \longrightarrow Y$ is said to be of type g-ql if, for every $n \in \mathbb{N}$ and every $x_1, \ldots, x_n \in D$, there exist $y_1, \ldots, y_n \in D$, not necessarily all different, such that for any subset $\{y_{i_1}, \ldots, y_{i_k}\} \subseteq \{y_1, \ldots, y_n\}$, $1 \leq k \leq n$, and for every $y \in \mathrm{co}\{y_{i_1}, \ldots, y_{i_k}\}$, we have

$$0 \in \mathrm{co}\{\psi(y, x_{i_1}), \ldots, \psi(y, x_{i_k})\}.$$

We have the following result.

Lemma 3.1.1. *Let X and Y be two real linear spaces, $D \subseteq X$ be a convex set and let $B : D \subseteq X \longrightarrow Y$ be of type g-ql. Then the operator $\psi : D \times D \longrightarrow Y$ which is defined by $\psi(x, y) = B(y) - B(x)$ is of type g-ql.*

Proof. Let $n \in \mathbb{N}$ and let $x_1, \ldots, x_n \in D$. Since $B : D \subseteq X \longrightarrow Y$ is of type g-ql , there exist $y_1, \ldots, y_n \in D$ such that for any subset $\{y_{i_1}, \ldots, y_{i_k}\} \subseteq \{y_1, \ldots, y_n\}$, $1 \leq k \leq n$ we have

$$B(\mathrm{co}\{y_{i_1}, \ldots, y_{i_k}\}) \subseteq \mathrm{co}\{B(x_{i_1}), \ldots, B(x_{i_k})\}.$$

Let us fix $k \in \{1, 2, \ldots, n\}$ and consider $y \in \mathrm{co}\{y_{i_1}, \ldots, y_{i_k}\}$. Then we have

$$B(y) \in \mathrm{co}\{B(x_{i_1}), \ldots, B(x_{i_k})\},$$

hence

$$0 \in \mathrm{co}\{B(x_{i_1}) - B(y), \ldots, B(x_{i_k}) - B(y)\} = \mathrm{co}\{\psi(y, x_{i_1}), \ldots, \psi(y, x_{i_k})\}.$$

□

Next we extend this concept in a Banach space context, with respect to an operator. For $x \in K$ we introduce the following notation:

$$A^+(x) = \{y \in X : \langle A(x), y \rangle \geq 0\}.$$

Definition 3.1.3. Let X be a real Banach space, let X^* be its topological dual space, let $D \subseteq X$ be a convex set and let $A : X \longrightarrow X^*$ be an operator. An operator $\psi : D \times D \longrightarrow X$ is said to be of type g-ql w.r.t. A if, for every $n \in \mathbb{N}$ and every $x_1, \ldots, x_n \in D$, there exist $y_1, \ldots, y_n \in D$,

not necessarily all different, such that for any subset $\{y_{i_1},\dots,y_{i_k}\}\subseteq\{y_1,\dots,y_n\}$, $1\le k\le n$, and for every $y\in \mathrm{co}\{y_{i_1},\dots,y_{i_k}\}$, we have

$$\mathrm{co}\{\psi(y,x_{i_1}),\dots,\psi(y,x_{i_k})\}\cap A^+(y)\neq\emptyset.$$

An operator $B:D\longrightarrow X$ is said to be of type g-ql w.r.t. A if the operator $\psi:D\times D\longrightarrow X$ which is defined by $\psi(x,y)=B(y)-B(x)$, for each $x,y\in D$ is of type g-ql w.r.t. A.

Remark 3.1.1. Let $A:X\longrightarrow X^*$ be an arbitrary operator. Since $0\in A^+(y)$ for each $y\in D$, we obtain that every operator of type g-ql $\psi:D\times D\longrightarrow X$ is of type g-ql w.r.t. A.

Next we provide an example to emphasize the importance of this concept.

Example 3.1.2. Let $A:\mathbb{R}\longrightarrow(0,\infty)$ be a function and let $f:\mathbb{R}\longrightarrow\mathbb{R}$ be an arbitrary map. Notice that for each $y\in\mathbb{R}$, $A^+(y)=[0,\infty)$. We show that f is of type g-ql with respect to A. To prove the claim consider $x_1,\dots,x_n\in\mathbb{R}$. For each $i\in\{1,\dots,n\}$, let $y_i=x_0\in\mathrm{argmin}\{f(x):x\in\{x_1,\dots,x_n\}\}$. Then obviously $\mathrm{co}\{y_j:j\in J\}=x_0$ for each $\emptyset\neq J\subseteq\{1,2,\dots,n\}$. Obviously, for $y=x_0$, we have $f(x_i)-f(y)\ge 0$, for all $i\in\{1,2,\dots,n\}$, hence

$$\mathrm{co}\{f(x_j)-f(y):j\in J\}\cap A^+(y)\neq\emptyset,$$

for every nonempty subset $J\subseteq\{1,2,\dots,n\}$.

3.1.3 Existence of the solutions of extended general variational inequalities

In this paragraph, we present some existence results of the solutions for the extended general variational inequalities that we discussed in Section 1. We need the following definition of a generalized KKM mapping (see [74, 211]).

Definition 3.1.4. Let E be a linear space and let $Y\subseteq E$. Let X be an arbitrary nonempty set. The mapping $G:X\rightrightarrows Y$ is called a GKKM mapping if for each finite subset $\{x_1,\dots,x_n\}\subseteq X$, there exist $y_1,\dots,y_n\in Y$ such that, for any subset $\{y_{i_1},\dots,y_{i_k}\}\subseteq\{y_1,\dots,y_n\}$, $1\le k\le n$, we have

$$\mathrm{co}\{y_{i_1},\dots,y_{i_k}\}\subseteq\bigcup_{j=1}^{k}G(x_{i_j})$$

The following extension of the classical KKM principle in Hausdorff topological linear spaces is due to Chang and Zhang (see [74]).

Theorem 3.1.6. *Let X be a nonempty subset of a Hausdorff topological linear space E and $G : X \rightrightarrows E$ be a GKKM mapping with nonempty, closed values. Then, the family $\{G(x) : x \in X\}$ has the finite intersection property, i.e.,*

$$\bigcap_{x \in S} G(x) \neq \emptyset,$$

for every finite subset S of X. Moreover, if there exists $x_0 \in X$ such that $G(x_0)$ is compact, then

$$\bigcap_{x \in X} G(x) \neq \emptyset.$$

Let us mention, that this theorem is actually a reformulation of Theorem 3.1 from [121]. In [147] Lin, Chuang and Yu proved the following more general result.

Theorem 3.1.7. *Let X be an arbitrary nonempty set and let Y be a closed nonempty subset of a topological linear space E. Let $G : X \rightrightarrows Y$ be a GKKM mapping with nonempty, closed values. Then, the family $\{G(x) : x \in X\}$ has the finite intersection property, i.e.,*

$$\bigcap_{x \in S} G(x) \neq \emptyset,$$

for every finite subset S of X. Moreover, if there exists a finite subset S of X such that $\bigcap_{x \in S} G(x)$ is compact, then

$$\bigcap_{x \in X} G(x) \neq \emptyset.$$

In what follows, let X and Y be two real Banach spaces and let X^* the topological dual of X. Recall that an operator $T : X \longrightarrow Y$ is called weak to $\|\cdot\|$-sequentially continuous at $x \in X$, if for every sequence $\{x_k\} \subseteq X$ that converges weakly to x, we have that $\{T(x_k)\} \subseteq Y$ converges to $T(x)$ in the topology of the norm of Y. An operator $T : X \longrightarrow Y$ is called weak to weak-sequentially continuous at $x \in X$, if for every sequence $\{x_k\} \subseteq X$ that converges weakly to x, we have that $\{T(x_k)\} \subseteq Y$ converges weakly to $T(x)$.

The next results will be very useful in the proof of our main existence result bellow.

Lemma 3.1.2. *If $P \subset Q \subset X$, where Q is weakly compact and P is weakly sequentially closed then P is weakly compact.*

Proof. Indeed, by Eberlein-Šmulian theorem, Q is weakly sequentially compact. Let $\{x_k\} \subseteq P$, hence $\{x_k\} \subseteq Q$, which is weakly sequentially compact. Hence, there exists $\{x_{k_n}\} \subseteq \{x_k\}$, weakly convergent to a point $x \in Q$. But obviously $\{x_{k_n}\} \subseteq P$, which is weakly sequentially

closed, hence $x \in P$. Thus P is weakly sequentially compact, and according to Eberlein-Šmulian theorem (see, for instance, [86]), P is weakly compact. □

Lemma 3.1.3. *Consider a bounded net $\{(x_i, x_i^*)\}_{i\in I} \subset X \times X^*$, and assume that one of the following conditions is fulfilled:*

a) $\{x_i\}_{i\in I}$ converges to x in the weak topology of X and $\{x_i^*\}_{i\in I}$ converges to x^* in the topology of norm of X^*.

b) $\{x_i\}_{i\in I}$ converges to x in the topology of norm of X and $\{x_i^*\}_{i\in I}$ converges to x^* in the weak* topology of X^*.

Then $\langle x_i^*, x_i\rangle \longrightarrow \langle x^*, x\rangle$.

Proof. We prove $a)$ the proof in case $b)$ is similar. Assume that $a)$ is satisfied. We have $|\langle x_i^* - x^*, x_i\rangle| \leq \|x_i^* - x^*\|\|x_i\|$, hence $\langle x_i^*, x_i\rangle - \langle x^*, x_i\rangle \longrightarrow 0$ which shows that $\langle x_i^*, x_i\rangle \longrightarrow \langle x^*, x\rangle$. □

Consider the set $K \subseteq X$ and let $A : K \longrightarrow X^*$, $a : K \longrightarrow X$ and $\psi : K \times K \longrightarrow X$ be given operators. Now, we are ready to state our first main existence result.

Theorem 3.1.8. *Let K be a weakly compact and convex set and let ψ be of type g-ql with respect to A. Assume that one of the following conditions is fulfilled.*

a) A is weak to $\|\cdot\|$-sequentially continuous and $\psi(\cdot, y)$ is weak to weak-sequentially continuous for each $y \in K$.

b) A is weak to weak-sequentially continuous and $\psi(\cdot, y)$ is weak to $\|\cdot\|$-sequentially continuous for each $y \in K$.

Then $VI(A, \psi, K)$ admits solutions.

Proof. Define the set-valued mapping $G : K \rightrightarrows K$ by

$$G(y) = \{x \in K : \langle A(x), \psi(x, y)\rangle \geq 0\}, \text{ for all } y \in K.$$

It is easy to see that the existence of the solution of $VI(A, \psi, K)$ is equivalent to $\bigcap_{y\in X} G(y) \neq \emptyset$. We show that G is a $GKKM$ mapping with weakly compact values hence, the assumptions of Theorem 3.1.6 are satisfied. Since ψ is of type g-ql with respect to A, according to Definition 3.1.3, $G(y) \neq \emptyset$. Now, we show that for each $y \in K$, $G(y)$ is weakly compact, thus

it is weakly closed as well. To show this, for $y \in K$ consider a sequence $\{x_k\} \subseteq G(y)$ that converges weakly to $x \in K$. We show that $x \in G(y)$.

If $a)$ is satisfied, then according to Lemma 3.1.3 $a)$ we have $\lim_{k \longrightarrow \infty} \langle A(x_k), \psi(x_k, y) \rangle = \langle A(x), \psi(x, y) \rangle$, and from $\langle A(x_k), \psi(x_k, y) \rangle \geq 0$ we get that $\langle A(x), \psi(x, y) \rangle \geq 0$.

If $b)$ is satisfied, then according to Lemma 3.1.3 $b)$ we have $\lim_{k \longrightarrow \infty} \langle A(x_k), \psi(x_k, y) \rangle = \langle A(x), \psi(x, y) \rangle$, and from $\langle A(x_k), \psi(x_k, y) \rangle \geq 0$ we get that $\langle A(x), \psi(x, y) \rangle \geq 0$.

Hence $x \in G(y)$, which means that $G(y)$ is weakly sequentially closed for all $y \in K$. Since K is weakly compact and $G(y) \subseteq K$, by Lemma 3.1.2 we obtain that $G(y)$ is weakly compact.

Now we prove that G is a GKKM mapping. Let $y_1, \ldots, y_n \in K$. Since $\psi : K \times K \longrightarrow X$ is of type g-ql with respect to A, for every finite subset $\{y_1, \ldots, y_n\}$ of K, there exist $z_1, \ldots, z_n \in K$ such that for every subset $\{z_{i_1}, \ldots, z_{i_k}\} \subseteq \{z_1, \ldots, z_n\}$, $1 \leq k \leq n$, and for every $z \in \mathrm{co}\{z_{i_1}, \ldots, z_{i_k}\}$, we have

$$\mathrm{co}\{\psi(z, y_{i_1}), \ldots, \psi(z, y_{i_k})\} \cap A^+(z) \neq \emptyset.$$

We show that

$$\mathrm{co}\{z_{i_1}, \ldots, z_{i_k}\} \subseteq \bigcup_{j=1}^{k} G(y_{i_j}),$$

consequently G is a GKKM mapping.

Suppose the contrary, that is, there exists $z \in \mathrm{co}\{z_{i_1}, \ldots, z_{i_k}\}$ such that $z \notin G(y_{i_j})$, for every $j \in \{1, 2, \ldots, k\}$. Hence, $\langle A(z), \psi(z, y_{i_j}) \rangle < 0$, for every $j \in \{1, 2, \ldots, k\}$.

Since $\mathrm{co}\{\psi(z, y_{i_j}) : j \in \{1, 2, \ldots, k\}\} \cap A^+(z) \neq \emptyset$, there exist $\lambda_{i_j} \geq 0$, $j \in \{1, 2, \ldots, k\}$ with $\sum_{j=1}^{k} \lambda_{i_j} = 1$ such that $\sum_{j=1}^{k} \lambda_{i_j} \psi(z, y_{i_j}) \in A^+(z)$, or equivalently

$$\left\langle A(z), \sum_{j=1}^{k} \lambda_{i_j} \psi(z, y_{i_j}) \right\rangle \geq 0.$$

On the other hand, by multiplying the inequalities $\langle A(z), \psi(z, y_{i_j}) \rangle < 0$ one by one with λ_{i_j}, $j \in \{1, 2, \ldots, k\}$ and summing up them from 1 to k, we obtain that

$$\left\langle A(z), \sum_{j=1}^{k} \lambda_{i_j} \psi(z, y_{i_j}) \right\rangle < 0,$$

a contradiction.

According to Theorem 3.1.6 $\bigcap_{y\in X} G(y) \neq \emptyset$, i.e., there exists $x \in K$, such that

$$\langle A(x), \psi(x,y)\rangle \geq 0 \text{ for all } y \in K.$$

□

As an immediate consequence we obtain the following result, a generalization of Theorem 4.1 from [132].

Theorem 3.1.9. *Let $K \subseteq X$ be a weakly compact and convex set and let a be of type g-ql. Assume that one of the following conditions is fulfilled.*

a) A is weak to $\|\cdot\|$-sequentially continuous and a is weak to weak-sequentially continuous.

b) A is weak to weak-sequentially continuous and a is weak to $\|\cdot\|$-sequentially continuous.

Then $VI_S(A,a,K)$ admits solutions.

Proof. Let $\psi(x,y) = a(y) - a(x)$ for each $x,y \in K$. Since a is of type g-ql according to Lemma 3.1.1 ψ is of type g-ql. Hence according to Remark 3.1.1, ψ is of type g-ql w.r.t. A. The conclusion follows immediately from Theorem 3.1.8. □

Remark 3.1.2. An anonymous referee emphasized the following extension of the problem $VI(A,\psi,K)$. Find an element $x \in K$ such that

$$\langle A(x), \psi(x,y)\rangle \geq 0, \text{ for all } y \in L,$$

where L is an arbitrary nonempty set and $\psi : K \times L \longrightarrow X$ is a given operator. Let us introduce the map $G : L \rightrightarrows K$, $G(y) = \{x \in K : \langle A(x), \psi(x,y)\rangle \geq 0\}$, for all $y \in L$.

An existence result of solution for this variational inequality, similar to Theorem 3.1.8, can be established using as argument Theorem 3.1.7 instead of Theorem 3.1.6.

The following result provides sufficient conditions for the existence of solutions of the problem $VI(A,\psi,K)$, without any continuity assumptions imposed on the operators A and a.

Theorem 3.1.10. *Let $K \subseteq X$ be a weakly compact and convex set. Suppose that the following conditions are satisfied:*

(a) ψ is of type g-ql with respect to A;

(b) If $\{x_k\} \subseteq K$ converges weakly to $x \in K$, then

$$\liminf_{k \longrightarrow \infty} \langle A(x_k), \psi(x_k, y) \rangle \leq \langle A(x), \psi(x, y) \rangle, \ \textit{for all} \ \ y \in K.$$

Then $VI(A, \psi, K)$ admits solutions.

Proof. Let us define the mapping $G : K \rightrightarrows K$ as in the proof of Theorem 3.1.8. We show that $G(y)$ is weakly sequentially closed for all $y \in K$, the rest of the proof is similar to the proof of Theorem 3.1.8 and will be omitted.
Since ψ is of type g-ql with respect to A, according to Definition 3.1.3, $G(y) \neq \emptyset$. For $y \in K$, consider a sequence $\{x_k\} \subseteq G(y)$ that converges weakly to $x \in K$. We show that $x \in G(y)$. Indeed, we have $\langle A(x_k), \psi(x_k, y) \rangle \geq 0$ for all $k \in \mathbb{N}$. Then according to (b) we have

$$0 \leq \liminf_{k \longrightarrow \infty} \langle A(x_k), \psi(x_k, y) \rangle \leq \langle A(x), \psi(x, y) \rangle.$$

Hence $0 \leq \langle A(x), \psi(x, y) \rangle$, which shows that $x \in G(y)$. Consequently $G(y)$ is weakly sequentially closed for all $y \in K$. □

The next result extends Theorem 3.2 from [207] and Theorem 4.3 from [132].

Theorem 3.1.11. *Let $K \subseteq X$ be a weakly compact and convex set. Suppose that the following conditions are satisfied:*

(a) a is of type g-ql ;

(b) If $\{x_k\} \subseteq K$ converges weakly to $x \in K$, then $\liminf_{k \longrightarrow \infty} \langle A(x_k), y) \rangle \leq \langle A(x), y) \rangle$ for all $y \in K$;

(c) The function $x \rightarrowtail \langle A(x), a(x) \rangle$, mapping K into $\mathbb{R}$ is sequentially weakly lower semi-continuous.

Then $VI_S(A, a, K)$ admits solutions.

Proof. It suffices to show that the condition (b) of Theorem 3.1.10 is satisfied by $\psi(x, y) = a(y) - a(x)$. To show this let $\{x_k\} \subseteq K$ be a sequence that converges weakly to $x \in K$. Then from (b) and (c), we have

$$\liminf_{k \longrightarrow \infty} \langle A(x_k), a(y) - a(x_k) \rangle \leq \liminf_{k \longrightarrow \infty} \langle A(x_k), a(y) \rangle - \liminf_{k \longrightarrow \infty} \langle A(x_k), a(x_k) \rangle \leq$$

$$\leq \langle A(x), a(y) \rangle - \langle A(x), a(x) \rangle = \langle A(x), a(y) - a(x) \rangle.$$

□

The next example shows, that the condition that the operator ψ is of type g-ql with respect to A is essential in the hypotheses of the previous theorems.

Example 3.1.3. Let us consider the operators $A : K \longrightarrow \mathbb{R}^2, A(x,y) = (1,x)$, and $\psi : K \times K \longrightarrow \mathbb{R}^2$, $\psi((x,y),(u,v)) = (uv - xy, y - v)$, where $K = [(-1,1),(1,-1)] \subseteq \mathbb{R}^2$. Then all the assumptions in the hypothesis of Theorem 3.1.8 are satisfied excepting the one, that ψ is of type g-ql with respect to A. We show that the conclusion of Theorem 3.1.8 fails, that is, $VI(A,\psi,K)$ has no solutions.

Obviously A is continuous, K is compact and convex, and $\psi((\cdot,\cdot),(u,v))$ is continuous for every $(u,v) \in K$. It is also obvious that $\psi((x,y),(x,y)) = (0,0)$ for all $(x,y) \in K$. This shows in particular, that the mapping G introduced in the proof of Theorem 3.1.8 has nonempty values.

We show that ψ is not of type g-ql with respect to A. Suppose the contrary and consider $(x,y) = (-1,1) \in K$ and $(u,v) = (1,-1) \in K$. Then there exist $(p_1,q_1),(p_2,q_2) \in K$ such that $\psi((p_1,q_1),(-1,1)) \cap A^+(p_1,q_1) \neq \emptyset$, $\psi((p_2,q_2),(1,-1)) \cap A^+(p_2,q_2) \neq \emptyset$ and for all $(p,q) \in [(p_1,q_1),(p_2,q_2)]$ we have

$$[\psi((p,q),(-1,1)), \psi((p,q),(1,-1))] \cap A^+(p,q) \neq \emptyset.$$

From $\psi((p_1,q_1),(-1,1)) \cap A^+(p_1,q_1) \neq \emptyset$ we obtain $\langle(-1-p_1q_1, q_1-1),(1,p_1)\rangle \geq 0$, hence $p_1 \leq -1$, and taking into account that $(p_1,q_1) \in K$, we get $(p_1,q_1) = (-1,1)$.

From $\psi((p_2,q_2),(1,-1)) \cap A^+(p_2,q_2) \neq \emptyset$ we obtain $\langle(-1-p_2q_2, q_2+1),(1,p_2)\rangle \geq 0$, hence $p_2 \geq 1$, and taking into account that $(p_2,q_2) \in K$, we get $(p_2,q_2) = (1,-1)$.

Hence, for all $(p,q) \in [(p_1,q_1),(p_2,q_2)] = K$ we have

$$[\psi((p,q),(-1,1)), \psi((p,q),(1,-1))] \cap A^+(p,q) \neq \emptyset.$$

Let $(p,q) = (0,0) \in K$. Then $[\psi((p,q),(-1,1)), \psi((p,q),(1,-1))] = [(-1,-1)(-1,1)]$ and $A^+(p,q) = (1,0)$, hence

$$[\psi((p,q),(-1,1)), \psi((p,q),(1,-1))] \cap A^+(p,q) = \emptyset, \text{ contradiction.}$$

We showed that ψ is not of type g-ql with respect to A, and it remained to show that $VI(A,\psi,K)$ has no solution. Let us suppose that there exists $(x,y) \in K$, a solution of the

problem $VI(A,\psi,K)$. Then for every $(u,v) \in K$ we have $\langle A(x,y), \psi((x,y),(u,v))\rangle \geq 0$ or equivalently $v(u-x) \geq 0$. For $u=1$, $v=-1$ we get $1-x \leq 0$, but $x \in [-1,1]$, hence $x=1$.

For $u=-1$, $v=1$ we get $1+x \leq 0$, but $x \in [-1,1]$, hence $x=-1$. Contradiction.

3.1.4 Coincidence points I.

In this paragraph we apply an existence result obtained in the previous section, to establish a coincidence point result involving operators of type g-ql. As particular case we obtain Brouwer's fixed point theorem. Let X and Y be two arbitrary sets and $f,g\colon X \longrightarrow Y$ be two given mappings. Recall that a point $x \in X$ is a coincidence point of f and g if $f(x)=g(x)$. Our coincidence point result is stated as follows.

Theorem 3.1.12. *Let H be a real Hilbert space identified with its dual. Let $K \subseteq H$ be a convex and weakly compact set, let $f : K \longrightarrow H$ be weak to weak sequentially continuous, $g : K \longrightarrow H$ be weak to norm sequentially continuous and of type g-ql. Assume that $f(K) \subseteq g(K)$. Then there exists $x_0 \in K$ such that $f(x_0) = g(x_0)$.*

Proof. Consider the operator $A : K \longrightarrow H, A(x) = g(x) - f(x)$. It can be easily verified that the operator A is weak to weak sequentially continuous.

Hence, the condition of Theorem 3.1.9 $b)$ is satisfied (with $a=g$), thus there exists $x_0 \in K$ such that $\langle A(x_0), g(y)-g(x_0)\rangle \geq 0$ for all $y \in K$, or, equivalently $\langle g(x_0)-f(x_0), g(y)-g(x_0)\rangle \geq 0$ for all $y \in K$.

Since $f(K) \subseteq g(K)$ there exists $y \in K$ such that $g(y) = f(x_0)$. Then we have $\langle g(x_0)-f(x_0), f(x_0)-g(x_0)\rangle \geq 0$, or, equivalently $-\|f(x_0)-g(x_0)\|^2 \geq 0$ which shows that $f(x_0) = g(x_0)$. □

It is well known that in finite dimensional spaces the weak and strong topologies coincide. As a corollary we obtain the following coincidence point result, a generalization of Brouwer fixed point theorem.

Corollary 3.1.1. *Let $K \subseteq \mathbb{R}^n$ be a convex and compact set and let $f,g : K \longrightarrow \mathbb{R}^n$ be continuous. Assume that $f(K) \subseteq g(K)$ and that g is of type g-ql. Then there exists $x_0 \in K$ such that $f(x_0) = g(x_0)$.*

As an immediate consequence we obtain Brouwer's fixed point theorem. Indeed, if we take $g : K \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $g(x) = x$, then g obviously is of type g-ql and $g(K) = K$. We have the following corollary.

Corollary 3.1.2. *Let $K \subseteq \mathbb{R}^n$ be a convex and compact set and let $f : K \longrightarrow \mathbb{R}^n$ be continuous. Assume that $f(K) \subseteq K$. Then there exists $x_0 \in K$ such that $f(x_0) = x_0$.*

3.2 Solution existence of general variational inequalities and coincidence points

In this section, by using a simple technique, we obtain several existence results of the solutions for general variational inequalities of Stampacchia type. We also show, that the existence of a coincidence point of two mappings is equivalent to the existence of the solution of a particular general variational inequality of Stampacchia type. As applications several coincidence and fixed point results are obtained.

We use these results to obtain some new coincidence point results in Hilbert spaces. Let us mention that the results from this section were partially published in [11]:[A. Amini-Harandi, S. László, *Solution existence of general variational inequalities and coincidence points*, Carpathian J. Math., **30**, 15-22 (2014).]

3.2.1 Preliminary notions and results

In what follows let X be a real Banach space and let X^* be the topological dual of X. We denote by $\langle x^*, x\rangle$ the value of the linear and continuous functional $x^* \in X^*$ in $x \in X$. Consider the set $K \subseteq X$, and let $A : K \longrightarrow X^*$ and $a : K \longrightarrow X$ be two given operators.

Recall that the general variational inequality of Stampacchia type $VI_S(A,a,K)$ (see [132]), consists in finding an element $x \in K$ such that $\langle A(x), a(y) - a(x)\rangle \geq 0,$ for all $y \in K$.

We denote by id_K the identity mapping on K, that is, $\mathrm{id}_K : K \longrightarrow K$, $\mathrm{id}_K(x) = x,$ for all $x \in K$. Obviously, when $a \equiv \mathrm{id}_K$, then $VI_S(A,a,K)$ reduces to Stampacchia variational inequality, $VI_S(A,K)$, i.e., find $x \in K$ such that $\langle A(x), y - x\rangle \geq 0,$ for all $y \in K$.

The next example shows, that even in finite dimensional case and even if the operators A and a are continuous, $VI_S(A,a,K)$ might have no solution. Nevertheless, in this case Stampacchia variational inequality admits solution which shows, that the known existence results of the solution for Stampacchia variational inequality cannot be extended to $VI_S(A,a,K)$ without some additional assumptions imposed on the operator a.

Example 3.2.1. (see Example 3.1 [134]) Let us consider the operator $A : K \longrightarrow \mathbb{R}^2$, $A(x,y) = (1,-x)$, where $K = [-1,1] \times [-1,1] \subseteq \mathbb{R}^2$ and let $a : K \longrightarrow K$, $a(x,y) = \left(x^2y, xy\right)$. Then obviously A and a are continuous, K is compact and convex but the general variational inequality problem of Stampacchia type admits no solution.

Let us suppose that there exists $(x,y) \in K$ a solution of the problem $VI_S(A,a,K)$. Then for every $(u,v) \in K$, we have $\langle A(x,y), a(u,v) - a(x,y)\rangle \geq 0$ or equivalently $uv(u-x) \geq 0$.

For $u = v = -1$ we get $-1 - x \geq 0$ but $x \in [-1,1]$, hence $x = -1$. For $u = 1$, $v = -1$ we get $x - 1 \geq 0$ but $x \in [-1,1]$, hence $x = 1$, contradiction. On the other hand $(x,y) = (-1,-1)$ is a solution of Stampacchia variational inequality $VI_S(A,K)$. Indeed, $\langle A(x,y),(u,v)-(x,y)\rangle = u+v+2 \geq 0$, for all $(u,v) \in K$.

Let Z and Y be two arbitrary sets and let $f,g\colon Z \longrightarrow Y$ be two given mappings. Recall that a point $x \in Z$ is a coincidence point of f and g if $f(x) = g(x)$. If $Z = Y$ and $g \equiv \mathrm{id}_Z$ then a coincidence point $x \in Z$ of f and g is called a fixed point of f, that is $f(x) = x$.

A considerable number of problems concerning on the existence of the solution of non-linear inequalities, arising in different areas of mathematics, can be treated as a coincidence point problem. For instance, the existence of the solution for the perturbed Hammerstein integral equation (see [192]),

$$g(t,x(t)) = \int_0^\infty k(s,t)f(s,x(s))ds,\ t \in \mathbb{R}_+,$$

where f,k and g are given measurable functions and $x \in L^1(\mathbb{R}_+)$ is unknown, is actually a coincidence point problem for the operators S and P defined on $L^1(\mathbb{R}_+)$ by

$$S(x)(t) := g(t,x(t)) \text{ and } P(x)(t) := \int_0^\infty k(s,t)f(s,x(s))ds.$$

Let $D \subseteq X$ be a convex set. Recall that the operator $T : D \longrightarrow X^*$ is called sign-continuous on D (see [33]) if, for every $x,y \in D$ and for all $\alpha \in (0,1)$ $\langle T(x+\alpha(y-x)),y-x\rangle \geq 0$ implies $\langle T(x),y-x)\rangle \geq 0$. The operator $T : D \longrightarrow X^*$ is called hemicontinuous, if for all $x,y \in D$ we have $T(x+\alpha(y-x)) \longrightarrow T(x)$, $\alpha \downarrow 0$ in the weak* topology of X^*. Observe that every hemicontinuous operator is also sign-continuous.

We say that T is quasimonotone, if for every $x,y \in D$ $\langle T(x),y-x\rangle > 0$ implies $\langle T(y),y-x\rangle \geq 0$. Let $t : D \longrightarrow X$ be another operator. We say that T is $t-$quasimonotone, if for every $x,y \in D$ $\langle T(x),t(y)-t(x)\rangle > 0$ implies $\langle T(y),t(y)-t(x)\rangle \geq 0$.

In what follows, we introduce the concept of an α-t inverse strongly monotone operator.

Definition 3.2.1. Let $T : D \subseteq X \longrightarrow X^*$ and $t : D \longrightarrow X$ be two given operators. We say that T is α-t inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle T(x)-T(y),t(x)-t(y)\rangle \geq \alpha\|T(x)-T(y)\|^2, \text{ for all } x,y \in D.$$

Observe when $t \equiv \mathrm{id}_D$ then T is actually α inverse strongly monotone (see [202]). Next we provide an example of α-t inverse strongly monotone operator.

Example 3.2.2. Let $D = [0,1] \subseteq \mathbb{R}$ and consider the functions $T, t : D \longrightarrow \mathbb{R}$, $T(x) = \ln(1+\sqrt{x})$ and $t(x) = \sqrt{x}$. Then T is α-t inverse strongly monotone, with $\alpha = 1$, but is not α inverse strongly monotone for any $\alpha > 0$.

To show that T is 1-t-inverse strongly monotone note that $(\ln(1+\sqrt{x}) - \ln(1+\sqrt{y})) \leq (\sqrt{x} - \sqrt{y})$ for all $x, y \in [0,1]$, $y \leq x$. Suppose now, that T is inverse strongly monotone, that is, there exists $\alpha > 0$ such that $\alpha(\ln(1+\sqrt{x}) - \ln(1+\sqrt{y})) \leq (x-y)$, for all $x, y \in [0,1]$, $y \leq x$. For $y = 0$ we obtain that $\frac{\ln(1+\sqrt{x})}{x} \leq \frac{1}{\alpha}$. On the other hand $\lim_{x \longrightarrow 0^+} \frac{\ln(1+\sqrt{x})}{x} = \infty$, which leads to contradiction.

Definition 3.2.2. Let D be a nonempty subset of a real Hilbert space H and let $T, t : D \longrightarrow H$ be two given mappings. Assume that $T(D) \subseteq t(D)$. We say that T is strict $t-$pseudocontractive on D, if there exists a real number k, with $0 \leq k < 1$, such that $\|T(x) - T(y)\|^2 \leq \|t(x) - t(y)\|^2 + k\|(t-T)(x) - (t-T)(y)\|^2$ for all $x, y \in D$.

Note that if $k = 0$ then T is $t-$nonexpansive, i.e. $\|T(x) - T(y)\| \leq \|t(x) - t(y)\|$ for all $x, y \in D$. If $t \equiv \mathrm{id}_D$ we obtain the notion of a strict pseudocontractive, respectively, nonexpansive mapping (see [202]).

Proposition 3.2.1. *If T is strict $t-$pseudocontractive on D with some $0 \leq k < 1$, then $t - T$ is α-t inverse strongly monotone with $\alpha = \frac{1-k}{2}$.*

Proof. Indeed, let $S = t - T$. Then for all $x, y \in D$, we have $\|(t-S)(x) - (t-S)(y)\|^2 \leq \|t(x) - t(y)\|^2 + k\|S(x) - S(y)\|^2$, or equivalently $\|t(x) - t(y)\|^2 - 2\langle S(x) - S(y), t(x) - t(y)\rangle + \|S(x) - S(y)\|^2 \leq \|t(x) - t(y)\|^2 + k\|S(x) - S(y)\|^2$. Hence $\langle S(x) - S(y), t(x) - t(y)\rangle \geq \frac{1-k}{2}\|S(x) - S(y)\|^2$ for all $x, y \in D$, which shows that $t - T$ is $\frac{1-k}{2}$-t inverse monotone. □

Let Z and Y be two arbitrary sets. Recall that the inverse of a mapping $f : Z \longrightarrow Y$ is defined as the set-valued mapping $f^{-1} : Y \rightrightarrows Z$, $f^{-1}(y) = \{z \in Z : f(z) = y\}$. A single valued selection of the set-valued map $F : Z \rightrightarrows Y$ is the single valued map $f : Z \longrightarrow Y$ satisfying $f(z) \in F(z)$ for all $z \in Z$.

3.2.2 Existence results of solution for general variational inequalities

In this paragraph, by using a simple technique, we obtain several existence results of solution for general variational inequalities. In what follows, unless is otherwise specified, X denotes a real Banach space, X^* denotes the topological dual of X

The following result ensures the equivalence between the existence of the solution of the general variational inequality of Stampacchia type and the existence of the solution of a particular Stampacchia variational inequality.

Lemma 3.2.1. *Let K be a nonempty subset of X and let $A : K \longrightarrow X^*$ and $a : K \longrightarrow X$ be two given operators. Let $b : a(K) \longrightarrow K$ be a single valued selection of a^{-1}. Then $u \in a(K)$ is a solution of $VI_S(A \circ b, a(K))$ if and only if, $b(u) \in K$ is a solution of $VI_S(A, a, K)$.*

Proof. Assume that $u \in a(K)$ is a solution of $VI_S(A \circ b, a(K))$ that is, $\langle (A \circ b)(u), v - u \rangle \geq 0$, for all $v \in a(K)$. Since for every $y \in K$ there exists $v \in a(K)$ such that $a(y) = v$, we obtain $\langle (A \circ b)(u), a(y) - u \rangle \geq 0$, for all $y \in K$. Observe that $a(b(u)) = u$. Thus $\langle A(b(u)), a(y) - a(b(u)) \rangle \geq 0$, for all $y \in K$, or equivalently, $b(u) \in K$ is a solution of $VI_S(A, a, K)$.

Conversely, assume that for $u \in a(K)$, $b(u) \in K$ is a solution of $VI_S(A, a, K)$, that is

$$\langle A(b(u)), a(y) - a(b(u)) \rangle \geq 0,$$

for all $y \in K$. Since for every $v \in a(K)$ there exists $y \in K$ such that $a(y) = v$, we obtain $\langle A(b(u)), v - a(b(u)) \rangle \geq 0$, for all $v \in a(K)$. Observe that $a(b(u)) = u$. Thus $\langle (A \circ b)(u), v - u \rangle \geq 0$, for all $v \in a(K)$, or equivalently, $u \in a(K)$ is a solution of $VI_S(A \circ b, a(K))$. □

Lemma 3.2.1 allows us to obtain existence results of solution for general variational inequalities of Stampacchia type by using some known results concerning on the existence of the solution of Stampacchia variational inequality. The next proposition is the single-valued version of the very general result from [33]. Let us denote the closed ball with center 0 and radius ρ, of the Banach space X, by $\overline{B}(0, \rho)$.

Proposition 3.2.2. *(see Theorem 2.1, [33]) Let K be a nonempty convex subset of X. Further, let $T : K \longrightarrow X^*$ be a quasimonotone operator such that the following conditions hold:*

(a) There exists $\rho > 0$, such that for all $x \in K \setminus \overline{B}(0, \rho)$ there exists $y \in K$ with $\|y\| < \|x\|$ and $\langle T(x), x - y \rangle \geq 0$.

(b) There exists $\rho' > \rho$ such that $K \cap \overline{B}(0, \rho')$ is nonempty weakly compact.

(c) T is sign continuous.

Then $VI_S(T, K)$ admits solutions.

Remark 3.2.1. Note that, in Proposition 3.2.2, the condition (b) is satisfied automatically if K is weakly compact or X is reflexive and K is closed, (a) is also satisfied automatically if K is bounded, while (c) is satisfied if T is hemicontinuous.

Theorem 3.2.1. *Let K be a nonempty subset of X. Further, let $a : K \longrightarrow X$ be an operator such that $a(K)$ is convex and let $A : K \longrightarrow X^*$ be a $a-$quasimonotone operator. Assume that the following conditions hold:*

(a) There exists $\rho > 0$, such that for all $x \in K$ with $a(x) \notin \overline{B}(0,\rho)$ there exists $y \in K$, satisfying $\|a(y)\| < \|a(x)\|$ and for all $x' \in K$, $a(x') = a(x)$ implies $\langle A(x'), a(x) - a(y)\rangle \geq 0$.

(b) There exists $\rho' > \rho$ such that $a(K) \cap \overline{B}(0,\rho')$ is nonempty weakly compact.

(c) For every $x, y \in K$ it holds: for all $z \in K$, $a(z) \in (a(x), a(y))$, $\langle A(z), a(y) - a(x)\rangle \geq 0$ implies $\langle A(x), a(y) - a(x)\rangle \geq 0$.

Then $VI_S(A,a,K)$ admits solutions.

Proof. According to Lemma 3.2.1 is enough to prove that $VI_S(A \circ b, a(K))$ admits a solution, where $b : a(K) \longrightarrow K$ is a single valued selection of a^{-1}.

Let $u, v \in a(K)$ such that $\langle (A \circ b)(u), v - u\rangle > 0$. Since $a(b(v)) = v$, respectively $a(b(u)) = u$, we have $\langle A(b(u)), a(b(v)) - a(b(u))\rangle > 0$.

But A is $a-$quasimonotone, hence, $\langle A(b(v)), a(b(v)) - a(b(u))\rangle \geq 0$ or equivalently $\langle (A \circ b)(v), v - u\rangle \geq 0$, which shows that $A \circ b$ is quasimonotone.

We show that the assumption (a) in the hypothesis of Proposition 3.2.2 involving the operator $A \circ b : a(K) \longrightarrow X^*$ is satisfied. Let $\rho > 0$, such that for all $x \in K$ with $a(x) \notin \overline{B}(0,\rho)$ there exists $y \in K$, satisfying $\|a(y)\| < \|a(x)\|$ and for all $x' \in K$, $a(x') = a(x)$ implies $\langle A(x'), a(x) - a(y)\rangle \geq 0$. In other words, for all $u \in a(K) \setminus \overline{B}(0,\rho)$ there exists $v \in a(K)$, satisfying $\|v\| < \|u\|$ and $\langle A(b(u)), u - v\rangle \geq 0$.

We show that $A \circ b$ is sign continuous. Let $u, v \in a(K)$ such that for all $\alpha \in (0,1)$, $\langle (A \circ b)(u + \alpha(v - u)), v - u\rangle \geq 0$. Since $a(b(v)) = v$, $a(b(u)) = u$ and $a(b(u + \alpha(v - u))) = u + \alpha(v - u) \in (a(b(u)), a(b(v)))$, we get that $\langle (A \circ b)(u + \alpha(v - u)), v - u\rangle \geq 0$ is equivalent to

$$\langle A(b(u + \alpha(v - u)), a(b(v)) - a(b(u))\rangle \geq 0,$$

which combined with (c) leads to $\langle A(b(u)), a(b(v) - a(b(u))\rangle \geq 0$, or, equivalently $\langle (A \circ b)(u), v - u\rangle \geq 0$. According to Proposition 3.2.2, $VI_S(A \circ b, a(K))$ admits a solution. □

Remark 3.2.2. Observe that the condition (a) is satisfied automatically if $a(K)$ is bounded, (b) is satisfied automatically if $a(K)$ is weakly compact or X is reflexive and $a(K)$ is closed and (c) is satisfied if the operator A is hemicontinuous.

Corollary 3.2.1. *Let K be a nonempty subset of X. Further, consider the operator $a : K \longrightarrow X$, such that $a(K)$ is convex and weakly compact (or X is reflexive and $a(K)$ is bounded and closed) and let $A : K \longrightarrow X^*$ be an $a-$quasimonotone, hemicontinuous operator. Then $VI_S(A,a,K)$ admits solutions.*

In what follows we obtain an existence result of solution for general variational inequality of Stampacchia type, without assuming any continuity property of the operator A. However, in this case we assume that A has a quite strong monotonicity property. The next result was established in [202].

Proposition 3.2.3. *(Proposition 2.1 [202]) Let K be a bounded, closed and convex subset of a real Hilbert space H and let $A : K \longrightarrow H$ be an α inverse strongly monotone operator. Then $VI_S(A,K)$ admits solutions.*

The following result concerning on the existence of the solution of general variational inequalities of Stampacchia type is an easy consequence of Proposition 3.2.3 and Lemma 3.2.1.

Theorem 3.2.2. *Let K be a subset of a real Hilbert space H and let $a : K \longrightarrow H$ be an operator and let $A : K \longrightarrow H$ be an α-a inverse strongly monotone operator. Assume that $a(K)$ is a bounded, closed and convex subset of H. Then $VI_S(A,a,K)$ admits solutions.*

Proof. According to Lemma 3.2.1 is enough to show that $VI_S(A \circ b, a(K))$ admits a solution, where $b : a(K) \longrightarrow K$ is a single valued selection of a^{-1}. Hence, according to Proposition 3.2.3 is enough to show that $A \circ b : a(K) \longrightarrow X^*$ is α inverse strongly-monotone.

Consider $b : a(K) \longrightarrow K$ a single valued selection of a^{-1}. For $u, v \in a(K)$ we have $\langle (A \circ b)(u) - (A \circ b)(v), u - v\rangle = \langle A(x) - A(y), a(x) - a(y)\rangle$, where $x = b(u)$, $y = b(v)$. According to the α-a inverse strongly monotonicity of A, there exists $\alpha > 0$, such that $\langle A(x) - A(y), a(x) - a(y)\rangle \geq \alpha\|A(x) - A(y)\|^2$, hence $\langle (A \circ b)(u) - (A \circ b)(v), u - v\rangle \geq \alpha\|A(x) - A(y)\|^2 = \alpha\|(A \circ b)(u) - (A \circ b)(v)\|^2$. The latter inequality shows that $A \circ b$ is α inverse strongly-monotone. $\square$

Remark 3.2.3. Obviously, if A is α inverse strongly monotone then it is $\frac{1}{\alpha}$ Lipschitz continuous due to the fact that $\|A(x) - A(y)\|\|x - y\| \geq \langle A(x) - A(y), x - y\rangle \geq \alpha\|A(x) - A(y)\|^2$. However this is not the case for α-a inverse strongly monotone operators as the next example shows.

Example 3.2.3. Let $A, a : [0,1] \longrightarrow \mathbb{R}$ be two operators given by

$$A(x) = \begin{cases} \ln(1+\sqrt{x}), \text{ if } x \in [0, e+1-2\sqrt{e}] \\ 1, \text{ if } x \in (e+1-2\sqrt{e}, 1], \end{cases} \quad \text{and } a(x) = \begin{cases} \sqrt{x}, \text{ if } x \in [0, e+1-2\sqrt{e}] \\ 1, \text{ if } x \in (e+1-2\sqrt{e}, 1]. \end{cases}$$

Then, obviously A is not continuous but it is $\alpha-$a inverse strongly monotone, with $\alpha = 1$.

Indeed, $\ln(1+\sqrt{x})-\ln(1+\sqrt{y}) \leq \sqrt{x}-\sqrt{y}$ for all $x,y \in [0,e+1-2\sqrt{e}]$, $y \leq x$, hence,

$$\langle A(x)-A(y), a(x)-a(y)\rangle \geq \|A(x)-A(y)\|^2 \text{ for all } x,y \in [0,e+1-2\sqrt{e}],\ y \leq x.$$

If $x,y \in (e+1-2\sqrt{e},1]$, then $A(x)-A(y)=0$, hence it remains to show the inequality for $x \in [0,e+1-2\sqrt{e}]$ and $y \in (e+1-2\sqrt{e},1]$. In this case $A(x)<A(y)$, consequently

$$\langle A(x)-A(y), a(x)-a(y)\rangle = (\ln(1+\sqrt{x})-1)(\sqrt{x}-1) \geq (\ln(1+\sqrt{x})-1)^2 = \|A(x)-A(y)\|^2.$$

Remark 3.2.4. While in Proposition 3.2.3 we assumed implicitly that the operator A is continuous, the previous example has shown that the conclusion of Theorem 3.2.2 remains true in a quite general setting. More precisely, we do not require any continuity property of the operator A.

3.2.3 Coincidence points II.

In this paragraph we obtain several coincidence point results. In the sequel H denotes a real Hilbert space identified with its dual. The next lemma shows that there is a strong connection between the coincidence points of two mappings and the solution of the problem $VI_S(A,a,K)$. More precisely, the following result ensures the equivalence between the existence of a coincidence point of two mappings and the existence of the solution of a particular general variational inequality of Stampacchia type.

Lemma 3.2.2. *Let $K \subseteq H$ and let $f,g : K \longrightarrow H$ be two given mappings. Assume that $f(K) \subseteq g(K)$. Then $x \in K$ is a solution of $VI_S(g-f,g,K)$ if, and only if, $f(x)=g(x)$.*

Proof. Indeed, let $x \in K$ be a solution of $VI_S(g-f,g,K)$. Hence, $\langle g(x)-f(x), g(y)-g(x)\rangle \geq 0$ for all $y \in K$. Since $f(K) \subseteq g(K)$, choose $y \in K$, such that $g(y)=f(x)$. Then $\langle g(x)-f(x), f(x)-g(x)\rangle \geq 0$, or equivalently $-\|f(x)-g(x)\|^2 \geq 0$, which leads to $f(x)=g(x)$.

Obviously, if $x \in K$ is a coincidence point of f and g then $\langle g(x)-f(x), g(y)-g(x)\rangle = 0$ for every $y \in K$, hence x is a solution of $VI_S(g-f,g,K)$. □

The following coincidence point result is obtained from Theorem 3.2.1, via Lemma 3.2.2.

Theorem 3.2.3. *Let $K \subseteq H$ and let $f,g : K \longrightarrow H$ be two given mappings such that $f(K) \subseteq g(K)$ and $g(K)$ is convex. Assume that the following conditions hold:*

(a) For every $x,y \in K$ $\langle g(x)-f(x), g(y)-g(x)\rangle > 0 \Longrightarrow \langle g(y)-f(y), g(y)-g(x)\rangle \geq 0$.

(b) There exists $\rho > 0$, such that for all $x \in K$ with $g(x) \notin \overline{B}(0,\rho)$ there exists $y \in K$, satisfying

$$\|g(y)\| < \|g(x)\| \text{ and } \forall x' \in K,\, g(x') = g(x) \Longrightarrow \langle g(x') - f(x'), g(x) - g(y)\rangle \geq 0.$$

(c) There exists $\rho' > \rho$ such that $g(K) \cap \overline{B}(0,\rho')$ is nonempty weakly compact.

(d) For every $x, y \in K$ it holds: $\forall z \in K$, $g(z) \in (g(x), g(y))$, $\langle g(z) - f(z), g(y) - g(x)\rangle \geq 0 \Longrightarrow \langle g(x) - f(x), g(y) - g(x)\rangle \geq 0$.

Then f and g have a coincidence point.

Proof. It can be easily verified that the operators $A : K \longrightarrow H, A(x) = g(x) - f(x)$ and $a : K \longrightarrow H, a(x) = g(x)$ satisfy the assumptions in the hypothesis of Theorem 3.2.1, hence, $VI_S(g - f, g, K)$ admits solutions. According to Lemma 3.2.2, f and g have a coincidence point. □

Corollary 3.2.2. *Let $K \subseteq H$ and let $f, g : K \longrightarrow H$ be two given mappings such that $f(K) \subseteq g(K)$ and $g(K)$ is convex, bounded and closed. Assume that the following conditions hold:*

(a) For every $x, y \in K$ $\langle g(x) - f(x), g(y) - g(x)\rangle > 0 \Longrightarrow \langle g(y) - f(y), g(y) - g(x)\rangle \geq 0$.

(b) $g - f$ is hemicontinuous.

Then f and g have a coincidence point.

Proof. The conclusion follows from Corollary 3.2.1 combined with Lemma 3.2.2, applied for the operators $A : K \longrightarrow H, A(x) = g(x) - f(x)$ and $a : K \longrightarrow H, a(x) = g(x)$. □

Corollary 3.2.3. *Let $K \subseteq H$ be convex, bounded and closed and let $f : K \longrightarrow H$ be a given mapping such that $f(K) \subseteq K$. Assume that the following conditions hold:*

(a) For every $x, y \in K$ $\langle x - f(x), y - x\rangle > 0 \Longrightarrow \langle y - f(y), y - x\rangle \geq 0$.

(b) f is hemicontinuous.

Then f has a fixed point.

The following coincidence point result is obtained from Theorem 3.2.2, via Lemma 3.2.2.

Theorem 3.2.4. *Let $K \subseteq H$ and let $f, g : K \longrightarrow H$ be two given mappings. Assume that there exists $\alpha > 0$, such that for all $x, y \in K$ one has*

$$(2\alpha - 1)\langle f(x) - f(y), g(x) - g(y)\rangle \geq (\alpha - 1)\|g(x) - g(y)\|^2 + \alpha\|f(x) - f(y)\|^2.$$

Assume further, that $f(K) \subseteq g(K)$ and $g(K)$ is bounded closed and convex. Then f and g have a coincidence point.

Proof. We show that the operators $A : K \longrightarrow H$, $A(x) = g(x) - f(x)$ and $a : K \longrightarrow H$, $a(x) = g(x)$ satisfy the assumptions of the hypothesis of Theorem 3.2.2.

From $(2\alpha - 1)\langle f(x) - f(y), g(x) - g(y)\rangle \geq (\alpha - 1)\|g(x) - g(y)\|^2 + \alpha\|f(x) - f(y)\|^2$ for all $x, y \in K$, we obtain $(2\alpha - 1)\langle f(x) - f(y), g(x) - g(y)\rangle + (1 - \alpha)\langle g(x) - g(y), g(x) - g(y)\rangle - \alpha\langle f(x) - f(y), f(x) - f(y)\rangle \geq 0$ for all $x, y \in K$, or equivalently

$$\langle (g - f)(x) - (g - f)(y), g(x) - g(y) - \alpha((g - f)(x) - (g - f)(y))\rangle \geq 0$$

for all $x, y \in K$, which shows that $g - f$ is α-g inverse monotone.

Hence, $VI_S(g - f, g, K)$ admits solutions. According to Lemma 3.2.2, f and g have a coincidence point. □

Remark 3.2.5. Observe, that if f is $g-$nonexpansive, then the condition

$$(2\alpha - 1)\langle f(x) - f(y), g(x) - g(y)\rangle \geq (\alpha - 1)\|g(x) - g(y)\|^2 + \alpha\|f(x) - f(y)\|^2$$

for all $x, y \in K$, holds with $\alpha = \frac{1}{2}$.

Remark 3.2.6. Note that according to Proposition 3.2.1, if f is strict $g-$pseudocontractive on K with some $0 \leq k < 1$, then $g - f$ is α-g inverse monotone, hence the condition

$$(2\alpha - 1)\langle f(x) - f(y), g(x) - g(y)\rangle \geq (\alpha - 1)\|g(x) - g(y)\|^2 + \alpha\|f(x) - f(y)\|^2, \forall x, y \in K,$$

in the hypothesis of Theorem 3.2.4 is also satisfied with $\alpha = \frac{1-k}{2}$. In conclusion the following statement holds.

Corollary 3.2.4. *Let $K \subseteq H$ and let $f, g : K \longrightarrow H$ be two given mappings and assume that f is strict $g-$pseudocontractive on K with some $0 \leq k < 1$. Assume further, that $f(K) \subseteq g(K)$ and $g(K)$ is bounded closed and convex. Then f and g have a coincidence point.*

As immediate consequences, by taking $g \equiv \mathrm{id}_K$, we obtain the following fixed point results.

Corollary 3.2.5. *Let $K \subseteq H$ be a bounded closed and convex set, let $f : K \longrightarrow H$ be a given mapping such that $f(K) \subseteq K$. Assume there exists $\alpha > 0$ such that $(2\alpha - 1)\langle f(x) - f(y), x - y\rangle \geq (\alpha - 1)\|x - y\|^2 + \alpha\|f(x) - f(y)\|^2$ for all $x, y \in K$. Then f has a fixed point.*

Corollary 3.2.6. *Let $K \subseteq H$ be bounded closed and convex set and let $f : K \longrightarrow H$ be a given mapping. Assume that f is strict pseudocontractive on K with some $0 \leq k < 1$. Assume further, that $f(K) \subseteq K$. Then f has a fixed point.*

3.3 Applications of general variational inequalities to coincidence point results

In this section we obtain some existence results of solution for general variational inequalities. As applications several coincidence and fixed point results are provided. Let us mention that the results from this section were partially published in [12]: [A. Amini-Harandi, S. László, *Applications of general variational inequalities to coincidence point results*, Publicationes Mathematicae Debrecen, **85**, 47-58 (2014).]

3.3.1 Preliminaries

Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Noor considered the following problem, called general variational inequality (see [168, 169, 172]). For two continuous mappings $T, g : H \longrightarrow H$, find $u \in g^{-1}(C)$ such that

$$\langle T(u), g(v) - g(u)\rangle \geq 0, \text{ for all } v \in g^{-1}(C).$$

Noor remarked (see [168]), that if $C^* = \{u \in H | \langle u, v\rangle \geq 0, \text{ for all } v \in C\}$ is a polar cone of the convex cone $C \subseteq H$, then the general variational inequality is equivalent to finding $u \in H$, such that $g(u) \in C$, $T(u) \in C^*$, $\langle T(u), g(u)\rangle = 0$, which is known as the general nonlinear complementarity problem.

If $C = H$, then the general variational inequality problem is equivalent to finding $u \in H$, such that $\langle T(u), g(v)\rangle = 0$, for all $g(v) \in H$ which is known as the weak formulation of the boundary value problem.

For a concrete third-order obstacle boundary value problem, that may be characterized by a general variational inequality see [170, 174].

The problem of general variational inequalities has been extended by Noor to the case when the operators involved are set-valued.

Let $C(H)$ be the family of all nonempty compact subsets of H. Let $T : H \longrightarrow C(H)$ be a set-valued operator, and let $g : H \longrightarrow H$ be a single-valued operator. Let K be a nonempty, closed and convex set in H. Consider the problem of finding $x \in H$, $g(x) \in K$, $u \in T(x)$ such that

$$\langle u, g(y) - g(x) \rangle \geq 0, \text{ for all } g(y) \in K.$$

This problem is called a multivalued variational inequality. It has been shown, that a wide class of multivalued odd-order and nonsymmetric, free, obstacle, moving equilibrium and optimization problems arising in pure and applied sciences can be studied via the multivalued variational inequality, (see [171]).

In what follows let X be a real Banach space and let X^* be the topological dual of X. We denote by $\langle x^*, x \rangle$ the value of the linear and continuous functional $x^* \in X^*$ in $x \in X$. Consider the set $K \subseteq X$, and let $A : K \longrightarrow X^*$ and $a : K \longrightarrow X$ be two given operators.

We deal with the following formulation of the general variational inequality problem, denoted by $VI_S(A, a, K)$ (see [132]). Find an element $x \in K$, such that

$$\langle A(x), a(y) - a(x) \rangle \geq 0, \text{ for all } y \in K.$$

We denote by id_K the identity mapping on K, that is,

$$\mathrm{id}_K : K \longrightarrow K, \mathrm{id}_K(x) = x, \text{ for all } x \in K.$$

Obviously, when $a \equiv \mathrm{id}_K$, then $VI_S(A, a, K)$ reduces to Stampacchia variational inequality, $VI_S(A, K)$ (see [199]), that is, find $x \in K$ such that

$$\langle A(x), y - x \rangle \geq 0, \text{ for all } y \in K.$$

3.3.2 Solution existence

In the sequel let X be a real Banach space and let X^* be the topological dual of X. Recall that the operator $T : D \subseteq X \longrightarrow X^*$ is called weak to $\|\cdot\|$-sequentially continuous at $x \in D$, if for every sequence $\big(x_n\big) \subseteq D$ that converges weakly to $x \in D$, the sequence $\big(T(x_n)\big) \subseteq X^*$ converges to $T(x) \in X^*$ in the topology of the norm of X^*. We say that T is weak to $\|\cdot\|$-sequentially continuous on $D \subseteq X$, if has this property at every point $x \in D$. Let us mention, that the compact operators between two Banach spaces, in particular the Fredholm integral operators, have this continuity property.

The operator $T : D \subseteq X \longrightarrow X$ is called weak to weak-sequentially continuous at $x \in D$,

if for every sequence $\big(x_n\big) \subseteq D$ that converges weakly to $x \in D$, the sequence $\big(T(x_n)\big) \subseteq X$ converges weakly to $T(x) \in X$. We say that T is weak to weak-sequentially continuous on $D \subseteq X$, if has this property at every point $x \in D$.

The following result was established in [132].

Proposition 3.3.1. *(Corollary 4.1 [132]) Let $A : K \subseteq X \longrightarrow X^*$ be a given operator. If A is weak to $\|\cdot\|$-sequentially continuous and K is weakly compact and convex, then Stampacchia variational inequality, $VI_S(A,K)$, admits solutions.*

Now we are able to provide a result concerning on existence of the solution of $VI_S(A,a,K)$. We need the following concept. Let Z and Y be two arbitrary sets. Recall that the inverse of a mapping $f : Z \longrightarrow Y$ is defined as the set-valued mapping $f^{-1} : Y \rightrightarrows Z$, $f^{-1}(y) = \{z \in Z : f(z) = y\}$. A single valued selection of the set-valued map $F : Z \rightrightarrows Y$ is the single valued map $f : Z \longrightarrow Y$ satisfying $f(z) \in F(z)$ for all $z \in Z$.

Theorem 3.3.1. *Let $K \subseteq X$ and let $A : K \longrightarrow X^*$ and $a : K \longrightarrow X$ be two given operators. Assume that $a(K)$ is weakly compact and convex. Assume further, that for every sequence $\big(x_n\big) \subseteq K$ the following condition is satisfied: if the sequence $\big(a(x_n)\big) \subseteq a(K)$ converges weakly to $a(x) \in a(K)$ then the sequence $\big(A(x_n)\big) \subseteq X^*$ is norm convergent to $A(x) \in X^*$. Then $VI_S(A,a,K)$ admits solutions.*

Proof. Consider $b : a(K) \longrightarrow K$ a single valued selection of a^{-1}. Let $\big(u_n\big) \subseteq a(K)$ a weakly convergent sequence to $u \in X$. Then due to the weak compactness of $a(K)$ we have $u \in a(K)$. We show that $(A \circ b)(u_n) \longrightarrow (A \circ b)(u)$, $n \longrightarrow \infty$. Since $\big(u_n\big) \subseteq a(K)$ there exists a sequence $\big(x_n\big) \subseteq K$ such that $u_n = a(x_n)$, $n \in \mathbb{N}$. Analogously $u = a(x)$ for some $x \in K$. Note that $a(b(u_n)) = u_n$, $n \in \mathbb{N}$ and $a(b(u)) = u$, hence the sequence $\big(a(b(u_n))\big)$ converges weakly to $a(b(u))$. According to the hypothesis of the theorem

$$(A \circ b)(u_n) \longrightarrow (A \circ b)(u),\, n \longrightarrow \infty.$$

Hence, the operator $A \circ b : a(K) \longrightarrow X^*$ is weak to $\|\cdot\|$-sequentially continuous. According to Proposition 3.3.1 there exists $u \in a(K)$ such that,

$$\langle (A \circ b)(u), v - u\rangle \geq 0, \text{ for all } v \in a(K).$$

Since for every $y \in K$ there exists $v \in a(K)$ such that $a(y) = v$, we obtain the following general nonlinear variational inequality (see [173]),

$$\langle (A \circ b)(u), a(y) - u\rangle \geq 0, \forall y \in K.$$

Observe that $a(b(u)) = u$. Thus,

$$\langle A(b(u)), a(y) - a(b(u))\rangle \geq 0, \text{ for all } y \in K,$$

or equivalently, $b(u) \in K$ is a solution of $VI_S(A, a, K)$. □

Remark 3.3.1. The condition: $\big(a(x_n)\big) \subseteq a(K)$ converges weakly to $a(x) \in a(K)$ then the sequence $\big(A(x_n)\big) \subseteq X^*$ is norm convergent to $A(x) \in X^*$, in the hypothesis of Theorem 3.3.1 implies that $a^{-1}(a(x)) \subseteq A^{-1}(A(x))$ for every $x \in K$. Indeed, let $x \in K$. Since $a(K)$ is weakly sequentially closed, there exists a sequence $\big(a(x_n)\big) \subseteq a(K)$ converging to $a(x)$ in the weak topology of X. But then the sequence $\big(A(x_n)\big)$ converges strongly to $A(x)$. Let $y \in a^{-1}(a(x))$. Then $a(y) = a(x)$ hence $\big(A(x_n)\big)$ converges strongly to $A(y)$. Therefore $A(y) = A(x)$, hence $y \in A^{-1}(A(x))$.

The next Corollary allows us to obtain the conclusion of Theorem 3.3.1 in conditions that can more easily be verified.

Corollary 3.3.1. *Assume that K is weakly compact, a is weak to weak-sequentially continuous and $a(K)$ is convex. Assume further, that for every sequence $\big(x_n\big) \subseteq K$ the following condition holds: if the sequence $\big(a(x_n)\big) \subseteq a(K)$ converges weakly to $a(x) \in a(K)$ then the sequence $\big(A(x_n)\big) \subseteq X^*$ is norm convergent to $A(x) \in X^*$. Then $VI_S(A, a, K)$ admits solutions.*

Proof. We show that $a(K)$ is weakly compact, and the conclusion follows from Theorem 3.3.1. By Eberlein-Šmulian theorem, (see, for instance, [86]) $a(K)$ is weakly compact if and only if is weakly sequentially compact. To prove that $a(K)$ is weakly sequentially compact, let $\big(u_n\big)$ be an arbitrary sequence in $a(K)$. Then there exists a sequence $\big(x_n\big) \subseteq K$ such that $u_n = a(x_n)$, $n \in \mathbb{N}$. We show that $\big(a(x_n)\big)$ has a weakly convergent subsequence in $a(K)$. Since $\big(x_n\big)$ is a sequence in the weakly compact set K, $\big(x_n\big)$ has a weakly convergent subsequence (note that by Eberlein-Šmulian theorem K is weakly sequentially compact, that is, every sequence in K has a convergent subsequence). Let $\big(x_{n_i}\big)$ be a subsequence of $\big(x_n\big)$ that is weakly convergent to $x \in K$. Since a is weak to weak sequentially continuous $\big(a(x_{n_i})\big)$ converges weakly to $a(x)$ and the proof is completed. □

Recall that the operator $T : D \subseteq X \longrightarrow X^*$ is called monotone (see [69, 70, 98, 157, 158]) if for all $x, y \in D$ one has $\langle T(x) - T(y), x - y\rangle \geq 0$.

We say that T is monotone relative to the operator $t : D \longrightarrow X$, if for all $x, y \in D$, we have $\langle T(x) - T(y), t(x) - t(y)\rangle \geq 0$.

Obviously, if $t \equiv \mathrm{id}_D$ we obtain the definition of monotonicity. T is called continuous on finite dimensional subspaces, if for every finite dimensional subspace $M \subseteq X$ the restriction

of T to $D \cap M$ is weak continuous, that is, for every sequence $\big(x_n\big) \subseteq D \cap M$ converging to $x \in M$ the sequence $\big(A(x_n)\big) \subseteq X^*$ converges to $A(x)$ in the weak topology of X^* (see [186]).

In what follows we present a well known existence results of solution for Stampacchia variational inequality $VI_S(A,K)$. The following classical result concerning on the existence of the solution of $VI(A,K)$, is due to Hartmann and Stampacchia (see [110]).

Proposition 3.3.2. *Let X be a reflexive Banach space, let K be a weakly compact convex nonempty subset of X. If $A : K \longrightarrow X^*$ is a monotone operator, continuous on finite dimensional subspaces then $VI_S(A,K)$ admits solutions.*

In what follows we provide a result concerning on the existence of the solution of general variational inequalities.

Theorem 3.3.2. *Assume that the Banach space X is reflexive. Let $A : K \subseteq X \longrightarrow X^*$ be monotone relative to $a : K \longrightarrow X$, where $a(K)$ is weakly compact and convex. Assume further, that for every finite dimensional subset $L \subseteq a(K)$ and for every sequence $\big(x_n\big) \subseteq K$, such that $a(x_n) \in L$ for every $n \in \mathbb{N}$, the following condition holds: if the sequence $\big(a(x_n)\big) \subseteq L$ converges to $a(x) \in a(K)$ then the sequence $\big(A(x_n)\big) \subseteq X^*$ is weakly convergent to $A(x) \in X^*$. Then $VI_S(A,a,K)$ admits solutions.*

Proof. Consider $b : a(K) \longrightarrow K$ a single valued selection of a^{-1} and let $u, v \in a(K)$. Then $\langle (A \circ b)(u) - (A \circ b)(v), u - v\rangle = \langle A(x) - A(y), a(x) - a(y)\rangle$, where $x = b(u)$, $y = b(v)$. Since A is monotone relative to a, we have $\langle A(x) - A(y), a(x) - a(y)\rangle \geq 0$, hence, the operator $A \circ b : a(K) \longrightarrow X^*$ is monotone.

Let M be a finite dimensional subspace of X and let $L = M \cap a(K)$. Let $\big(u_n\big) \subseteq L$ be a sequence convergent to $u \in a(K)$. Since M is finite dimensional subspace it is closed. Hence, according to the weak compactness of $a(K)$ we get that $u \in L$. We have to show that the sequence $\big((A \circ b)(u_n)\big) \subseteq X^*$ converges to $(A \circ b)(u) \in X^*$ in the weak topology of X^*. Since $\big(u_n\big) \subseteq a(K)$ there exists $\big(x_n\big) \subseteq K$ such that $u_n = a(x_n)$, $n \in \mathbb{N}$. Analogously $u = a(x)$ for some $x \in K$. Since $b : a(K) \longrightarrow K$ is a single valued selection of a^{-1}, observe that $a(b(u_n)) = u_n \in L$, $n \in \mathbb{N}$ and $a(b(u)) = u \in L$. Hence $\big(a(b(u_n))\big)$ converges to $a(b(u))$.

According to the hypothesis of the theorem the sequence $\big(A(b(u_n))\big) \subseteq X^*$ converges weakly to $A(b(u)) \in X^*$ when $n \longrightarrow \infty$, which shows that $A \circ b$ is continuous on finite dimensional subspaces. Hence, according to Proposition 3.3.2 there exists $u \in a(K)$ such that,

$$\langle (A \circ b)(u), v - u\rangle \geq 0, \text{ for all } v \in a(K).$$

Since for every $y \in K$ there exists $v \in a(K)$ such that $a(y) = v$, we obtain

$$\langle (A \circ b)(u), a(y) - u \rangle \geq 0, \forall y \in K.$$

Observe that $a(b(u)) = u$. Thus,

$$\langle A(b(u)), a(y) - a(b(u)) \rangle \geq 0, \text{ for all } y \in K,$$

or equivalently, $b(u) \in K$ is a solution of $VI_S(A, a, K)$. □

Remark 3.3.2. Observe, that if K is weakly compact and a is weak to weak-sequentially continuous then, according to the proof of Corollary 3.3.1, $a(K)$ is weakly compact, hence we have the following corollary.

Corollary 3.3.2. *Let the Banach space X be reflexive. Assume that K is weakly compact, $a(K)$ is convex and a is weak to weak-sequentially continuous. Let A be monotone relative to a and assume further, that for every finite dimensional subset $L \subseteq a(K)$ and for every sequence $(x_n) \subseteq K$, such that $a(x_n) \in L$ for every $n \in \mathbb{N}$, the following condition holds: if the sequence $(a(x_n)) \subseteq L$ converges to $a(x) \in a(K)$ then the sequence $(A(x_n)) \subseteq X^*$ is weakly convergent to $A(x) \in X^*$. Then $VI_S(A, a, K)$ admits solutions.*

3.3.3 Coincidence points III.

In this paragraph we obtain several coincidence point results for two given mappings by using the existence results of the solution for general variational inequalities established in the previous section. Let X and Y be two arbitrary sets and let $f, g \colon X \longrightarrow Y$ be two given mappings. Recall that a point $x \in X$ is a coincidence point of f and g if $f(x) = g(x)$. If $X = Y$ and $g \equiv \mathrm{id}_X$ then a coincidence point $x \in X$ of f and g is called a fixed point of f, that is $f(x) = x$. A considerable number of problems concerning on the existence of the solution of nonlinear inequalities, arising in different areas of mathematics, can be treated as a coincidence point or fixed point problem (see, for instance, [192]).

In the sequel H denotes a real Hilbert space identified with its dual.

The following coincidence point result is an easy consequence of Theorem 3.3.1.

Theorem 3.3.3. *Let $K \subseteq H$ and let $f, g : K \longrightarrow H$ be two given mappings. Assume that $f(K) \subseteq g(K)$ and $g(K)$ is weakly compact and convex. Assume further, that for every sequence $(x_n) \subseteq K$ the following condition holds: if the sequence $(g(x_n)) \subseteq g(K)$ converges*

weakly to $g(x) \in g(K)$ then the sequence $\big(g(x_n) - f(x_n)\big) \subseteq H$ converges to $g(x) - f(x) \in H$ in the topology of the norm of H. Then f and g have a coincidence point.

Proof. It can be easily verified that the operators $A : K \longrightarrow H, A(x) = g(x) - f(x)$ and $a : K \longrightarrow H, a(x) = g(x)$ satisfy the assumptions of the hypothesis of Theorem 3.3.1. Hence, $VI_S(g - f, g, K)$ admits solutions.

Let $x \in K$ be a solution of $VI_S(g - f, g, K)$. Then we have $\langle g(x) - f(x), g(y) - g(x)\rangle \geq 0$ for all $y \in K$. Since $f(K) \subseteq g(K)$ choose $y \in K$ such that $g(y) = f(x)$. Then $\langle g(x) - f(x), f(x) - g(x)\rangle \geq 0$, or equivalently $-\|f(x) - g(x)\|^2 \geq 0$, which leads to $f(x) = g(x)$. □

Corollary 3.3.3. *Let $K \subseteq H$ be a weakly compact set, let $f : K \longrightarrow H$ and $g : K \longrightarrow H$ be two mappings, such that g is weak to weak-sequentially continuous and $g(K)$ is convex. Assume that for every sequence $\big(x_n\big) \subseteq K$ the following condition holds: if the sequence $\big(g(x_n)\big) \subseteq g(K)$ converges weakly to $g(x) \in g(K)$ then the sequence $\big(g(x_n) - f(x_n)\big) \subseteq H$ is norm convergent to $g(x) - f(x) \in H$. If $f(K) \subseteq g(K)$ then f and g have a coincidence point.*

Proof. According to the proof of Corollary 3.3.1, $g(K)$ is weakly compact. The conclusion follows from Theorem 3.3.3. □

The next corollary provides sufficient conditions for the existence of a fixed point of a given mapping.

Corollary 3.3.4. *Let $K \subseteq H$ be a weakly compact and convex set, let $f : K \longrightarrow H$ be a given mapping such that $f(K) \subseteq K$. Assume that for every sequence $\big(x_n\big) \subseteq K$ the following condition holds: if the sequence $\big(x_n\big) \subseteq K$ converges weakly to $x \in K$ then the sequence $\big(x_n - f(x_n)\big) \subseteq H$ is norm convergent to $x - f(x) \in H$. Then f has a fixed point.*

Proof. The conclusion follows from Theorem 3.3.3 by taking $g \equiv \mathrm{id}_K$. □

In finite dimensional Hilbert spaces the weak topology and the topology of the norm are equivalent, hence we have the following corollaries.

Corollary 3.3.5. *Let $K \subseteq \mathbb{R}^n$ and let $f, g : K \longrightarrow \mathbb{R}^n$ be two given mappings. Assume that $f(K) \subseteq g(K)$ and $g(K)$ is compact and convex. Assume further, that for every sequence $\big(x_n\big) \subseteq K$ the following condition holds: if the sequence $\big(g(x_n)\big) \subseteq g(K)$ converges to $g(x) \in g(K)$ then the sequence $\big(f(x_n)\big) \subseteq \mathbb{R}^n$ converges to $f(x) \in \mathbb{R}^n$. Then f and g have a coincidence point.*

The next result can be viewed as a coincidence point version of Brouwer fixed point theorem.

Corollary 3.3.6. *Let $K \subseteq \mathbb{R}^n$ be a compact set, let $f, g : K \longrightarrow \mathbb{R}^n$ be two given mappings, such that g is continuous, $g(K)$ is convex and $f(K) \subseteq g(K)$. Assume that for every sequence $\big(x_n\big) \subseteq K$ the following condition holds: if the sequence $\big(g(x_n)\big)$ converges to $g(x) \in g(K)$ then the sequence $\big(f(x_n)\big)$ converges to $f(x) \in \mathbb{R}^n$. Then there exists $x_0 \in K$ such that $f(x_0) = g(x_0)$.*

The following coincidence point result is obtained via Theorem 3.3.2.

Theorem 3.3.4. *Let $K \subseteq H$ and let $f, g : K \longrightarrow H$ be two given mappings, such that*

$$\|g(x) - g(y)\|^2 \geq \langle f(x) - f(y), g(x) - g(y)\rangle \text{ for all } x, y \in K.$$

Assume that $f(K) \subseteq g(K)$ and $g(K)$ is weakly compact and convex. Assume further, that for every finite dimensional subset $L \subseteq g(K)$ and for every sequence $\big(x_n\big) \subseteq K$, such that $g(x_n) \in L$ for every $n \in \mathbb{N}$, the following condition holds: if the sequence $\big(g(x_n)\big) \subseteq L$ converges to $g(x) \in g(K)$ then the sequence $\big(f(x_n)\big) \subseteq H$ converges to $f(x) \in H$ in the weak topology of H. Then f and g have a coincidence point.

Proof. We show that the operators $A : K \longrightarrow H$, $A(x) = g(x) - f(x)$ and $a : K \longrightarrow H$, $a(x) = g(x)$ satisfy the assumptions in the hypothesis of Theorem 3.3.2.

Let $L \subseteq g(K)$ be a finite dimensional subset . It is obvious that if the sequence $\big(g(x_n)\big) \subseteq L$ converges to $g(x) \in g(K)$ and the sequence $\big(f(x_n)\big) \subseteq H$ converges to $f(x) \in H$ in the weak topology of H then the sequence $\big(g(x_n) - f(x_n)\big) \subseteq H$ converges to $g(x) - f(x) \in H$ in the weak topology of H.

From $\|g(x) - g(y)\|^2 \geq \langle f(x) - f(y), g(x) - g(y)\rangle$ for all $x, y \in K$ we obtain $\langle g(x) - g(y), g(x) - g(y)\rangle \geq \langle f(x) - f(y), g(x) - g(y)\rangle$ for all $x, y \in K$, or equivalently

$$\langle (g - f)(x) - (g - f)(y), g(x) - g(y)\rangle \geq 0$$

for all $x, y \in K$, which shows that $g - f$ is monotone relative to g.

Hence, according to Theorem 3.3.2, $VI_S(g - f, g, K)$ admits solutions.

Let $x \in K$ be a solution of $VI_S(g - f, g, K)$. Then we have $\langle g(x) - f(x), g(y) - g(x)\rangle \geq 0$ for all $y \in K$. Since $f(K) \subseteq g(K)$ choose $y \in K$ such that $g(y) = f(x)$. Then $\langle g(x) - f(x), f(x) - g(x)\rangle \geq 0$, or equivalently $-\|f(x) - g(x)\|^2 \geq 0$, which leads to $f(x) = g(x)$. □

As an immediate consequence we obtain the following fixed point result.

Corollary 3.3.7. *Let $K \subseteq H$ be a weakly compact and convex set, let $f : K \longrightarrow H$ be a given mapping such that $f(K) \subseteq K$. Assume that f is continuous on finite dimensional subspaces and $\|x-y\|^2 \geq \langle f(x)-f(y), x-y\rangle$ for all $x, y \in K$. Then f has a fixed point.*

Proof. The conclusion follows from Theorem 3.3.4 by taking $g \equiv \mathrm{id}_K$. □

Remark 3.3.3. Note that if f is $g-$nonexpansive, that is $\|f(x)-f(y)\| \leq \|g(x)-g(y)\|$ for all $x, y \in K$, then the condition $\|g(x)-g(y)\|^2 \geq \langle f(x)-f(y), g(x)-g(y)\rangle$ for all $x, y \in K$ in the hypothesis of Theorem 3.3.4 is satisfied, since for all $x, y \in K$ we have

$$\|g(x)-g(y)\|^2 \geq \|f(x)-f(y)\|\|g(x)-g(y)\| \geq \langle f(x)-f(y), g(x)-g(y)\rangle.$$

It particular, if f is nonexpansive, then the condition $\|x-y\|^2 \geq \langle f(x)-f(y), x-y\rangle$ for all $x, y \in K$ in Corollary 3.3.7 is satisfied, since in this case we have

$$\|x-y\|^2 \geq \|f(x)-f(y)\|\|x-y\| \geq \langle f(x)-f(y), x-y\rangle \text{ for all } x, y \in K.$$

3.4 Multivalued variational inequalities and coincidence point results

A considerable number of results that guarantee the existence of coincidence points for pairs of mappings are based on some generalizations of Banach contraction principle (see for instance [38, 43] and the references therein). In this section we obtain some coincidence point results in Hilbert spaces without making use of any generalized contraction mapping. Our results are based on the existence of solutions of some variational inequalities involving operators belonging to a class, the class of operators of type ql that was recently introduced in [132]. Moreover, we show by an example that our results fail outside of this class.

Let X and Y be two arbitrary sets and $f\colon X \longrightarrow Y$, $T : X \rightrightarrows Y$ be two given mappings. We say that a point $x \in X$ is a coincidence point of f and T if $f(x) \in T(x)$.

Coincidence theory (the study of coincidence points) is, in most settings, a generalization of fixed point theory, the study of points x with $x \in T(x)$. Indeed, a fixed point is the special case obtained from the coincidence point by letting $X = Y$ and taking f to be the identity mapping. We show in the last subsection, that Kakutani fixed point theorem is a particular case of our coincidence point results.

Let us mention that the results from this section were partially published in [137]:[S. László, *Multivalued variational inequalities and coincidence point results*, J. Math. Anal.

Appl. **404**, 105-114 (2013).]

3.4.1 Multivalued variational inequalities

Let H be a real Hilbert space and let $C(H)$ be the family of all nonempty compact subsets of H. Let $T : H \longrightarrow C(H)$ be a set-valued operator, and let $f : H \longrightarrow H$ be a single-valued operator. Let K be a nonempty, closed, and convex set in H. Consider the problem of finding $x \in H$, $f(x) \in K$, $u \in T(x)$ such that

$$\langle u, f(y) - f(x)\rangle \geq 0, \forall f(y) \in K.$$

This problem is called a multivalued variational inequality. It has been shown, that a wide class of multivalued odd-order and nonsymmetric free, obstacle, moving equilibrium and optimization problems arising in pure and applied sciences can be studied via the multivalued variational inequality, (see [171]).

In what follows we extend this problem to Banach spaces. Let X be a real Banach space and X^* be the topological dual of X. We denote by $\langle x^*, x\rangle$ the value of the linear and continuous functional $x^* \in X^*$ in $x \in X$. Let $K \subseteq X$ be nonempty and let $T : K \rightrightarrows X^*$ and $f : K \longrightarrow X$ be given operators. Consider the following problems. Find an element $x \in K$, such that

$$\forall y \in K \, \exists u \in T(x) : \langle u, f(y) - f(x)\rangle \geq 0, \tag{3. 1}$$

$$\exists u \in T(x) : \forall y \in K \, \langle u, f(y) - f(x)\rangle \geq 0, \tag{3. 2}$$

$$\forall y \in K \, \forall v \in T(y) : \langle v, f(y) - f(x)\rangle \geq 0. \tag{3. 3}$$

It can be easily observed that if T is single valued, then (3.1), respectively (3.2) reduce to the general variational inequality of Stampacchia type, $VI_S(T, f, K)$, which consists in finding an element $x \in K$, such that $\langle T(x), f(y) - f(x)\rangle \geq 0$, for all $y \in K$, (see [132]). Let us denote by $S_w(T, f, K)$, respectively by $S(T, f, K)$ the set of solutions of (3.1), respectively the set of solutions of (3.2).

It is also obvious that if T is single valued, then (3.3) reduces to the general variational inequality of Minty type, $VI_M(T, f, K)$, which consists in finding an element $x \in K$, such that $\langle T(y), f(y) - f(x)\rangle \geq 0$, for all $y \in K$, (see [132]). Let us denote by $M(T, f, K)$ the set of solutions of (3.3).

In order to continue our analysis we need the following notion. Let X_1, respectively X_2 be Hausdorff topological spaces and let $T : X_1 \rightrightarrows X_2$ be a set-valued operator with nonempty values. T is said to be upper semicontinuous if, for every $x_0 \in X_1$ and for every open set N containing $T(x_0)$, there exists a neighborhood M of x_0 such that $T(M) \subseteq N$.

We have the following characterization of upper semicontinuity (see [166]).

Lemma 3.4.1. *If T is compact-valued, then T is upper semicontinuous if and only if, for every net $(x_i) \subseteq X_1$ such that $x_i \longrightarrow x_0 \in X_1$ and for every $z_i \in T(x_i)$, there exist $z_0 \in T(x_0)$ and a subnet (z_{i_j}) of (z_i) such that $z_{i_j} \longrightarrow z_0$, (see also [90]). If X_1, respectively X_2 are metric spaces, instead of nets one can consider sequences (see [178]).*

Let X and Y be two Banach spaces. Recall that an operator $T : X \longrightarrow Y$ is called weak to norm-sequentially continuous at $x \in X$, if for every sequence (x_n) that converges weakly to x, we have that $T(x_n)$ converges to $T(x)$ in the topology of the norm of Y. An operator $T : X \rightrightarrows Y$ is said to be weak to weak* upper semicontinuous if, for every $x_0 \in X$ and for every open set $N \subseteq Y$, in the weak* topology of Y, containing $T(x_0)$, there exists a neighborhood M of x_0, in the weak topology of X, such that $T(M) \subseteq N$.

The following results will be very useful in the proof of our main existence results in the next section.

Lemma 3.4.2. *If $P \subset Q \subset X$, where Q is weakly compact and P is weakly sequentially closed then P is weakly compact.*

Proof. Indeed, by Eberlein-Šmulian theorem, (see, for instance, [86]), Q is weakly sequentially compact. Let $(x_n) \subseteq P$, hence $(x_n) \subseteq Q$, which is weakly sequentially compact. Hence, there exists $(x_{n_k}) \subseteq (x_n)$, weakly convergent to a point $x \in Q$. But obviously $(x_{n_k}) \subseteq P$, which is weakly sequentially closed, hence $x \in P$. Thus P is weakly sequentially compact and according to Eberlein-Šmulian theorem P is weakly compact. □

Lemma 3.4.3. *Consider a bounded net $((x_i, x_i^*))_{i \in I} \subset X \times X^*$, and assume that one of the following conditions is fulfilled:*

a) $x_i \rightharpoonup x$, i.e. the net (x_i) converges to x in the weak topology of X, and $x_i^* \longrightarrow x^*$, i.e. the net (x_i^*) converges to x^* in the topology of norm of X^*.

b) $x_i \longrightarrow x$, i.e. the net (x_i) converges to x in the topology of norm of X, and $x_i^* \rightharpoonup^* x^*$, i.e. the net (x_i^*) converges to x^* in the weak* topology of X^*.

Then $\langle x_i^*, x_i \rangle \longrightarrow \langle x^*, x \rangle$.

Proof. Indeed, by the triangle inequality we have $|\langle x_i^* - x^*, x_i\rangle| \leq \|x_i^* - x^*\|\|x_i\|$ if $a)$ is satisfied and $|\langle x_i^*, x_i - x\rangle| \leq \|x_i^*\|\|x_i - x\|$ if $b)$ is satisfied, hence $\langle x_i^*, x_i\rangle - \langle x^*, x_i\rangle \longrightarrow 0$, or $\langle x_i^*, x_i\rangle - \langle x_i^*, x\rangle \longrightarrow 0$, which shows that $\langle x_i^*, x_i\rangle \longrightarrow \langle x^*, x\rangle$. □

3.4.2 A Ky Fan type result

In what follows we present the notion of KKM mapping, which was initially introduced by Knaster, Kuratowski and Mazurkiewicz, and will be very useful in establishing the existence of the solutions for the problems $(3.1) - (3.3)$.

Let X be a real linear space and let $D \subseteq X$. Recall that the convex hull of the set D is defined as the set

$$\mathrm{co}(D) := \left\{ \sum_{i=1}^{n} \lambda_i x_i : x_i \in D, \sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geq 0, \text{ for all } i \in \{1, 2, \ldots, n\}, n \in \mathbb{N} \right\}.$$

Definition 3.4.1. *(Knaster-Kuratowski-Mazurkiewicz) Let X be a Hausdorff topological real linear space and let $M \subseteq X$. The set-valued mapping $G : M \rightrightarrows X$ is called a KKM mapping, if for every finite number of elements $x_1, x_2, \ldots, x_n \in M$ one has*

$$\mathrm{co}\{x_1, x_2, \ldots, x_n\} \subseteq \bigcup_{i=1}^{n} G(x_i).$$

The following result is due to Ky Fan (see [88]).

Lemma 3.4.4. *Let X be a Hausdorff topological real linear space, $M \subseteq X$ and $G : M \rightrightarrows X$ be a KKM mapping. If $G(x)$ is closed for every $x \in M$, and there exists $x_0 \in M$, such that $G(x_0)$ is compact, then*

$$\bigcap_{x \in M} G(x) \neq \emptyset.$$

In what follows we give a generalization of this result, which will be used in the proof of Theorem 3.4.3 in the next section.

Lemma 3.4.5. *Let X be a Banach space, let $M \subseteq X$ be a nonempty set, and let $G : M \rightrightarrows X$ be a KKM mapping. If $G(x)$ is weakly sequentially closed for every $x \in M$, and there exists $x_0 \in M$, such that $G(x_0)$ is weakly compact, then*

$$\bigcap_{x \in M} G(x) \neq \emptyset.$$

Proof. We show that the family of sets $\{G(x) : x \in M\}$ has the finite intersection property, that is, for every finite set $N \subseteq M$ we have $\bigcap_{x\in N} G(x) \neq \emptyset$.

Let $x_1, x_2, \ldots, x_n \in M$ and let $E = \mathrm{co}\{e_1, e_2, \ldots, e_n\}$, where e_i, $i \in \{1, 2, \ldots, n\}$ are those unit vectors in $\mathbb{R}^n$ which form the canonical base.

Consider the mapping $F : \{e_1, e_2, \ldots, e_n\} \rightrightarrows E$,

$$F(e_i) = \left\{ \sum_{j=1}^{n} \lambda_j e_j : \lambda_j \geq 0,\, j \in \{1, 2, \ldots, n\}, \sum_{j=1}^{n} \lambda_j = 1,\, \sum_{j=1}^{n} \lambda_j x_j \in G(x_i) \right\}.$$

Obviously, for all $i \in \{1, 2, \ldots, n\}$, $F(e_i) \subseteq E$, hence is bounded. We show that for all $i \in \{1, 2, \ldots, n\}$, $F(e_i)$ is closed.

Indeed, let $(y_k) \subseteq F(e_i)$ be a sequence and assume that $y_k \longrightarrow y$, $k \longrightarrow \infty$. We show that $y \in F(e_i)$. It is obvious that for all $k \geq 1$ we have

$$y_k = \sum_{j=1}^{n} \lambda_j^k e_j, \text{ where } \lambda_j^k \geq 0, \text{ for all } j \in \{1, 2, \ldots, n\}, \sum_{j=1}^{n} \lambda_j^k = 1, \text{ and } \sum_{j=1}^{n} \lambda_j^k x_j \in G(x_i).$$

Note that $y \in \mathbb{R}^n$ can be written in the form $y = \sum_{j=1}^{n} \alpha_j e_j$, hence we have $\lambda_j^k \longrightarrow \alpha_j$, $k \longrightarrow \infty$, for all $j \in \{1, 2, \ldots, n\}$. Since $\lambda_j^k \geq 0$, for all $j \in \{1, 2, \ldots, n\}$ and $\sum_{j=1}^{n} \lambda_j^k = 1$, we obtain that $\alpha_j \geq 0$, for all $j \in \{1, 2, \ldots, n\}$ and $\sum_{j=1}^{n} \alpha_j = 1$.

Since $G(x_i)$ is weakly sequentially closed, we have

$$\sum_{j=1}^{n} \lambda_j^k x_j \rightharpoonup \sum_{j=1}^{n} \alpha_j x_j \in G(x_i),\, k \longrightarrow \infty,$$

where $\rightharpoonup$ denotes the convergence in the weak topology of X. Hence $y \in F(e_i)$, which combined with its boundedness implies that $F(e_i)$ is compact.

We show next, that F is a KKM mapping. Let $r \in \mathrm{co}\{e_{i_1}, \ldots, e_{i_k}\}$, $r = \sum_{j=1}^{k} \lambda_j e_{i_j}$, where $\lambda_j \geq 0$, $j \in \{1, 2, ..., k\}$ and $\sum_{j=1}^{k} \lambda_j = 1$. Then $\sum_{j=1}^{k} \lambda_j x_{i_j} \in \mathrm{co}\{x_{i_1}, \ldots, x_{i_k}\}$, and since G is a KKM mapping, we obtain that

$$\sum_{j=1}^{k} \lambda_j x_{i_j} \in \bigcup_{j=1}^{k} G(x_{i_j}).$$

Hence, there exists $s \in \{1, 2, ..., k\}$ such that $\sum_{j=1}^{k} \lambda_j x_{i_j} \in G(x_{i_s})$, consequently $r \in F(e_{i_s})$, which shows that $\mathrm{co}\{e_{i_1}, \ldots, e_{i_k}\} \subseteq \bigcup_{j=1}^{k} F(e_{i_j})$.

Hence, F is a KKM mapping and according to Ky Fan's Lemma $\bigcap_{i=1}^{n} F(e_i) \neq \emptyset$, which

leads to $\bigcap_{i=1}^{n} G(x_i) \neq \emptyset$.

Since $G(x)$ is weakly sequentially closed for all $x \in M$, and $G(x_0)$ is weakly compact, we have $G(x) \cap G(x_0) \subseteq G(x_0)$ weakly sequentially closed for all $x \in M$. According to Lemma 3.4.2, $G(x) \cap G(x_0)$ is weakly compact for all $x \in M$, which particulary shows that $G(x) \cap G(x_0)$ is weakly closed for all $x \in M$.

According for the first part of the proof $\{G(x) \cap G(x_0) \subseteq G(x_0) : x \in M\}$ has the finite intersection property, and by the weak compactness of $G(x_0)$ we obtain

$$\bigcap_{x \in M} (G(x) \cap G(x_0)) \neq \emptyset$$

which shows, that

$$\bigcap_{x \in M} G(x) \neq \emptyset.$$

□

Let X be a real linear space. For $x, y \in X$ let us denote by $[x,y] := \{z = (1-t)x + ty : t \in [0,1]\}$ the closed line segment with the endpoints x respectively y. The open line segment with the endpoints x respectively y is defined by $(x,y) := [x,y] \setminus \{x,y\} = \{z = (1-t)x + ty : t \in (0,1)\}$. We need the following notion introduced in [132].

Definition 3.4.2. *Let X and Y be two real linear spaces. We say that the operator $T : D \subseteq X \longrightarrow Y$ is of type ql, if for every $x, y \in D$, and every $z \in [x,y] \cap D$, one has $T(z) \in [T(x), T(y)]$.*

This concept is a generalization of the notion of monotonicity of a real valued function of one real variable. It was also shown (see [132]), that this notion may be viewed as a generalization of the concept of a linear or affine operator as well, even more, when an operator takes its values in $\mathbb{R}$ then it is of type ql if, and only if, is quasilinear. To provide a nontrivial example of an operator of type ql we need the following result from [132].

Proposition 3.4.1. *Let X, Y, Z be real linear spaces, $D \subseteq X$, $C \subseteq Y$ and let $A : D \longrightarrow Y$, $B : C \longrightarrow Z$, $A(D) \subseteq C$ be two operator of type ql. Then $B \circ A : D \longrightarrow Z$ is of type ql.*

In virtute of Proposition 3.4.1, one can easily obtain ql type operators from existing ones as the next example shows.

Example 3.4.1. *Let $L : C[a,b] \longrightarrow C[a,b]$, be the odd power of a Fredholm integral operator with separable kernel, that is*

$$L(u)(x) = \left(\int_a^b k(x,s)u(s)ds \right)^{2n+1}, n \in \mathbb{N},$$

where the kernel $k(x,s) = \alpha(x)\beta(s)$, $\alpha, \beta \in C[a,b]$. *Then* L *is an operator of type ql.*

Obviously L is neither linear nor affine. Let $r : \mathbb{R} \longrightarrow \mathbb{R}$, $r(t) = t^{2n+1}$, $n \in \mathbb{N}$ and let $p : C[a,b] \longrightarrow \mathbb{R}$, $p(u) = \int_a^b u(s)ds$. Then r is a monotone real valued function, hence is of type ql. Obviously p is of type ql since is linear, consequently, according to Proposition 3.4.1, the operator $l : C[a,b] \longrightarrow \mathbb{R}, l(u) = (r \circ p)(u)$ is of type ql.

Next we prove that for every $u, v \in C[a,b]$ and $w \in [u,v]$, there exists $\lambda \in [0,1]$, such that $L(w)(x) = (1-\lambda)L(u)(x) + \lambda L(v)(x)$, for all $x \in [a,b]$.

We have

$$L(w)(x) = \left(\int_a^b \alpha(x)\beta(s)w(s)ds\right)^{2n+1} = \alpha^{2n+1}(x)l(\beta w).$$

Since l is of type ql and $\beta w \in [\beta u, \beta v]$, we have $l(\beta w) \in [l(\beta u), l(\beta v)]$, hence there exists $\lambda \in [0,1]$, such that $l(\beta w) = (1-\lambda)l(\beta u) + \lambda l(\beta v)$. Consequently,

$$L(w)(x) = \alpha^{2n+1}(x)((1-\lambda)l(\beta u) + \lambda l(\beta v)) = (1-\lambda)L(u)(x) + \lambda L(v)(x),$$

for all $x \in [a,b]$, or, equivalently $L(w) \in [L(u), L(v)]$, which shows that L is of type ql.

In our next result we need the following continuity property of operators. Let X be a topological real linear space, let Y be a topological space, and let $T : D \subseteq X \longrightarrow Y$ be an operator. We say, that T is continuous on line segments in D, if its restriction on every line segment in D is continuous. The following result extends Lemma 3.1 form [132].

Lemma 3.4.6. *Let* X *be a topological real linear space and let* Y *be a Hausdorff topological real linear space, let* $D \subseteq X$ *convex, and* $A : D \longrightarrow Y$ *an operator continuous on line segments and of type ql. Then for every* $x, y \in D$ *one has* $A([x,y]) = [A(x), A(y)]$.

Proof. Consider $x, y \in D$, $x \neq y$. If $A(x) = A(y)$ then, there is nothing to prove. Suppose that $A(x) \neq A(y)$, and let $u \in (A(x), A(y))$. We show that there exists $z \in [x,y]$ with $A(z) = u$.

Let us consider the function $\varphi : [0,1] \longrightarrow [A(x), A(y)]$, $\varphi(t) = A((1-t)x + ty)$. Obviously φ is continuous. We show that there exists $t \in [0,1]$ such that $\varphi(t) = u$. Let $M := \{t \in [0,1] : \varphi(t) \in [A(x), u]\}$, and let $s := \sup M$. We show that $\varphi(s) = u$.

Suppose that $\varphi(s) \in [A(x), u)$. Since φ is continuous, there exists $\varepsilon_0 > 0$ such that $\varphi(t) \in [A(x), u)$ for all $t \in (s - \varepsilon_0, s + \varepsilon_0)$. Let $t_0 \in (s - \varepsilon_0, s + \varepsilon_0)$, $t_0 > s$. Then $t_0 \in M$, since $\varphi(t_0) \in [A(x), u]$, hence $t_0 \leq s$, contradiction.

Suppose that $\varphi(s) \in (u, A(y))$. Since φ is continuous, there exists $\varepsilon_0 > 0$ such that $\varphi(t) \in (u, A(y)]$ for all $t \in (s-\varepsilon_0, s+\varepsilon_0)$. Since $s = \sup M$, there exists $t_0 \in M$ such that $s < t_0 + \frac{\varepsilon_0}{2}$. But then $t_0 \in (s-\varepsilon_0, s+\varepsilon_0)$, hence $t_0 \notin M$, contradiction. □

We will also use the following result from [132].

Theorem 3.4.1. *Let X and Y be two real linear spaces, let $D \subseteq X$ be convex, and let $T : D \longrightarrow Y$ be an operator of type ql. Then for every $n \in \mathbb{N}$, every $x_1, x_2, \ldots, x_n \in D$, and every $x \in co\{x_1, x_2, \ldots, x_n\}$, we have $T(x) \in co\{T(x_1), T(x_2), \ldots, T(x_n)\}$.*

Moreover, we need the following definition, (see [170]).

Definition 3.4.3. *Let X be a real Banach space, X^* be its topological dual, and let $T : D \subseteq X \rightrightarrows X^*$ and $f : D \longrightarrow X$ be given operators. We say that T is f-pseudomonotone, if for all $x, y \in D$, $u \in T(x)$, $v \in T(y)$, $\langle u, f(y) - f(x)\rangle \geq 0 \implies \langle v, f(y) - f(x)\rangle \geq 0$.*

Remark 3.4.1. *Observe that this notion is actually the generalization of the notion of pseudomonotonicity of multivalued operators that was initially introduced by J. C. Yao in [208]. Indeed, in the case when f is the identity, that is $f(x) = x$, for all $x \in D$, the notion of f-pseudomonotonicity of multivalued operators reduce to the the concept of pseudomonotonicity for multivalued operators.*

3.4.3 The existence of solutions

In this section, by making use of Lemma 3.4.4, respectively Lemma 3.4.5, we establish some existence results of the solution for the variational inequality problems (3. 1)-(3. 3). By an example we show that the condition that the operator f, involved in these variational inequalities, is of type ql, is essential in obtaining these results.

In what follows, unless is otherwise specified, we assume that X is a real Banach space and X^* is the topological dual of X.

Theorem 3.4.2. *Let K be a nonempty, weakly compact and convex subset of X, and consider the set-valued map $T : K \rightrightarrows X^*$. Further, let $f : K \longrightarrow X$ be an operator of type ql. Assume that one of the following conditions is fulfilled.*

a) f is weak to weak-sequentially continuous on K, T is weak to norm upper semicontinuous on K and $T(x)$ is compact for every $x \in K$.

b) f is weak to norm-sequentially continuous on K, T weak to weak* upper semicontinuous on K and $T(x)$ is weak* compact for every $x \in K$.

Then, $S_w(T,f,K) \neq \emptyset$. If in addition T is f-pseudomonotone, then $M(T,f,K) \neq \emptyset$.

Proof. We prove that the mapping $G : K \rightrightarrows K$,

$$G(y) := \{x \in K : \exists u \in T(x) \text{ such that } \langle u, f(y) - f(x)\rangle \geq 0\}$$

satisfies the assumptions of Ky Fan's lemma.

Let $y_1, y_2, ..., y_n \in K$ and $y \in \text{co}\{y_1, y_2, ..., y_n\}$. Let us suppose that $y \notin \bigcup_{i=1}^n G(y_i)$. Then, for all $u \in T(y)$, we have

$$\langle u, f(y_1) - f(y)\rangle < 0, \langle u, f(y_2) - f(y)\rangle < 0, \ldots, \langle u, f(y_n) - f(y)\rangle < 0.$$

Since f is of type ql and $y \in \text{co}\{y_1, y_2, ..., y_n\}$, according to Theorem 3.4.1 we have $f(y) \in \text{co}\{f(y_1), f(y_2), ..., f(y_n)\}$, hence, there exists $\lambda_i \geq 0$, $i = \overline{1,n}$ where $\sum_{i=1}^n \lambda_i = 1$, such that

$$f(y) = \sum_{i=1}^n \lambda_i f(y_i).$$

For every fixed $u \in T(y)$ we have

$$0 > \sum_{i=1}^n \lambda_i \langle u, f(y_i) - f(y)\rangle = \sum_{i=1}^n \langle u, \lambda_i (f(y_i) - f(y))\rangle = 0, \text{ contradiction.}$$

Hence, G is a KKM mapping. We show next that $G(y)$ is weakly compact for all $y \in K$. Obviously $G(y) \neq \emptyset$, since for all $y \in K$ we have $y \in G(y)$. For $y \in K$ consider a sequence $(x_n) \subseteq G(y)$ that converges weakly to $x \in K$. Hence, there exists $u_n \in T(x_n)$ such that $\langle u_n, f(y) - f(x_n)\rangle \geq 0$.

Assume that a) holds. From Lemma 3.4.1 we obtain that the sequence (u_n) contains a subsequence (u_{n_k}) that converges to a $v \in T(x)$ in the norm topology of X^*. Since f is weak to weak-sequentially continuous we obtain that $f(x_{n_k})$ converges to $f(x)$, $k \longrightarrow \infty$ in the weak topology of X. According to Lemma 3.4.3 a) we have $\langle u_{n_k}, f(y) - f(x_{n_k})\rangle \longrightarrow \langle v, f(y) - f(x)\rangle$, $k \longrightarrow \infty$, and from $\langle u_{n_k}, f(y) - f(x_{n_k})\rangle \geq 0$ for all $k \in \mathbb{N}$, we get that $\langle v, f(y) - f(x)\rangle \geq 0$.

Assume that b) holds. From Lemma 3.4.1 we obtain that the sequence (u_n) contains a subsequence (u_{n_k}) that converges to a $v \in T(x)$ in the weak* topology of X^*. Since f is weak to norm-sequentially continuous we obtain that $f(x_{n_k}) \longrightarrow f(x)$, $k \longrightarrow \infty$. According to Lemma 3.4.3 b) we have $\langle u_{n_k}, f(y) - f(x_{n_k})\rangle \longrightarrow \langle v, f(y) - f(x)\rangle$, $k \longrightarrow \infty$, and from $\langle u_{n_k}, f(y) - f(x_{n_k})\rangle \geq 0$ for all $k \in \mathbb{N}$, we get that $\langle v, f(y) - f(x)\rangle \geq 0$.

Hence, $x \in G(y)$, which means that $G(y)$ is weakly sequentially closed for all $y \in K$. But $G(y) \subseteq K$ and K is weakly compact, thus according to Lemma 3.4.2, $G(y)$ is weakly compact for all $y \in K$, which obviously implies that is weakly closed as well.

Hence, G is a KKM mapping that satisfies the assumptions of Ky Fan's lemma, consequently, $\bigcap_{y \in K} G(y) \neq \emptyset$. In other word, there exists $x \in K$ such that for every $y \in K$ there exists $u \in T(x)$ satisfying $\langle u, f(y) - f(x) \rangle \geq 0$. This shows that $S_w(T, f, K) \neq \emptyset$.

Assume now, that T is f-pseudomonotone. Let $x \in S_w(T, f, K)$. Then, for all $y \in K$ there exists $u \in T(x)$, such that $\langle u, f(y) - f(x) \rangle \geq 0$, and from the f-pseudomonotonicity of T we obtain, that $\langle v, f(y) - f(x) \rangle \geq 0$ for all $v \in T(y)$, hence $x \in M(T, f, K)$. □

As an immediate consequence we obtain a result different from those that have been established in [132].

Corollary 3.4.1. *Let K be a nonempty, weakly compact and convex subset of X, and consider the operators $T : K \longrightarrow X^*$ and $f : K \longrightarrow X$. Assume further, that the operator f is of type ql and that one of the following conditions is fulfilled.*

a) f is weak to weak-sequentially continuous on K, T is weak to norm-sequentially continuous on K.

b) f is weak to norm-sequentially continuous on K, T weak to weak*-sequentially continuous on K.

Then, the general variational inequality of Stampacchia type, $VI_S(T, f, K)$, admits solutions. If in addition T is f-pseudomonotone, then the general variational inequality of Minty type, $VI_M(T, f, K)$, admits solutions.

Remark 3.4.2. According to the proof of Theorem 3.4.2, $G(y)$ is weakly compact for every $y \in K$. On the other hand, Ky Fan's lemma requires the existence of only one element $y \in K$ with this property, hence, the conditions of Theorem 3.4.2 may be weakened.

In reflexive Banach spaces we have the following result.

Theorem 3.4.3. *Let X be a reflexive Banach space and let $K \subseteq X$ be a nonempty, weakly sequentially closed and convex subset. Consider the set-valued map $T : K \rightrightarrows X^*$ and let $f : K \longrightarrow X$ be an operator of type ql. Assume that there exists $y_0 \in K$ such that*

$$\liminf_{\|x\| \longrightarrow \infty, x \in K} \inf_{u \in T(x)} \langle u, f(x) - f(y_0) \rangle > 0.$$

Moreover, assume that one of the following conditions is fulfilled.

a) *f is weak to weak-sequentially continuous on K, T is weak to norm upper semicontinuous on K and $T(x)$ is compact for every $x \in K$.*

b) *f is weak to norm-sequentially continuous on K, T weak to weak* upper semicontinuous on K and $T(x)$ is weak* compact for every $x \in K$.*

Then $S_w(T, f, K) \neq \emptyset$. If in addition T is f-pseudomonotone, then $M(T, f, K) \neq \emptyset$.

Proof. Let us define the mapping $G : K \rightrightarrows K$ as in the proof of Theorem 3.4.2. According to the proof of Theorem 3.4.2 $G(y)$ is weakly sequentially closed for all $y \in K$. We show that $G(y_0)$ is weakly compact. The rest of the proof is similar to the proof of Theorem 3.4.2 and will be omitted.

We prove that $G(y_0)$ is bounded. Indeed, supposing the contrary we obtain, that there exists $(x_k) \subseteq G(y_0)$, such that $\|x_k\| \longrightarrow \infty$, $k \longrightarrow \infty$. But, since $x_k \in G(y_0)$ for all $k \in \mathbb{N}$, we have that for all $k \in \mathbb{N}$,

$$\inf_{u \in T(x_k)} \langle u, f(x_k) - f(y_0) \rangle \leq 0.$$

Hence,

$$\liminf_{\|x_k\| \longrightarrow \infty} \inf_{u \in T(x_k)} \langle u, f(x_k) - f(y_0) \rangle \leq 0,$$

which contradicts the assumptions of the theorem. Thus, we have $G(y_0)$ is bounded and weakly sequentially closed. But then, there exists $N > 0$, such that $G(y_0) \subseteq B_N$, where B_N denotes the closed ball centered in 0 with radius N. Since X is reflexive, it is known that B_N is weakly compact. From here, in virtute of Lemma 3.4.2 we conclude that $G(y_0)$ is weakly compact. According to Lemma 3.4.5 one has

$$\bigcap_{x \in K} G(x) \neq \emptyset.$$

□

As an immediate consequence we obtain some results that have already been partially established in [132].

Corollary 3.4.2. *Let X be a reflexive Banach space. Let K be a nonempty, weakly sequentially closed and convex subset of X, and consider the operators $T : K \longrightarrow X^*$ and $f : K \longrightarrow X$, where the operator f is of type ql. Assume that there exists $y_0 \in K$ such that*

$$\liminf_{\|x\| \longrightarrow \infty, x \in K} \langle T(x), f(x) - f(y_0) \rangle > 0.$$

Further, assume that one of the following conditions is fulfilled.

a) f *is weak to weak-sequentially continuous on* K, T *is weak to norm-sequentially continuous on* K.

b) f *is weak to norm-sequentially continuous on* K, T *weak to weak***-sequentially continuous on* K.

Then, the general variational inequality of Stampacchia type, $VI_S(T,f,K)$, *admits solutions. If in addition* T *is* f*-pseudomonotone, then the general variational inequality of Minty type,* $VI_M(T,f,K)$, *admits solutions.*

Remark 3.4.3. *The method based on Ky Fan's lemma, used in the proof of Theorem 3.4.2 to establish the solutions existence of some variational inequalities, is well known in the literature, see, for instance, [81, 129, 134, 145, 149, 208, 209].*

The following concepts were introduced by Ky Fan, see [89]. Let X and Y be arbitrary sets. A function $h : X \times Y \longrightarrow \mathbb{R}$ is said to be convexlike on X, if for any $u_1, u_2 \in X$ and $t \in (0,1)$, there exists $u_0 \in X$ such that for all $y \in Y$, one has

$$h(u_0, y) \leq th(u_1, y) + (1-t)h(u_2, y).$$

Similarly, h is said to be concavelike on Y, if for any $v_1, v_2 \in Y$ and $t \in (0,1)$, there exists $v_0 \in Y$ such that for all $x \in X$, one has

$$h(x, v_0) \geq th(x, v_1) + (1-t)h(x, v_2).$$

These concepts are used in Fan's minimax theorem.

Theorem 3.4.4. *(Ky Fan) Let* X *be a compact space,* Y *a set, and* $h : X \times Y \longrightarrow \mathbb{R}$ *a function that is concavelike on* Y, *convexlike on* X, *and for each* $y \in Y$ *the function* $x \mapsto h(x,y)$ *is lower semicontinuous on* X. *Then*

$$\sup_{y \in Y} \min_{x \in X} h(x,y) = \min_{x \in X} \sup_{y \in Y} h(x,y).$$

In what follows, we present a result that ensures the existence of the solution for (3. 2).

Theorem 3.4.5. *Let* K *be nonempty, weakly compact and convex subset of* X, *and consider the set-valued map* $T : K \rightrightarrows X^*$. *Let* $f : K \longrightarrow X$ *be an operator of type ql. Assume that one of the following conditions is fulfilled.*

a) *f is weak to weak-sequentially continuous on K, T is weak to norm upper semicontinuous on K and $T(x)$ is compact and convex for every $x \in K$.*

b) *f is weak to norm-sequentially continuous on K, T weak to weak* upper semicontinuous on K and $T(x)$ is weak* compact and convex for every $x \in K$.*

Then, $S(T, f, K) \neq \emptyset$.

Proof. According to Theorem 3.4.2, whenever a) or b) holds we have $S_w(T, f, K) \neq \emptyset$. Let $x \in S_w(T, f, K)$. We prove that $x \in S(T, f, K)$. Indeed, by supposing the contrary, we obtain that for every $u \in T(x)$ there exists $y \in K$ such that $\langle u, f(y) - f(x)\rangle < 0$ hence, $\min_{y\in K}\langle u, f(y) - f(x)\rangle < 0$. But then, since $T(x)$ is compact (in the norm topology or in the weak* topology of X^*), we obtain that

$$\max_{u\in T(x)} \min_{y\in K}\langle u, f(y) - f(x)\rangle < 0. \tag{3. 4}$$

Consider the function $h : K \times T(x) \longrightarrow \mathbb{R}$, $h(y, u) = \langle u, f(y) - f(x)\rangle$. We show, that h satisfies the assumptions of Fan's minimax theorem. Since $T(x)$ is convex we have that $h(y, \cdot) : T(x) \longrightarrow \mathbb{R}$ is concave for every $y \in K$, hence h is concavelike on $T(x)$. Since f is of type ql, according to Lemma 3.4.6, for every $y_1, y_2 \in K$ we have $f([y_1, y_2]) = [f(y_1), f(y_2)]$, hence, for every $t \in (0, 1)$ we have $tf(y_1) + (1-t)f(y_2) = f(y_0)$, for some $y_0 \in [y_1, y_2]$. Thus, for every $y_1, y_2 \in K$ and $t \in (0, 1)$, there exists $y_0 \in [y_1, y_2]$, such that $\langle u, f(y_0) - f(x)\rangle = \langle u, tf(y_1) + (1-t)f(y_2) - f(x)\rangle$ for every $u \in T(x)$, that is $h(u, y_0) = th(u, y_1) + (1-t)h(u, y_2)$. Hence, h is convexlike on K. Obviously $h(\cdot, u)$ is weakly sequentially continuous on K for all $u \in T(x)$, hence according to Theorem 7.1.2 from [130], $h(\cdot, u)$ is weakly lower semicontinuous. According to Theorem 3.4.4, we have

$$\max_{u\in T(x)} \min_{y\in K} h(y, u) = \min_{y\in K} \max_{u\in T(x)} h(y, u).$$

In the same time, according to (3.4)

$$\max_{u\in T(x)} \min_{y\in K}\langle u, f(y) - f(x)\rangle = \max_{u\in T(x)} \min_{y\in K} h(y, u) < 0.$$

On the other hand, from $x \in S_w(T, f, K)$, we have that for every $y \in K$

$$\max_{u\in T(x)} \langle u, f(y) - f(x)\rangle = \max_{u\in T(x)} h(y, u) \geq 0,$$

which leads to

$$\min_{y\in K}\max_{u\in T(x)} h(y,u)\geq 0,$$

contradiction. □

Remark 3.4.4. *Obviously, if $x\in S(T,f,K)$ then $x\in S_w(T,f,K)$. Hence, under the assumptions of Theorem* 3.4.5*, $S_w(T,f,K)=S(T,f,K)$.*

The next example shows that the assumption that the operator f is of type ql is essential in the previous results even in finite dimension.

Example 3.4.2. *Let $K=[-1,1]\times[-1,1]\subseteq\mathbb{R}^2$ and consider the operators $T:K\rightrightarrows\mathbb{R}^2$,*

$$T(x,y)=[-2,-1-x^2y^2]\times\{xy\} \text{ and } f:K\longrightarrow\mathbb{R}^2,\ f(x,y)=(-xy,xy).$$

Then all the assumptions of Theorem 3.4.2 and Theorem 3.4.5 are satisfied excepting the one that f is of type ql, hence their conclusion fail.

Indeed, it can easily be verified that K is compact and convex and $T(x,y)$ is compact and convex for all $(x,y)\in K$. We show that T is upper semicontinuous.

Consider the sequence $(x_n,y_n)\subseteq K$, $(x_n,y_n)\longrightarrow(x,y)\in K$ and let $(u_n,v_n)\in T(x_n,y_n)$, $n\in\mathbb{N}$. Then

$$-2\leq u_n\leq -1-x_n^2y_n^2\leq -1,\ v_n=x_ny_n.$$

Since (u_n) is bounded contains a convergent subsequence (u_{n_k}). Assume that $u_{n_k}\longrightarrow u$, $k\longrightarrow\infty$. From $-2\leq u_{n_k}\leq -1-x_{n_k}^2y_{n_k}^2$ we have $-2\leq u\leq -1-x^2y^2$. Obviously $v_{n_k}\longrightarrow xy$, $k\longrightarrow\infty$, hence $(u_{n_k},v_{n_k})\longrightarrow(u,xy)\in T(x,y)$. According to Lemma 3.4.1, T is upper semicontinuous.

Note that f is continuous. We show that f is not of type ql. Indeed, we have $f(-1,-1)=f(1,1)=(-1,1)$ and for $(0,0)\in[(-1,-1),(1,1)]$ we obtain

$$f(0,0)=(0,0)\notin[f(-1,-1),f(1,1)]=(-1,1).$$

This shows that f is not of type ql.

Suppose that there exist $(x,y)\in K$, $(u,v)\in T(x,y)$, such that $\langle(u,v),f(p,q)-f(x,y)\rangle\geq 0$ for all $(p,q)\in K$.

Let $(p,q)=(1,1)$. Then $\langle(u,xy),(-1+xy,1-xy)\rangle\geq 0$, or equivalently $(1-xy)(xy-u)\geq 0$. But obviously $1-xy\geq 0$, thus $xy-u\geq 0$.

Let $(p,q) = (1,-1)$. Then $\langle (u,xy),(1+xy,-1-xy)\rangle \geq 0$, or equivalently $(1+xy)(u-xy) \geq 0$. But obviously $1+xy \geq 0$, thus $u-xy \geq 0$.

Hence, we must have $u = xy$ but then $xy \in [-2,-1-x^2y^2]$ which leads to $(xy)^2+xy+1 \leq 0$, impossible.

3.4.4 Coincidence points IV.

In this paragraph we apply the existence results of solutions for the variational inequality problems (3. 1)-(3. 3) that we have obtained in previous paragraphs, to establish some coincidence point results involving operators of type ql. As a particular case we obtain Kakutani's fixed point theorem. For other works where by using some similar techniques to those presented in this paragraph, coincidence, respectively fixed point results were obtained, we refer the reader to [104, 129, 134, 149]. Everywhere in the sequel H denotes a real Hilbert space, identified with its dual. Recall that the range of the set-valued operator $F : K \subseteq H \rightrightarrows H$ is the set

$$R(F) := \bigcup_{x\in K} F(x).$$

Our first coincidence point result is stated as follows.

Theorem 3.4.6. *Let K be a nonempty, weakly compact and convex subset of H, and consider the weak to weak upper semicontinuous set-valued map $F : K \rightrightarrows H$ with weakly compact and convex values. Let $f : K \longrightarrow H$ be a weak to norm continuous operator which is of type ql. Assume that $R(F) \subseteq f(K)$. Then, there exists $x \in K$ such that $f(x) \in F(x)$.*

Proof. Consider the set-valued mapping $T : K \rightrightarrows H$, $T(x) = f(x) - F(x)$, which is obviously nonempty, convex and weakly compact valued. We show that T is weak to weak upper semicontinuous. Indeed, according to Lemma 3.4.1, it is enough to show, that for every weak-convergent net $(x_i) \subseteq K$, that is $x_i \rightharpoonup x^0 \in K$, and for every net $z_i \in T(x_i)$, there exists $z^0 \in T(x^0)$ and a subnet $(z_{i_j}) \subseteq (z_i)$, such that $z_{i_j} \rightharpoonup z^0$, that is (z_{i_j}) converges to z^0 in the weak topology of H.

Let $(x_i) \subseteq K, x_i \rightharpoonup x^0 \in K$ and let $z_i \in T(x_i)$. Then $z_i = f(x_i) - y_i$, where $y_i \in F(x_i)$. Since F is weakly compact valued and weak to weak upper semicontinuous, according to Lemma 3.4.1 there exists $y^0 \in F(x^0)$, and a subnet $(y_{i_j}) \subseteq (y_i)$ such that $y_{i_j} \rightharpoonup y^0$. But then $z_{i_j} \rightharpoonup f(x^0) - y^0 \in T(x^0)$, hence T is weak to weak upper semicontinuous.

According to Theorem 3.4.5 b), $S(T,f,K) \neq \emptyset$. Let $x \in S(T,f,K)$, i.e. there exists $u \in T(x)$ such that $\langle u, f(y) - f(x)\rangle \geq 0$ for all $y \in K$. But $u = f(x) - v$ for some $v \in F(x)$, hence,

for all $y \in K$ one has

$$\langle f(x) - v, f(y) - f(x) \rangle \geq 0.$$

Since $R(F) \subseteq f(K)$, let $y \in K$, such that $f(y) = v$. Then we obtain

$$\langle f(x) - v, v - f(x) \rangle \geq 0,$$

or equivalently

$$-\|v - f(x)\|^2 \geq 0,$$

which leads to $f(x) = v \in F(x)$. □

As an immediate consequence, in finite dimension we obtain a result that can be viewed as an extension of Kakutani's fixed point theorem.

Corollary 3.4.3. *Let $K \subseteq \mathbb{R}^n$ be a nonempty, compact and convex set, and consider the upper semicontinuous set-valued map $F : K \rightrightarrows \mathbb{R}^n$ with compact and convex values. Moreover, let $f : K \longrightarrow \mathbb{R}^n$ be a continuous operator which is of type ql. Assume that $R(F) \subseteq f(K)$. Then, there exists $x \in K$ such that $f(x) \in F(x)$.*

Obviously if $K \subseteq \mathbb{R}^n$ then $\mathrm{id}_K : K \longrightarrow K$, $\mathrm{id}_K(x) = x$ is of type ql (as a restriction of a linear operator), and $R(F) \subseteq f(K)$ is equivalent to $R(F) \subseteq K$. If K is compact, then from $R(F) \subseteq K$ we obtain that $T(x) \subseteq K$ for all $x \in K$, hence if $T(x)$ is closed then it is also compact. Next we state Kakutani's fixed point theorem (see [119]).

Corollary 3.4.4. *Let K be a nonempty, compact and convex subset of $\mathbb{R}^n$. Let $F : K \rightrightarrows K$ be an upper semicontinuous set-valued map on K with the property that $F(x)$ is non-empty, closed, and convex for all $x \in K$. Then F has a fixed point.*

The next coincidence point result can be obtained from Theorem 3.4.5 a). Unfortunately in this case we cannot assume directly a type of upper semicontinuity for the set-valued operator F as we did in Theorem 3.4.6.

Theorem 3.4.7. *Let $K \subseteq H$ be a nonempty, weakly compact and convex set, and consider the set-valued map $F : K \rightrightarrows H$ with weakly compact and convex values. Further, let $f : K \longrightarrow H$ be a weak to weak-sequentially continuous operator which is of type ql. Assume that $R(F) \subseteq f(K)$ and that the map $f - F$ is weak to norm upper semicontinuous. Then, there exists $x \in K$ such that $f(x) \in F(x)$.*

Proof. Consider the map $T : K \rightrightarrows H$, $T(x) = f(x) - F(x)$. By making use of Theorem 3.4.5 a), similarly to the proof of Theorem 3.4.6 we obtain that there exists $x \in K$ such that $f(x) \in F(x)$. □

In virtue of weak to weak-sequential continuity of the map $\mathrm{id}_K : K \longrightarrow K$, $\mathrm{id}_K(x) = x$, as a corollary we have the following fixed point result.

Corollary 3.4.5. *Let $K \subseteq H$ be a nonempty, weakly compact and convex set, and consider the set-valued map $F : K \rightrightarrows K$ with weakly compact and convex values. Assume that the map $\mathrm{id}_K - F$ is weak to norm upper semicontinuous. Then F has a fixed point.*

4

Equilibrium and minimax problems on dense sets

4.1 Densely defined equilibrium problems

Equilibrium problems play an important role in nonlinear analysis especially due to their implications in mathematical economics, and, due to its key applications, Ky Fan's minimax inequality [87] is considered to be the most notable existence result in this field.

More recently, A. Kristály and Cs. Varga [129] were able to prove two set-valued versions of Ky Fan's inequality. Their results guarantee the existence of solutions to set-valued equilibrium problems, that they introduced, on the trail of Browder's study of variational inclusions [68].

The present section is devoted to set-valued equilibrium problems, as defined in [129]. More precisely, we show that the hypotheses of the existence result of Kristály and Varga (Theorems 2.1 and 2.2 in [129]) can be weakened in the sense that the convexity and continuity assumptions must not hold on the whole domain, but just on a special type of dense subset of it, that we call *self segment-dense* [143].

This new concept is related to, but different from, that of a segment-dense set introduced by Dinh The Luc [149] in the context of densely quasimonotone, respectively densely pseudomonotone operators. In one dimension, the concepts of a segment-dense set, respectively a self segment-dense set, are equivalent to the concept of a dense set. Nevertheless, in dimension greater than one, self segment-dense subsets enjoy certain special properties, characterized by Lemma 4.1.2, which play a crucial role in obtaining our existence results.

We explore the role of self segment-dense sets in the context of equilibrium problems

both with and without compactness assumptions, and show how our abstract results can be applied.

The two applications, that we have in mind, concern the theory of economic equilibrium and game theory. In fact, we prove a result of Debreu-Gale-Nikaïdo type [84, 96, 167], that states the existence of an economic equilibrium, even if the constraint imposed by Walras' law holds only on a self segment-dense subset of the price simplex. Our second result proves the existence of Nash equilibria for non-cooperative n-person games under assumptions that are more general than those of the classical theory [29].

In an infinite dimensional real Hilbert space, it is known that the unit sphere is dense in the unit ball with respect to the weak topology, but, as we will see, it is not self segment-dense. This is a typical example of a dense set, that is not self segment-dense. Using this example, we argue that it is not enough to impose the convexity and continuity assumptions on a dense subset of the domain, and that it is essential to work with a self segment-dense subset.

Let us mention that the results from this section were partially published in [144]:[S. László, A. Viorel, *Densely defined equilibrium problems*. J. Optim. Theory Appl. **166**, 52-75 (2015)].

4.1.1 Preliminaries

In what follows, X and Y denote Hausdorff topological spaces. For a non-empty set $D \subseteq X$, we denote by $\operatorname{int} D$ its interior and by $\operatorname{cl} D$ its closure. We say that $P \subseteq D$ is dense in D iff $D \subseteq \operatorname{cl} P$, and that $P \subseteq X$ is closed with respect to D iff $\operatorname{cl} P \cap D = P \cap D$.

Let $T : X \rightrightarrows Y$ be a set-valued operator. We denote by $D(T) = \{x \in X : T(x) \neq \emptyset\}$ its domain and by $R(T) = \bigcup_{x \in D(T)} T(x)$ its range. The graph of the operator T is the set $G(T) = \{(x,y) \in X \times Y : y \in T(x)\}$.

Recall that T is said to be *upper semicontinuous* at $x \in D(T)$ iff for every open set $N \subseteq Y$ containing $T(x)$, there exists a neighbourhood $M \subseteq X$ of x such that, one has $T(M) \subseteq N$. T is said to be *lower semicontinuous* at $x \in D(T)$ iff for every open set $N \subseteq Y$ satisfying $T(x) \cap N \neq \emptyset$, there exists a neighbourhood $M \subseteq X$ of x such that, for every point $y \in M \cap D(T)$ one has $T(y) \cap N \neq \emptyset$. T is upper semicontinuous (lower semicontinuous) on $D(T)$ iff it is upper semicontinuous (lower semicontinuous) at every $x \in D(T)$.

With T as before and $V \subseteq Y$, let us introduce the following sets

$$T^-(V) := \{x \in X : T(x) \cap V \neq \emptyset\},$$

and

$$T^{+}(V) := \{x \in X : T(x) \subseteq V\},$$

called the inverse image of V, respectively the core of V.

Remark 4.1.1. Let $T : X \rightrightarrows Y$ be a set valued map. The following characterizations of lower semicontinuity, respectively upper semicontinuity [30] can easily be proved.

(i) T is lower semicontinuous at $x \in D(T)$ if and only if, for every net $(x_\alpha) \subseteq D(T)$ such that $x_\alpha \longrightarrow x$ and for every $x^* \in T(x)$, there exists a net $x^*_\alpha \in T(x_\alpha)$ such that $x^*_\alpha \longrightarrow x^*$.

(ii) T is upper semicontinuous at $x \in D(T)$ if and only if, for every net $(x_\alpha) \subseteq D(T)$ such that $x_\alpha \longrightarrow x$ and every open set $V \subseteq Y$ such that $T(x) \subseteq V$, one has that $T(x_\alpha) \subseteq V$, for sufficiently large α.

(iii) T is lower semicontinuous if and only if, for all closed set $V \subseteq Y$, one has that $T^{+}(V)$ is closed in X.

(iv) T is upper semicontinuous if and only if, for all closed set $V \subseteq Y$, one has that $T^{-}(V)$ is closed in X.

Obviously, when T is single-valued, then upper semicontinuity and also lower semicontinuity become the usual notion of continuity.

For a function $f : X \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ we denote by $\operatorname{dom} f$ its domain, that is $\operatorname{dom} f = \{x \in X : f(x) \in \mathbb{R}\}$.

We say that f is upper semicontinuous at $x_0 \in \operatorname{dom} f$ iff, for every $\varepsilon > 0$, there exists a neighbourhood U of x_0 such that $f(x) \leq f(x_0) + \varepsilon$ for all $x \in U$. The function f is called upper semicontinuous iff it is upper semicontinuous at every point of its domain.

Furthermore, we say that f is lower semicontinuous at $x_0 \in \operatorname{dom} f$ iff, for every $\varepsilon > 0$, there exists a neighbourhood U of x_0 such that $f(x) \geq f(x_0) - \varepsilon$ for all $x \in U$. The function f is called lower semicontinuous iff it is lower semicontinuous at every point of its domain.

Remark 4.1.2. Let $f : X \longrightarrow \overline{\mathbb{R}}$ be a function. Then, we have the following characterizations of the lower semicontinuity, respectively the upper semicontinuity of f:

(i) f is upper semicontinuous at x_0 iff, $\limsup_{x^\alpha \to x_0} f(x^\alpha) \leq f(x_0)$, where (x^α) is a net converging to x_0.

(ii) f is lower semicontinuous at x_0 iff, $\liminf_{x^\alpha \to x_0} f(x^\alpha) \geq f(x_0)$, where (x^α) is a net converging to x_0.

(iii) f is upper semicontinuous on X iff, the superlevel set $\{x \in X : f(x) \geq a\}$ is a closed set for every $a \in \mathbb{R}$.

(iv) f is lower semicontinuous on X iff, the sublevel set $\{x \in X : f(x) \leq a\}$ is a closed set for every $a \in \mathbb{R}$.

4.1.2 Set-Valued Equilibrium Problems

Let X be a real normed space, let $K \subseteq X$ be a nonempty set and let $F : K \times K \rightrightarrows \mathbb{R}$ be a set valued map. According to [129], a set-valued equilibrium problem consists in finding $x_0 \in K$ such that

$$F(x_0, y) \geq 0, \forall y \in K.$$

Here, $F(x_0, y) \geq 0$ means that $u \geq 0$ for all $u \in F(x_0, y)$, or, in other words, that

$$F(x_0, y) \subseteq [0, \infty[= \mathbb{R}_+.$$

A different set-valued equilibrium problem, also formulated in [129], consists in finding an $x_0 \in K$ such that

$$F(x_0, y) \cap \mathbb{R}_- \neq \emptyset, \forall y \in K.$$

For the the convenience of the reader, we recall the original existence results of Kristály and Varga regarding the two set-valued equilibrium problems.

Theorem 4.1.1. *Let X be a real normed space, K be a nonempty, convex and compact subset of X, and $F : K \times K \rightrightarrows \mathbb{R}$ a set valued map satisfying*

(i) $\forall y \in K, x \longrightarrow F(x, y)$ *is lower semicontinuous on K,*

(ii) $\forall x \in K, y \longrightarrow F(x, y)$ *is convex on K,*

(iii) $\forall x \in K, F(x, x) \geq 0$.

Then, there exists an element $x_0 \in K$ such that

$$F(x_0, y) \geq 0, \forall y \in K.$$

Theorem 4.1.2. *Let X be a real normed space, K be a nonempty, convex and compact subset of X, and $F : K \times K \rightrightarrows \mathbb{R}$ a set valued map satisfying*

(i) $\forall y \in K, x \longrightarrow F(x, y)$ *is upper semicontinuous on K,*

(ii) $\forall x \in K$, $y \longrightarrow F(x,y)$ *is concave on* K,

(iii) $\forall x \in K$, $F(x,x) \cap \mathbb{R}_- \neq \emptyset$.

Then, there exists an element $x_0 \in K$ *such that*

$$F(x_0,y) \cap \mathbb{R}_- \neq \emptyset, \forall y \in K.$$

Obviously these results hold not only in real normed spaces, but also in Hausdorff topological vector spaces. The convexity of a set-valued map $F : D \subseteq X \rightrightarrows \mathbb{R}$, where X is Hausdorff topological vector space, is understood in sense that, for all $x_1, x_2, \ldots, x_n \in D$ and $\lambda_i \geq 0$, $i \in \{1,2,\ldots,n\}$, $\sum_{i=1}^n \lambda_i = 1$ such that $\sum_{i=1}^n \lambda_i x_i \in D$, one has

$$\sum_{i=1}^n \lambda_i F(x_i) \subseteq F\left(\sum_{i=1}^n \lambda_i x_i\right). \tag{4. 1}$$

Here, the usual Minkowski sum of sets is meant by the summation sign. To define concavity in the same setting, one replaces the last inclusion by

$$\sum_{i=1}^n \lambda_i F(x_i) \supseteq F\left(\sum_{i=1}^n \lambda_i x_i\right). \tag{4. 2}$$

Note that, in the definition of these notions, we do not assume that D is convex.

The classical single-valued equilibrium problem [88], described by a function $\varphi : K \times K \longrightarrow \mathbb{R}$, consists in finding $x_0 \in K$ such that

$$\varphi(x_0,y) \geq 0, \forall y \in K.$$

We recall the famous existence result of Ky Fan.

Theorem 4.1.3. *Let* K *be a nonempty, convex and compact subset of the Hausdorff topological vector space* X *and let* $\varphi : K \times K \longrightarrow \mathbb{R}$ *be a function satisfying*

(i) $\forall y \in K$, *the function* $x \longrightarrow \varphi(x,y)$ *is upper semicontinuous on* K,

(ii) $\forall x \in K$, *the function* $y \to \varphi(x,y)$ *is quasiconvex on* K,

(iii) $\forall x \in K$, $\varphi(x,x) \geq 0$.

Then, there exists an element $x_0 \in K$ *such that*

$$\varphi(x_0,y) \geq 0, \forall y \in K.$$

In subsequent sections, the notion of a KKM map and the well-known intersection Lemma due to Ky Fan [88] will be needed.

Definition 4.1.1. (Knaster-Kuratowski-Mazurkiewicz) Let X be a Hausdorff topological vector space and let $M \subseteq X$. The application $G : M \rightrightarrows X$ is called a KKM application iff, for every finite number of elements $x_1, x_2, \ldots, x_n \in M$, one has

$$\mathrm{co}\{x_1, x_2, \ldots, x_n\} \subseteq \bigcup_{i=1}^{n} G(x_i).$$

Lemma 4.1.1. *Let X be a Hausdorff topological vector space, $M \subseteq X$ and $G : M \rightrightarrows X$ be a KKM application. If $G(x)$ is closed for every $x \in M$, and there exists $x_0 \in M$, such that $G(x_0)$ is compact, then*

$$\bigcap_{x \in M} G(x) \neq \emptyset.$$

4.1.3 Self Segment-Dense Sets

Let X be a Hausdorff topological vector space. We will use the following notations for the open, respectively closed, line segments in X with the endpoints x and y

$$\begin{aligned} (x,y) &:= \big\{z \in X : z = x + t(y-x),\, t \in]0,1[\big\}, \\ [x,y] &:= \big\{z \in X : z = x + t(y-x),\, t \in [0,1]\big\}. \end{aligned}$$

In [149], Definition 3.4, The Luc has introduced the notion of a so-called *segment-dense* set. Let $V \subseteq X$ be a convex set. One says that the set $U \subseteq V$ is segment-dense in V iff, for each $x \in V$, there can be found $y \in U$ such that x is a cluster point of the set $[x,y] \cap U$.

In what follows, we present a denseness notion [143], which is slightly different from the concept of The Luc presented above, but which is better suited for our needs.

Definition 4.1.2. Consider the sets $U \subseteq V \subseteq X$ and assume that V is convex. We say that U is self segment-dense in V iff, U is dense in V and

$$\forall x, y \in U, \text{ the set } [x,y] \cap U \text{ is dense in } [x,y].$$

Remark 4.1.3. Obviously, in one dimension the concepts of a segment-dense set respectively a self segment-dense set are equivalent to the concept of a dense set.

In what follows, we provide an essential example of a self segment-dense set.

Example 4.1.1. ([146], Example 3.1) Let V be the two dimensional Euclidean space $\mathbb{R}^2$ and define U to be the set

$$U := \{(p,q) \in \mathbb{R}^2 : p \in \mathbb{Q}, q \in \mathbb{Q}\},$$

where $\mathbb{Q}$ denotes the set of all rational numbers. Then, clearly U is dense in $\mathbb{R}^2$. On the other hand, U is not segment-dense in $\mathbb{R}^2$, since for $x = (0, \sqrt{2}) \in \mathbb{R}^2$ and for every $y = (p,q) \in U$, one has $[x,y] \cap U = \{y\}$. It can easily be observed that U is self segment-dense in $\mathbb{R}^2$, since for every $x, y \in U$ $x = (p,q)$, $y = (r,s)$ we have $[x,y] \cap U = \{(p+t(r-p), q+t(s-q)) : t \in [0,1] \cap \mathbb{Q}\}$, which is obviously dense in $[x,y]$.

To further circumscribe the notion of a self segment-dense set we provide an example of a subset that is dense but not self segment-dense.

Example 4.1.2. Let X be an infinite dimensional real Hilbert space, it is known that the unit sphere $\{x \in X : \|x\| = 1\}$ is dense with respect to the weak topology in the unit ball $\{x \in X : \|x\| \leq 1\}$, but it is obviously not self segment-dense since any segment with end-points on the sphere does not intersect the sphere in any other points.

Remark 4.1.4. Note that every dense convex subset of a Banach space is self segment-dense. In particular dense subspaces and dense affine subsets are self segment-dense.

4.1.4 Self Segment-Dense Sets and Equilibrium Problems

In this paragraph, by making use of the concept of a self segment-dense set, we obtain existence results for set-valued equilibrium problems. Ky Fan's Lemma is used in the proof of Theorem 4.1.4 and Theorem 4.1.6, the main results of this section, in order to establish the existence of solutions to equilibrium problems. This approach is well known in the literature, see, for instance, [129, 137, 145, 149, 208, 209].

The following lemma gives an interesting characterization of self segment-dense sets and will be used in the sequel. If X is a Hausdorff locally convex topological vector space, then the origin has a local base of convex, balanced and absorbent sets, and recall, that the set

$$\operatorname{core} D = \{u \in D | \ \forall x \in X \ \exists \delta > 0 \text{ such that } \forall \varepsilon \in [0,\delta] : u + \varepsilon x \in D\}$$

is called the algebraic interior (or core) of $D \subseteq X$ [210].

If D is convex with nonempty interior, then $\operatorname{int} D = \operatorname{core} D$ [210].

Lemma 4.1.2. *Let X be a Hausdorff locally convex topological vector space, let $V \subseteq X$ be a convex set and let $U \subseteq V$ a self segment-dense set in V. Then, for all finite subset*

$\{u_1, u_2, \ldots, u_n\} \subseteq U$ *one has*

$$\mathrm{cl}(\mathrm{co}\{u_1, u_2, \ldots, u_n\} \cap U) = \mathrm{co}\{u_1, u_2, \ldots, u_n\}.$$

Proof. We prove the statement by classical induction. For $n = 2$, by using the self segment-denseness of U in V we have that, for every $u_1, u_2 \in U$,

$$\mathrm{cl}(\mathrm{co}\{u_1, u_2\} \cap U) = \mathrm{cl}([u_1, u_2] \cap U) = [u_1, u_2] = \mathrm{co}\{u_1, u_2\}.$$

Assume that the statement holds for every $u_1, u_2, \ldots, u_{n-1} \in U$, and we show that is also true for all $u_1, u_2, \ldots, u_n \in U$.

For this let us fix $u_1, u_2, \ldots, u_{n-1} \in U$ and let $u_n \in U$. Obviously, one should take u_n such that $\mathrm{co}\{u_1, u_2, \ldots, u_n\} \neq \mathrm{co}\{u_1, u_2, \ldots, u_{n-1}\}$. In this case

$$\mathrm{co}\{u_1, u_2, \ldots, u_n\} = \bigcup_{u \in \mathrm{co}\{u_1, u_2, \ldots, u_{n-1}\}} [u, u_n].$$

We must show that $\mathrm{co}\{u_1, u_2, \ldots, u_n\} \cap U$ is dense in $\mathrm{co}\{u_1, u_2, \ldots, u_n\}$.

Assume the contrary, that is, there exists $s \in \mathrm{co}\{u_1, u_2, \ldots, u_n\}$ and a neighbourhood S of s such that $S \cap \mathrm{co}\{u_1, u_2, \ldots, u_n\}$ contains no points from U. Obviously, we can take $S = s + G$, where G is an open, balanced and convex neighbourhood of the origin. Note that we have $s = u_n + t(u - u_n)$ for some $t \in [0, 1]$, $u \in \mathrm{co}\{u_1, u_2, \ldots, u_{n-1}\}$. On the other hand, since $u_n \in U$, we have that $s \neq u_n$, hence $t \neq 0$.

Assume now, that $s = u$, that is $t = 1$. Then, $s \in \mathrm{co}\{u_1, u_2, \ldots, u_{n-1}\}$.

Since $\mathrm{co}\{u_1, u_2, \ldots, u_{n-1}\} \cap U$ is dense in $\mathrm{co}\{u_1, u_2, \ldots, u_{n-1}\}$, we obtain that every neighbourhood of s contains elements from $\mathrm{co}\{u_1, u_2, \ldots, u_{n-1}\} \cap U$, in particular the intersection $S \cap \mathrm{co}\{u_1, u_2, \ldots, u_{n-1}\}$ contains elements from U, which contradicts our assumption.

Consequently, $s = u_n + t(u - u_n)$ for some $t \in]0, 1[$ and $u \in \mathrm{co}\{u_1, u_2, \ldots, u_{n-1}\}$. By the induction hypothesis, we have that $\mathrm{co}\{u_1, u_2, \ldots, u_{n-1}\} \cap U$ is dense in $\mathrm{co}\{u_1, u_2, \ldots, u_{n-1}\}$, hence, there exists a net $(u^\alpha) \subseteq \mathrm{co}\{u_1, u_2, \ldots, u_{n-1}\} \cap U$ such that $\lim u^\alpha = u$. Thus, for $u + G$, a neighbourhood of u, there exists α_0 such that
$u^\alpha \in u + G$ for all $\alpha > \alpha_0$.

We show next, that for $u^\alpha \in u + G$ we have $s^\alpha = u_n + t(u^\alpha - u_n) \in s + G$.

Indeed $(u^\alpha - u_n) \in (u - u_n) + G$, and since G is balanced and $t \in]0, 1[$, we have that $t(u^\alpha - u_n) \in t(u - u_n) + G$.

Hence, $s^\alpha = u_n + t(u^\alpha - u_n) \in u_n + t(u - u_n) + G = s + G$. Note that $s + G$ is open and convex, hence $s + G = \mathrm{core}(s + G)$, which shows that $s^\alpha \in \mathrm{core}(s + G)$. Therefore, there exists

$\delta > 0$ such that $s^\alpha + \varepsilon(u_n - u^\alpha), s^\alpha + \varepsilon(u^\alpha - u_n) \in s + G$ for all $\varepsilon \in [0, \delta]$. Obviously, one can take $\delta \leq \min\{t, 1-t\}$. In this case, we have $t - \varepsilon, t + \varepsilon \in [0,1]$ for all $\varepsilon \in [0, \delta]$, hence

$$s^\alpha + \varepsilon(u_n - u^\alpha) = u_n + (t - \varepsilon)(u^\alpha - u_n) \in [u^\alpha, u_n],$$

and

$$s^\alpha + \varepsilon(u^\alpha - u_n) = u_n + (t + \varepsilon)(u^\alpha - u_n) \in [u^\alpha, u_n],$$

for all $\varepsilon \in [0, \delta]$. Thus,

$$s^\alpha \in [s^\alpha + \varepsilon(u^\alpha - u_n), s^\alpha + \varepsilon(u_n - u^\alpha)] \subseteq [u^\alpha, u_n]$$

for all $\varepsilon \in [0, \delta]$. Since $u^\alpha, u_n \in U$ and U is self segment-dense, obviously

$$[s^\alpha + \varepsilon(u^\alpha - u_n), s^\alpha + \varepsilon(u_n - u^\alpha)] \cap U \neq \emptyset$$

for all $\varepsilon \in]0, \delta]$, which leads to

$$(s + G) \cap \mathrm{co}\{u_1, u_2, \ldots, u_n\} \cap U \neq \emptyset,$$

which yields a contradiction. □ □

Remark 4.1.5. In Lemma 4.1.2, the assumption that U is self segment-dense cannot be replaced by the denseness of U as next example shows.

Example 4.1.3. Let V be the unit ball in $\mathbb{R}^3$, let A be the interior of a square with vertices $(-1,0,0), (0,-1,0), (1,0,0)$ respectively $(0,1,0)$. Here, the interior of the square is meant relative to the plane of the square. Then, obviously $U = V \setminus A$ is dense in V, but not self segment-dense in V, since, for instance, for $u_1 = (\frac{3}{5}, \frac{3}{5}, 0) \in U$ and $u_2 = (-\frac{3}{5}, -\frac{3}{5}, 0) \in U$, the set $[u_1, u_2] \cap U$ is not dense in $[u_1, u_2]$. This also shows that $\mathrm{cl}(\mathrm{co}\{u_1, u_2\} \cap U) \neq \mathrm{co}\{u_1, u_2\}$.

Next result gives an important application of self segment-dense sets in the framework of equilibrium problems presented above.

Theorem 4.1.4. *Let X be a Hausdorff locally convex topological vector space, let K be a nonempty, convex and compact subset of X, let $D \subseteq K$ be a self segment-dense set, and let $F : K \times K \rightrightarrows \mathbb{R}$ be a set valued map satisfying*

(i) $\forall y \in D, x \longrightarrow F(x,y)$ is lower semicontinuous on K,

(ii) $\forall x \in K$, $y \longrightarrow F(x,y)$ *is lower semicontinuous on* $K \setminus D$,

(iii) $\forall x \in D$, $y \longrightarrow F(x,y)$ *is convex on* D,

(iv) $\forall x \in D$, $F(x,x) \geq 0$.

Then, there exists an element $x_0 \in K$ *such that*

$$F(x_0,y) \geq 0, \forall y \in K.$$

Proof. Consider the map

$$G : D \rightrightarrows K, G(y) = \{x \in K : F(x,y) \geq 0\}.$$

We show that $\bigcap_{y \in D} G(y) \neq \emptyset$, or, in other words, that there exists $x_0 \in K$ such that

$$F(x_0,y) \geq 0, \forall y \in D.$$

We start by showing that $G(y)$ is closed for all $y \in D$. Indeed, for a fixed $y \in D$ we have $G(y) = f_y^+(\mathbb{R}_+)$, where $f_y : K \rightrightarrows \mathbb{R}$, $f_y(x) = F(x,y)$. From (i) we have that f_y is lower semicontinuous on K, and since $\mathbb{R}_+$ is closed, according to Remark 4.1.1, $f_y^+(\mathbb{R}_+)$ is closed. Hence $G(y) \subseteq K$ is closed for all $y \in D$, and by the compactness of K we get that $G(y)$ is compact for every $y \in D$.

Next, we show that G is a KKM mapping. In fact, we prove by a contradiction argument that, for given arbitrary $y_1, y_2, \ldots, y_n \in D$, one has

$$\mathrm{co}\{y_1, y_2, \ldots, y_n\} \cap D \subseteq \bigcup_{i=1}^{n} G(y_i).$$

So, assume the contrary, that there exist $\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$, $\sum_{i=1}^n \lambda_i = 1$ such that $\sum_{i=1}^n \lambda_i y_i \in D$ and

$$\sum_{i=1}^{n} \lambda_i y_i \notin \bigcup_{i=1}^{n} G(y_i).$$

This is equivalent with $F(\sum_{i=1}^n \lambda_i y_i, y_i) \cap]-\infty, 0[\neq \emptyset$, $\forall i \in \{1, 2, \ldots, n\}$, and hence,

$$\sum_{i=1}^{n} \lambda_i F\left(\sum_{i=1}^{n} \lambda_i y_i, y_i\right) \cap]-\infty, 0[\neq \emptyset.$$

Since by assumption (iii) $\forall x \in D$ the mapping $y \longrightarrow F(x,y)$ is convex on D, we have

$$\sum_{i=1}^{n} \lambda_i F\left(\sum_{i=1}^{n} \lambda_i y_i, y_i\right) \subseteq F\left(\sum_{i=1}^{n} \lambda_i y_i, \sum_{i=1}^{n} \lambda_i y_i\right) \geq 0,$$

or equivalently

$$\sum_{i=1}^{n} \lambda_i F\left(\sum_{i=1}^{n} \lambda_i y_i, y_i\right) \subseteq [0,\infty[,$$

which contradicts our initial assumption. Consequently,

$$\mathrm{co}\{y_1, y_2, \ldots, y_n\} \cap D \subseteq \bigcup_{i=1}^{n} G(y_i),$$

holds true, and leads to

$$\mathrm{cl}(\mathrm{co}\{y_1, y_2, \ldots, y_n\} \cap D) \subseteq \mathrm{cl}\left(\bigcup_{i=1}^{n} G(y_i)\right).$$

Furthermore, since $G(y_i)$ is closed for all $i \in \{1, 2, \ldots, n\}$, we have

$$\mathrm{cl}\left(\bigcup_{i=1}^{n} G(y_i)\right) = \bigcup_{i=1}^{n} G(y_i).$$

On the other hand, according to Lemma 4.1.2 we have

$$\mathrm{cl}(\mathrm{co}\{y_1, y_2, \ldots, y_n\} \cap D) = \mathrm{co}\{y_1, y_2, \ldots, y_n\},$$

so

$$\mathrm{co}\{y_1, y_2, \ldots, y_n\} \subseteq \bigcup_{i=1}^{n} G(y_i).$$

Hence, G is a KKM map.

Thus, according to Ky Fan's Lemma, $\bigcap_{y \in D} G(y) \neq \emptyset$. In other words, there exists $x_0 \in K$ such that $F(x_0, y) \geq 0$ for all $y \in D$.

At this point we make use of the assumption (ii) to extend the previous statement to the whole set K. Consider $y \in K \setminus D$; since D is dense in K, there exists a net $(y^\alpha) \subseteq D$ such that $\lim y^\alpha = y$. Now, due to the assumption (ii) and Remark 4.1.1, for every $y^* \in F(x_0, y)$ there exists a net $y^*_\alpha \in F(x_0, y^\alpha)$ such that $\lim y^*_\alpha = y^*$. However, obviously $y^*_\alpha \geq 0$, hence $y^* \geq 0$, and finally $F(x_0, y) \geq 0$, $\forall y \in K$. $\square$ $\square$

In the above Theorem, one can replace F by $-F$ and obtain a result concerning the opposite inequalities. Note that the convexity and lower semicontinuity of a set-valued map F is equivalent to the convexity and lower semicontinuity of $-F$.

Theorem 4.1.5. *Let X be a Hausdorff locally convex topological vector space, let K be a nonempty, convex and compact subset of X, let $D \subseteq K$ be a self segment-dense set, and let $F : K \times K \rightrightarrows \mathbb{R}$ be a set valued map satisfying*

(i) $\forall y \in D, x \longrightarrow F(x,y)$ *is lower semicontinuous on K,*

(ii) $\forall x \in K, y \longrightarrow F(x,y)$ *is lower semicontinuous on $K \setminus D$,*

(iii) $\forall x \in D, y \longrightarrow F(x,y)$ *is convex on D,*

(iv) $\forall x \in D, F(x,x) \leq 0$.

Then, there exists an element $x_0 \in K$ such that

$$F(x_0,y) \leq 0, \forall y \in K.$$

By similar methods to those used in the proof of Theorem 4.1.4 one can obtain a result concerning the second set-valued equilibrium problem. The following theorem holds.

Theorem 4.1.6. *Let X be a Hausdorff locally convex topological vector space, let K be a nonempty, convex and compact subset of X, let $D \subseteq K$ be a self segment-dense set, and let $F : K \times K \rightrightarrows \mathbb{R}$ be a set valued map satisfying*

(i) $\forall y \in D, x \longrightarrow F(x,y)$ *is upper semicontinuous on K,*

(ii) $\forall x \in K, y \longrightarrow F(x,y)$ *is upper semicontinuous on $K \setminus D$,*

(iii) $\forall x \in D, y \longrightarrow F(x,y)$ *is concave on D,*

(iv) $\forall x \in D, F(x,x) \cap \mathbb{R}_+ \neq \emptyset$.

Then, there exists an element $x_0 \in K$ such that

$$F(x_0,y) \cap \mathbb{R}_+ \neq \emptyset, \forall y \in K.$$

Proof. Consider the map $G : D \rightrightarrows K$, $G(y) = \{x \in K : F(x,y) \cap \mathbb{R}_+ \neq \emptyset\}$. We show that $G(y)$ is closed for all $y \in D$. Indeed, for a fixed $y \in D$ we have $G(y) = f_y^-(\mathbb{R}_+)$, where $f_y : K \rightrightarrows \mathbb{R}$, $f_y(x) = F(x,y)$. From (i) we have that f_y is upper semicontinuous on K, and since $\mathbb{R}_+$ is closed, according to Remark 4.1.1, $f_y^-(\mathbb{R}_+)$ is closed. Hence $G(y) \subseteq K$ is closed for all $y \in D$, and by the compactness of K we get that $G(y)$ is compact for every $y \in D$.

Following the proof of Theorem 4.1.4, it can be shown that $\bigcap_{y \in D} G(y) \neq \emptyset$, that is, there exists $x_0 \in K$ such that

$$F(x_0,y) \cap \mathbb{R}_+ \neq \emptyset, \forall y \in D.$$

Now, let us fix $y \in K \setminus D$ and assume that $F(x_0,y) \subseteq]-\infty,0[$. Since the set-valued function $F(x_0,.)$ is upper semicontinuous at y we obtain that there exists an open neighbourhood U of y, such that $F(x_0,U) \subseteq]-\infty,0[$. However, D is dense in K, hence there exists $u \in U$ such that $u \in D$, so $F(x_0,u) \cap \mathbb{R}_+ \neq \emptyset$, a contradiction. Thus,

$$F(x_0,y) \cap \mathbb{R}_+ \neq \emptyset, \forall y \in K$$

must hold true. □

The reminder of this subsection is concerned with the single-valued equilibrium problem.

Let $K \subseteq X$ be a subset and let $f : K \longrightarrow \mathbb{R}$. We say that f is convex on K, respectively concave on K iff, for all $x_1, x_2, \ldots, x_n \in K$ and $\lambda_i \geq 0$, $i \in \{1,2,\ldots,n\}$, $\sum_{i=1}^n \lambda_i = 1$ such that $\sum_{i=1}^n \lambda_i x_i \in K$ one has

$$\sum_{i=1}^n \lambda_i f(x_i) \geq f\left(\sum_{i=1}^n \lambda_i x_i\right), \text{ respectively } \sum_{i=1}^n \lambda_i f(x_i) \leq f\left(\sum_{i=1}^n \lambda_i x_i\right).$$

Note that, in these definitions, we do not assume the convexity of K. We have the following existence result for the single valued equilibrium problem.

Theorem 4.1.7. *Let X be a Hausdorff locally convex topological vector space, let K be a nonempty, convex and compact subset of X, let $D \subseteq K$ be a self segment-dense set and let $\varphi : K \times K \longrightarrow \mathbb{R}$ a function satisfying*

(i) $\forall y \in D$ *the function* $x \longrightarrow \varphi(x,y)$ *is upper semicontinuous on* K,

(ii) $\forall x \in K$, $y \longrightarrow \varphi(x,y)$ *is upper semicontinuous on* $K \setminus D$,

(iii) $\forall x \in D$, *the function* $y \to \varphi(x,y)$ *is convex on* D,

(iv) $\forall x \in D$, $\varphi(x,x) \geq 0$.

Then, there exists an element $x_0 \in K$ such that

$$\varphi(x_0, y) \geq 0, \forall y \in K.$$

Proof. We give only an outline of the proof, since the ideas are similar to those used in the proof of Theorem 4.1.4.

We consider the map

$$G : D \rightrightarrows K,\ G(y) = \{x \in K : \varphi(x, y) \geq 0\}.$$

Observe that for a fixed $y \in D$, the set $G(y)$ is the superlevel set $\{x \in K : \varphi_y(x) \geq 0\}$ of the function $\varphi_y : K \to \mathbb{R}$, $\varphi_y(x) = \varphi(x, y)$. Due to the assumption (i) and Remark 4.1.2, we have that $G(y)$ is closed for all $y \in D$.

Furthermore, from assumptions (iii), (iv) and Lemma 4.1.2 we obtain that G is a KKM application. Then, according to Ky Fan's Lemma, $\bigcap_{y \in D} G(y) \neq \emptyset$. Hence, there exists $x_0 \in K$ such that $\varphi(x_0, y) \geq 0$ for all $y \in D$.

Finally, if $y \in K \setminus D$, by the denseness of D in K, there exists a net $(y^\alpha) \subseteq D$ such that $\lim y^\alpha = y$. At this point, the assumption (ii), $\varphi(x_0, y)$ the upper semicontinuity of $\varphi(x_0, y)$ on $K \setminus D$, assures that $0 \leq \limsup_{y^\alpha \to y} \varphi(x_0, y^\alpha) \leq \varphi(x_0, y)$. Thus we have $\varphi(x_0, y) \geq 0$ for all $y \in K$.

□

The above result has also a complementary formulation, in which convexity is replaced by concavity and the inequalities have opposite direction.

Theorem 4.1.8. *Let X be a Hausdorff locally convex topological vector space, let K be a nonempty, convex and compact subset of X, let $D \subseteq K$ be a self segment-dense set and let $\varphi : K \times K \longrightarrow \mathbb{R}$ a function satisfying*

(i) $\forall y \in D$ *the function* $x \longrightarrow \varphi(x, y)$ *is lower semicontinuous on* K,

(ii) $\forall x \in K$, $y \longrightarrow \varphi(x, y)$ *is lower semicontinuous on* $K \setminus D$,

(iii) $\forall x \in D$, *the function* $y \to \varphi(x, y)$ *is concave on* D,

(iv) $\forall x \in D$, $\varphi(x, x) \leq 0$.

Then, there exists an element $x_0 \in K$ such that

$$\varphi(x_0, y) \leq 0, \forall y \in K.$$

Proof. Apply Theorem 4.1.7 to the function $-\varphi$. □

In what follows, we show that the assumption that D is self segment-dense, in the hypotheses of the previous theorems, is essential and it cannot be replaced by the denseness of D.

Indeed, let us consider the Hilbert space of square-summable sequences l_2, and let $K = \{x \in l_2 : \|x\| \leq 1\}$ be its unit ball, while $D = \{x \in l_2 : \|x\| = 1\}$ is the unit sphere. It is well known that l_2, endowed with the weak topology, is a Hausdorff locally convex topological vector space, and by Banach-Alaoglu Theorem, K is compact in this topology. Furthermore, we have seen in Example 4.1.2 that D is dense, but not self segment-dense in K.

In this setting we define the single-valued map

$$\varphi : K \times K \to \mathbb{R},\ \varphi(x,y) := \langle x,y\rangle - 1,$$

which has the following properties:

(a) for all $y \in K$, $x \longrightarrow \varphi(x,y)$ is continuous on K,

(b) for all $x \in K$, $y \longrightarrow \varphi(x,y)$ is continuous on K,

(c) for all $x \in K$, $y \longrightarrow \varphi(x,y)$ is affine, hence convex and also concave on K,

(d) $\varphi(x,x) = 0$ for all $x \in D$.

Now, consider the operators $F_1, F_2 : K \times K \rightrightarrows \mathbb{R}$,

$$F_1(x,y) := [\varphi(x,y), \infty[,$$

and

$$F_2(x,y) :=]-\infty, \varphi(x,y)].$$

Obviously, $F_1(x,x) = [0,\infty[$ for all $x \in D$, while $F_2(x,x) =]-\infty, 0]$ for all $x \in D$, and it can easily be shown that F_1, respectively F_2 satisfy the conditions (i)-(iii) in Theorem 4.1.4, respectively Theorem 4.1.6 (even some stronger assumptions, since we can take everywhere $x, y \in K$).

We see that F_1, F_2, respectively φ satisfy all the assumptions of Theorem 4.1.4, Theorem 4.1.6, respectively Theorem 4.1.7, except the assumption that D is self segment-dense (here D is only dense) and also that the conclusions of the above mentioned theorems fail, since for $y = 0 \in K$ one has

$$F_1(x,y) = [-1,\infty[,\ \forall x \in K,$$

$$F_2(x,y) =]-\infty,-1], \forall x \in K,$$

respectively

$$\varphi(x,y) = -1, \forall x \in K.$$

4.1.5 Densely Defined Equilibrium Problems on Noncompact Sets

The compactness of the domain K in the hypotheses of the existence theorems in the previous subsection is a rather strong condition. So, a natural question is whether similar existence results can be obtained without a compactness assumption. In this context, one can observe that the KKM mappings built in the proofs of the mentioned results are compact valued, while Ky Fan's Lemma requires the existence of a single point, where the KKM map must be compact valued. Motivated by this observation, in what follows we replace the compactness assumption by the closedness of the domain K in order to obtain existence results for the equilibrium problems presented so far.

Theorem 4.1.9. *Let X be a Hausdorff locally convex topological vector space, let K be a nonempty, convex and closed subset of X, let $D \subseteq K$ be a self segment-dense set, and let $F : K \times K \rightrightarrows \mathbb{R}$ be a set valued map satisfying*

(i) $\forall y \in D, x \longrightarrow F(x,y)$ *is lower semicontinuous on K,*

(ii) $\forall x \in K, y \longrightarrow F(x,y)$ *is lower semicontinuous on $K \setminus D$,*

(iii) $\forall x \in D, y \longrightarrow F(x,y)$ *is convex on D,*

(iv) $\forall x \in D, F(x,x) \geq 0$,

(v) $\exists K_0 \subseteq X$ *compact and* $y_0 \in D$, *such that* $F(x,y_0) \cap]-\infty,0[\neq \emptyset, \forall x \in K \setminus K_0$.

Then, there exists an element $x_0 \in K$ such that

$$F(x_0,y) \geq 0, \forall y \in K.$$

Proof. Consider the map

$$G : D \rightrightarrows K, G(y) = \{x \in K : F(x,y) \geq 0\}.$$

According to the proof of Theorem 4.1.4, $G(y)$ is closed for all $y \in D$. We show that $G(y_0)$ is compact, and the rest of the proof is similar to the proof of Theorem 4.1.4. It is thus enough

to show that $G(y_0) \subseteq K_0$. Assume the contrary, that there exists $z \in G(y_0)$ such that $z \notin K_0$. Then, $F(z, y_0) \geq 0$ which contradicts (v). $\square$

The following results can be proved analogously.

Theorem 4.1.10. *Let X be a Hausdorff locally convex topological vector space, let K be a nonempty, convex and closed subset of X, let $D \subseteq K$ be a self segment-dense set, and let $F : K \times K \rightrightarrows \mathbb{R}$ be a set valued map satisfying*

(i) $\forall y \in D, x \longrightarrow F(x,y)$ *is upper semicontinuous on K,*

(ii) $\forall x \in K, y \longrightarrow F(x,y)$ *is upper semicontinuous on $K \setminus D$,*

(iii) $\forall x \in D, y \longrightarrow F(x,y)$ *is concave on D,*

(iv) $\forall x \in D, F(x,x) \cap \mathbb{R}_+ \neq \emptyset$,

(v) $\exists K_0 \subseteq X$ *compact and* $y_0 \in D$, *such that* $F(x,y_0) \cap \mathbb{R}_+ = \emptyset, \forall x \in K \setminus K_0$.

Then, there exists an element $x_0 \in K$ such that

$$F(x_0, y) \cap \mathbb{R}_+ \neq \emptyset, \forall y \in K.$$

Theorem 4.1.11. *Let X be a Hausdorff locally convex topological vector space, let K be a nonempty, convex and closed subset of X, let $D \subseteq K$ be a self segment-dense set and let $\varphi : K \times K \longrightarrow \mathbb{R}$ a function satisfying*

(i) $\forall y \in D$, *the function* $x \longrightarrow \varphi(x,y)$ *is upper semicontinuous on K,*

(ii) $\forall x \in K, y \longrightarrow \varphi(x,y)$ *is upper semicontinuous on $K \setminus D$,*

(iii) $\forall x \in D$, *the function* $y \to \varphi(x,y)$ *is convex on D,*

(iv) $\forall x \in D, \varphi(x,x) \geq 0$,

(v) $\exists K_0 \subseteq X$ *compact and* $y_0 \in D$, *such that* $\varphi(x,y_0) < 0, \forall x \in K \setminus K_0$.

Then, there exists an element $x_0 \in K$ such that

$$\varphi(x_0, y) \geq 0, \forall y \in K.$$

The condition (v) in Theorems 4.1.9, 4.1.10 and 4.1.11 seems to be not so easy to verify. However, it is well known that, in a reflexive Banach space X, the closed ball, with radius $r > 0$, $\overline{B}_r := \{x \in X : \|x\| \leq r\}$, is weakly compact. Therefore, if we endow the reflexive Banach space X with the weak topology, condition (v) in Theorem 4.1.9, Theorem 4.1.10, respectively Theorem 4.1.11 becomes :

$$\exists r > 0 \text{ and } y_0 \in D, \text{ such that, for all } x \in K \text{ satisfying } \|x\| > r,$$

one has that

$$F(x, y_0) \cap]-\infty, 0[\neq \emptyset, \;\; F(x, y_0) \cap \mathbb{R}_+ = \emptyset, \text{ respectively } \varphi(x, y_0) < 0.$$

Furthermore, in this setting, condition (v) in the hypotheses of Theorems 4.1.9, 4.1.10 and 4.1.11 can be weakened by assuming that there exists $r > 0$ such that, for all $x \in K$ satisfying $\|x\| > r$, there exists some $y_0 \in K$ with $\|y_0\| < \|x\|$, and for which the appropriate condition

(i) $F(x, y_0) \cap]-\infty, 0[\neq \emptyset$,

(ii) $F(x, y_0) \cap \mathbb{R}_+ = \emptyset$,

(iii) $\varphi(x, y_0) < 0$,

holds.

More precisely, we have the following result.

Theorem 4.1.12. *Let X be a reflexive Banach space, let K be a nonempty, convex and closed subset of X, let $D \subseteq K$ be a self segment-dense set in the weak topology of X, and let $F : K \times K \rightrightarrows \mathbb{R}$ be a set valued map satisfying*

(i) $\forall y \in D, x \longrightarrow F(x, y)$ *is weak lower semicontinuous on K,*

(ii) $\forall x \in K, y \longrightarrow F(x, y)$ *is weak lower semicontinuous on $K \setminus D$,*

(iii) $\forall x \in K, y \longrightarrow F(x, y)$ *is convex on K,*

(iv) $\forall x \in K, F(x, x) \geq 0,$ *and* $F(x, x) \supseteq \{0\}$.

(v) $\exists r > 0$, *such that, for all $x \in K$, $\|x\| > r$, there exists $y_0 \in K$ with $\|y_0\| < \|x\|$ such that* $F(x, y_0) \cap]-\infty, 0] \neq \emptyset$.

Then, there exists an element $x_0 \in K$ such that

$$F(x_0, y) \geq 0, \forall y \in K.$$

Proof. Let $r > 0$ such that (v) holds and let $r_1 > r$. Let $K_0 = K \cap \overline{B}_{r_1}$. Since K is convex and closed it is also weakly closed, $\overline{B}_{r_1}$ is weakly compact, hence K_0 is convex and weakly compact. According to Theorem 4.1.4, there exists $x_0 \in K_0$ such that $F(x_0, y) \geq 0$ for all $y \in K_0$. Next, we show that there exists $z_0 \in K_0, \|z_0\| < r_1$ such that $F(x_0, z_0) \supseteq \{0\}$. Indeed, if $\|x_0\| < r_1$, then let $z_0 = x_0$, and the conclusion follows by (iv). If $\|x_0\| = r_1 > r$, then by (v) we have that there exists $z_0 \in K$ with $\|z_0\| < \|x_0\|$ such that $F(x_0, z_0) \cap]-\infty, 0] \neq \emptyset$. On the other hand, since $z_0 \in K_0$ we have $F(x_0, z_0) \geq 0$, hence $\{0\} \subseteq F(x_0, z_0)$.

Let $y \in K$. Then, there exists $\lambda \in [0,1]$ such that $\lambda z_0 + (1-\lambda)y \in K_0$. Therefore $F(x_0, \lambda z_0 + (1-\lambda)y) \geq 0$, and by (iii) we obtain

$$\lambda F(x_0, z_0) + (1-\lambda)F(x_0, y) \subseteq F(x_0, \lambda z_0 + (1-\lambda)y) \subseteq [0, \infty[.$$

Since $\{0\} \subseteq F(x_0, z_0)$, we have $F(x_0, y) \subseteq [0, \infty[$. □

Similar results can be obtained for the other two equilibrium problems studied in this paper. However, if one compares Theorem 4.1.12 with Theorem 4.1.4 or Theorem 4.1.9, one observes that conditions (iii) and (iv) have been considerably changed. This is due the fact that condition (v) in Theorem 4.1.12 with the assumptions (iii) and (iv) of Theorem 4.1.4 or Theorem 4.1.9 does not assure the existence of a solution when K is closed but not compact.

Our purpose is to overcome this situation by replacing (v) with a condition that assures the existence of a solution under the original assumptions (iii) and (iv). In fact, we show that, if $\forall x \in K, y \longrightarrow F(x, y)$ is convex on D, respectively $\forall x \in D, F(x, x) \geq 0$, instead of (iii), respectively (iv) in the previous theorem, then we can replace (v) with:

$\exists r > 0$, such that, for all $x \in K$ satisfying $\|x\| \leq r$, there exists $y_0 \in D$ with $\|y_0\| < r$, such that $\{0\} \subseteq F(x, y_0)$.

The following result holds.

Theorem 4.1.13. *Let X be a reflexive Banach space, let K be a nonempty, convex and closed subset of X, let $D \subseteq K$ be a self segment-dense set in the weak topology of X, and let $F : K \times K \rightrightarrows \mathbb{R}$ be a set valued map satisfying*

(i) $\forall y \in D, x \longrightarrow F(x, y)$ *is weak lower semicontinuous on K,*

(ii) $\forall x \in K, y \longrightarrow F(x, y)$ *is weak lower semicontinuous on $K \setminus D$,*

(iii) $\forall x \in K$, $y \longrightarrow F(x,y)$ *is convex on D,*

(iv) $\forall x \in D$, $F(x,x) \geq 0$,

(v) $\exists r > 0$, *such that, for all* $x \in K$, $\|x\| \leq r$, *there exists* $y_0 \in D$ *with* $\|y_0\| < r$, *such that* $\{0\} \subseteq F(x,y_0)$.

Then, there exists an element $x_0 \in K$ *such that*

$$F(x_0,y) \geq 0, \forall y \in K.$$

Proof. Let $r > 0$, such that (v) holds and consider the weakly compact set $K_0 = K \cap \overline{B}_r$. According to Theorem 4.1.4 there exists $x_0 \in K_0$ such that $F(x_0,y) \geq 0$, $\forall y \in K_0$.

According to (v), there exists $z_0 \in D$ with $\|z_0\| < r$ such that $\{0\} \subseteq F(x_0,z_0)$.

Now let $z \in D \setminus K_0$. Then, in virtue of self segment denseness of D in K, there exists $\lambda \in]0,1[$ such that $\lambda z_0 + (1-\lambda)z \in K_0 \cap D$. According to (iii)

$$F(x_0, \lambda z_0 + (1-\lambda)z) \supseteq \lambda F(x_0,z_0) + (1-\lambda)F(x_0,z),$$

but $F(x_0, \lambda z_0 + (1-\lambda)z) \geq 0$ and $\{0\} \subseteq F(x_0,z_0)$, which leads to $F(x_0,z) \geq 0$.

Hence $F(x_0,z) \geq 0$ for all $z \in D$. Let $y \in K \setminus D$. Since D is dense in K, there exists a net $(y^\alpha) \subset D$ such that $\lim y^\alpha = y$, where the limit is taken in the weak topology of X. According to (ii) $F(x_0,\cdot)$ is weakly lower semicontinuous at y. Now, due to Remark 4.1.1, for every $y^* \in F(x_0,y)$ there exists a net $y^*_\alpha \in F(x_0,y^\alpha)$ such that $\lim y^*_\alpha = y^*$. However, obviously $y^*_\alpha \geq 0$, hence $y^* \geq 0$, and finally $F(x_0,y) \geq 0$, $\forall y \in K$. □

Concerning the weaker set-valued equilibrium problem a similar result holds.

Theorem 4.1.14. *Let X be a reflexive Banach space, let K be a nonempty, convex and closed subset of X, let* $D \subseteq K$ *be a self segment-dense set in the weak topology of X, and let* $F : K \times K \rightrightarrows \mathbb{R}$ *be a set valued map satisfying*

(i) $\forall y \in D$, $x \longrightarrow F(x,y)$ *is weak upper semicontinuous on K,*

(ii) $\forall x \in K$, $y \longrightarrow F(x,y)$ *is weak upper semicontinuous on* $K \setminus D$,

(iii) $\forall x \in K$, $y \longrightarrow F(x,y)$ *is concave on D,*

(iv) $\forall x \in D$, $F(x,x) \cap \mathbb{R}_+ \neq \emptyset$.

(v) $\exists r > 0$, *such that for all* $x \in K$, $\|x\| \leq r$, *there exists* $y_0 \in D$ *with* $\|y_0\| < r$ *and* $F(x, y_0) \leq 0$.

Then, there exists an element $x_0 \in K$ *such that*

$$F(x_0, y) \cap \mathbb{R}_+ \neq \emptyset, \forall y \in K.$$

Proof. Let $r > 0$, such that (v) holds and consider the weakly compact set $K_0 = K \cap \overline{B}_r$. According to Theorem 4.1.6 there exists $x_0 \in K_0$ such that $F(x_0, y) \cap \mathbb{R}_+ \neq \emptyset$, $\forall y \in K_0$.

According to (v), there exists $z_0 \in D$ with $\|z_0\| < r$ such that $F(x_0, z_0) \leq 0$.

Now let $z \in D \setminus K_0$. Then, in virtue of self segment denseness of D in K, there exists $\lambda \in]0,1[$ such that $\lambda z_0 + (1-\lambda)z \in K_0 \cap D$. According to (iii)

$$F(x_0, \lambda z_0 + (1-\lambda)z) \subseteq \lambda F(x_0, z_0) + (1-\lambda)F(x_0, z),$$

which leads to $F(x_0, z) \cap \mathbb{R}_+ \neq \emptyset$.

Hence $F(x_0, z) \cap \mathbb{R}_+ \neq \emptyset$ for all $z \in D$. Let $y \in K \setminus D$. According to (ii) $F(x_0, \cdot)$ is weakly upper semicontinuous at y, hence, for any open set $V \subseteq \mathbb{R}$, such that $F(x_0, y) \subseteq V$, there exists an open neighbourhood U of y such that, for any $u \in U$ one has $F(x_0, u) \subseteq V$. Since D is dense in K we have $D \cap U \neq \emptyset$.

Assume that $F(x_0, y) \subseteq]-\infty, 0[$. Then, take $V =]-\infty, 0[$ and let $z \in U \cap D$. Then, $F(x_0, z) \subseteq]-\infty, 0[$ which contradicts the fact that $F(x_0, z) \cap \mathbb{R}_+ \neq \emptyset$. □

For sake of completeness, we also give sufficient conditions for the solution existence of densely defined single valued equilibrium problem in a reflexive Banach space setting.

Theorem 4.1.15. *Let* X *be a reflexive Banach space, let* K *be a nonempty, convex and closed subset of* X, *let* $D \subseteq K$ *be a self segment-dense set in the weak topology of* X, *and let* $\varphi : K \times K \longrightarrow \mathbb{R}$ *a function satisfying*

(i) $\forall y \in D$ *the function* $x \longrightarrow \varphi(x, y)$ *is weak upper semicontinuous on* K,

(ii) $\forall x \in K$, $y \longrightarrow \varphi(x, y)$ *is weak upper semicontinuous on* $K \setminus D$,

(iii) $\forall x \in K$, *the function* $y \to \varphi(x, y)$ *is convex on* D,

(iv) $\forall x \in D$, $\varphi(x, x) \geq 0$.

(v) $\exists r > 0$, *such that, for all* $x \in K$, $\|x\| \leq r$, *there exists* $y_0 \in D$ *with* $\|y_0\| < r$ *and* $\varphi(x, y_0) = 0$.

Then, there exists an element $x_0 \in K$ such that

$$\varphi(x_0, y) \geq 0, \forall y \in K.$$

Proof. Let $r > 0$ such that (v) holds and consider the set $K_0 = K \cap \overline{B}_r$, which obviously is weakly compact. According to Theorem 4.1.7, there exists $x_0 \in K_0$ such that $\varphi(x_0, y) \geq 0, \forall y \in K_0$. According to (v), there exists $z_0 \in D \cap K_0$, $\|z_0\| < r$ such that $\varphi(x_0, z_0) = 0$. Consider $z \in D \setminus K_0$. Since D is self segment dense in K, we obtain that there exists $\lambda \in]0,1[$ such that $\lambda z_0 + (1-\lambda)z \in D \cap K_0$. Hence, from (iii) we have $\varphi(x_0, \lambda z_0 + (1-\lambda)z) \leq \lambda\varphi(x_0, z_0) + (1-\lambda)\varphi(x_0, z)$, or equivalently $(1-\lambda)\varphi(x_0, z) \geq \varphi(x_0, \lambda z_0 + (1-\lambda)z) \geq 0$.

The latter relation shows that $\varphi(x_0, z) \geq 0$ for all $z \in D$. Finally, if $y \in K \setminus D$, by the denseness of D in K there exists a net $(y^\alpha) \subseteq D$ such that $\lim y^\alpha = y$, where the limit is taken in the weak topology of X. At this point, the assumption (ii), $\varphi(x_0, y) \geq 0$ for all $y \in D$ and the upper semicontinuity of $\varphi(x_0, y)$ on $K \setminus D$, assures that

$$0 \leq \limsup_{y^\alpha \to y} \varphi(x_0, y^\alpha) \leq \varphi(x_0, y).$$

Thus, we have $\varphi(x_0, y) \geq 0$ for all $y \in K$. □

4.1.6 A Generalized Debreu-Gale-Nikaïdo Theorem

As an application of the set-valued equilibrium results in the previous sections, we present a Debreu-Gale-Nikaïdo-type theorem, which extends the famous classical result in economic equilibrium theory by requiring that the collective Walras law holds not on the entire price simplex, but on a self segment-dense subset of it. For the original results we refer to [84] Section 5.6(1), the Principal Lemma in [96] and to Theorem 16.6 in [167].

Consider the simplex

$$M^n := \left\{ x \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i = 1 \right\}$$

and the set valued map $C : M^n \rightrightarrows \mathbb{R}^n$. Assume that $G(C)$, the graph of C, is closed and C has nonempty, bounded and convex values. According to Debreu-Gale-Nikaïdo theorem, if for all $(x, y) \in G(C)$ we have $\langle y, x \rangle \geq 0$ (Walras law), then there exists $x_0 \in M^n$ such that

$$C(x_0) \cap \mathbb{R}^n_+ \neq \emptyset.$$

In what follows, we extend this result by weakening the conditions imposed on C and

assuming that Walras' law holds only on D, a self segment-dense subset of M^n. Hence, we consider the set-valued map with nonempty, compact and convex values $C : M^n \rightrightarrows \mathbb{R}^n$. We will use the following notation

$$\forall y \in \mathbb{R}^n, \quad \sigma(C(x), y) := \sup_{z \in C(x)} \langle z, y \rangle$$

and we say that C is *upper hemi-continuous* if the map $x \to \sigma(C(x), y)$ is upper semi-continuous for all $y \in \mathbb{R}^n$. We have the following result.

Theorem 4.1.16. *Let $C : M^n \rightrightarrows \mathbb{R}^n$ be a set-valued map with nonempty, compact and convex values, and let D be a self segment-dense subset of M^n. If*

(i) *C is upper hemi-continuous regarding D, i.e., $\forall y \in D,\ x \longrightarrow \sigma(C(x), y)$ is upper semi-continuous on M^n,*

(ii) $\forall x \in D$, $\sigma(C(x), x) \geq 0$,

then there exists $x_0 \in M^n$, such that $C(x_0) \cap \mathbb{R}^n_+ \neq \emptyset$.

Proof. We consider the set-valued map $F : M^n \times M^n \rightrightarrows \mathbb{R}$

$$F(x, y) =]-\infty, \sigma(C(x), y)]$$

and prove that it satisfies the hypotheses of Theorem 4.1.6.

In view of Remark 4.1.1, for all $y \in D,\ x \longrightarrow F(x, y)$ is upper semi-continuous on M^n if, for any sequence $(x_k) \in M^n$, $x_k \longrightarrow x$ and any $b \in R$ such that $\sigma(C(x), y) < b$, we can show that $\sigma(C(x_k), y) < b$ for k sufficiently large. However, this holds true, since the upper hemi-continuity of C together with Remark 4.1.2, guarantees that

$$\limsup_{x_k \longrightarrow x} \sigma(C(x_k), y) \leq \sigma(C(x), y).$$

To see that

$$\forall x \in M^n, y \longrightarrow F(x, y) \text{ is upper semi-continuous on } M^n \backslash D,$$

holds true, we rely again on Remark 4.1.1, and also on the fact that $C(x)$ is compact and convex. Since $\sigma(C(x), y)$, the support function of $C(x)$, is continuous we have that

$$\sigma(C(x), y_k) \longrightarrow \sigma(C(x), y).$$

For any $x \in D$, the concavity of the set-valued map $y \longrightarrow F(x,y)$ follows from the convexity of single-valued map $y \longrightarrow \sigma(C(x), y)$, which is the pointwise supremum of a family of affine functions.

Finally, the condition that

$$\forall x \in D, F(x,x) \cap \mathbb{R}_+ \neq \emptyset,$$

is exactly Walras' law in our hypothesis (ii). So, based on Theorem 4.1.6, we conclude that there exists $x_0 \in M^n$ such that

$$F(x_0, y) \cap \mathbb{R}^n_+ \neq \emptyset, \forall y \in M^n,$$

or in other words,

$$\sigma(C(x_0), y) \geq 0, \forall y \in M^n. \tag{4. 3}$$

On the other hand, the above inequality is equivalent to

$$\sigma\left(C(x_0) - \mathbb{R}^n_+, y\right) \geq 0, \forall y \in \mathbb{R}^n, \tag{4. 4}$$

since

$$\sigma\left(-\mathbb{R}^n_+, y\right) = \begin{cases} 0, & \text{if } y \in \mathbb{R}^n_+ \\ +\infty, & \text{if } y \notin \mathbb{R}^n_+. \end{cases}$$

At this point, we need the fact that $C(x_0) - \mathbb{R}^n_+$ is closed and convex to conclude from (4. 4), that $0 \in C(x_0) - \mathbb{R}^n_+$, that is

$$C(x_0) \cap \mathbb{R}^n_+ \neq \emptyset.$$

□

4.1.7 Non-Cooperative Equilibrium in $n-$Person Games

Following the approach of Aubin, we consider a n-person game in normal (strategic) form [29] and we denote by E^i the strategy set of each player i, $i \in \{1, \ldots, n\}$, while $E = \prod_{i=1}^n E^i$ is the set of multistrategies $x = \left(x^1, \ldots, x^n\right)$.

In the absence of cooperation, from the perspective of player i, the set of multistrategies can be regarded as a product between the set E^i of strategies that he controls, and the set of

strategies $\widehat{x}^i = \left(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n\right)$ of all other players

$$E = E^i \times \widehat{E}^i.$$

The behavior of each player is defined by a loss function $f^i : E \to \mathbb{R}$ with associated decision rules

$$C^i : \widehat{E}^i \to E^i, \;\; C^i\left(\widehat{x}^i\right) = \left\{ x^i \in E^i : f^i\left(x^i, \widehat{x}^i\right) = \inf_{y^i \in E^i} f^i\left(y^i, \widehat{x}^i\right) \right\}.$$

A non-cooperative equilibrium (or Nash equilibrium) is a fixed point of the set-valued map

$$C : E \rightrightarrows E, \;\; C(x) = \prod_{i=1}^{n} C^i\left(\widehat{x}^i\right).$$

As shown in [29], Nash equilibria can be characterized using the map

$$\varphi : E \times E \to \mathbb{R}, \varphi(x,y) = \sum_{i=1}^{n} \left(f^i\left(x^i, \widehat{x}^i\right) - f^i\left(y^i, \widehat{x}^i\right) \right).$$

Lemma 4.1.3 ([29]). *A multistrategy $x_0 \in E$ is a non-cooperative equilibrium if and only if*

$$\varphi(x_0, y) \leq 0, \quad \forall y \in E.$$

Now we can verify the existence of non-cooperative equilibria under convexity assumptions formulated on self segment-dense subsets of the strategy sets. This generalizes the classical result of Nash ([29] Theorem 12.2) by allowing that the convexity is violated on small sets.

Theorem 4.1.17. *Suppose that for any $i \in \{1, \ldots, n\}$, the sets E^i are convex and compact and let $D^i \subseteq E^i$ be self segment-dense subsets of E^i. Assume further that for every $i \in \{1, \ldots, n\}$ the following hold:*

(i) f^i is lower semicontinuous on E,

(ii) for all $y^i \in D^i$ the map $\widehat{x}^i \longrightarrow f^i(y^i, \widehat{x}^i)$ is upper semicontinuous on $\widehat{E}^i$,

(iii) for all $\widehat{x}^i \in \widehat{E}^i$ the map $y^i \longrightarrow f^i(y^i, \widehat{x}^i)$ is upper semicontinuous on $E^i \setminus D^i$,

iv) for all $\widehat{x}^i \in \widehat{D}^i$ the map $y^i \longrightarrow f^i(y^i, \widehat{x}^i)$ is convex on D^i.

Then, there exists a non-cooperative equilibrium.

Proof. The theorem follows from Theorem 4.1.8.

We consider the set E and the function φ defined above. The set E, being a product of compact and convex sets is itself compact and convex, meanwhile the set $D = \prod_{i=1}^{n} D^i$ is self segment-dense in E.

The assumptions $(i) - (iv)$ assure that the hypotheses $(i) - (iii)$ of Theorem 4.1.8 are satisfied. Moreover, we have that $\varphi(x,x) = 0$, for any $x \in E$. So, the characterization of Nash equilibria together with the mentioned abstract result guarantee the existence of a non-cooperative equilibrium point. □

4.1.8 Perspectives and Conclusions

Set-valued versions of Ky-Fan's inequality are related to a wide class of problems in diverse fields such as:

a) fixed point and coincidence theorems,

b) variational inequalities,

c) vector optimization or

d) game theory.

Already in their original article, Kristály and Varga [129], showed how set-valued equilibrium results apply to fixed point theory and to variational inclusions. Subsequently, many scientists explored other applications.

By introducing the concept of a self segment-dense subset, we are able to relax the convexity and continuity assumptions of Kristály and Varga, and it seems legitimate to ask ourselves about the consequences that the new concept can have in the related fields mentioned above.

Partially, this line of inquiry is carried out in Subsections 4.1.6 and 4.1.7 of this work, where by means of our Ky Fan-Type results we can prove: 1) The existence of an economic equilibrium when the constraint imposed by Walras' law holds just on a self segment-dense subset of the price simplex. 2) The existence of a Nash equilibrium for a non-cooperative n-person game under the assumption that the loss function of each player is convex on a self segment-dense subset of the set of strategies, not on the whole set. However, these are just basic applications, that can be further refined and extended both by considering more general frameworks (non-cooperative games with constraints, for example) and by analyzing specific problems, which are relevant to the applied scientist (e.g., the Cournot duopoly).

4.2 Vector equilibrium problems on dense sets

Starting with the pioneering work of Giannessi [99], several extensions of the scalar equilibrium problem to the vector case have been considered. These vector equilibrium problems, much like their scalar counterpart, offer a unified framework for treating vector optimization, vector variational inequalities or cone saddle point problems, to name just a few [13–16, 102, 103].

In this section, we obtain some existence results of the solution for two vector equilibrium problems. The conditions that we consider are imposed on a special type of dense set that we call self-segment-dense [143]. Our conditions, which ensure the solution existence of the weak vector equilibrium problems and strong vector equilibrium problems, weaken considerably the existing conditions from the literature. More precisely, we assume some continuity, diagonal and convexity properties on the vector bifunction involved in these problems, not on the whole domain of the bifunction, but rather on a self-segment-dense subset of it. The special case, when the domain of this vector bifunction is a closed subset of a reflexive Banach space, is also investigated. We show that these results fail, if we replace the self-segment-denseness of the subset mentioned above by its usual denseness property. Finally, we apply our results to vector optimization and vector variational inequalities.

Let us mention that the results from this section were partially published in [139]:[S. László, *Vector Equilibrium Problems on Dense Sets*, J. Optim. Theory Appl., **170**, 437-457 (2016)].

4.2.1 Preliminaries

Let X be a real Hausdorff, locally convex topological vector space. For a non-empty set $D \subseteq X$, we denote by $\operatorname{int} D$ its interior, by $\operatorname{cl} D$ its closure, by $\operatorname{co} D$ its convex hull and by $\operatorname{span} D$ the subspace of X generated by D. We say that $P \subseteq D$ is dense in D iff $D \subseteq \operatorname{cl} P$. Recall that a set $C \subseteq X$ is a cone iff $tc \in C$ for all $c \in C$ and $t > 0$. The cone C is called a convex cone iff $C + C = C$. The cone C is called a pointed cone iff $C \cap (-C) = \{0\}$. Note that a closed, convex and pointed cone C induces a partial ordering on X, that is $z_1 \leq z_2 \Leftrightarrow z_2 - z_1 \in C$. In the sequel, when we use $\operatorname{int} C$, we tacitly assume that the cone C has nonempty interior. It is an easy exercise to show that $C + C \setminus \{0\} = C \setminus \{0\}$ and $\operatorname{int} C + C = \operatorname{int} C$. A well known example of a closed, convex and pointed cone, with nonempty interior, is the non-negative orthant of $\mathbb{R}^n$, that is,

$$\mathbb{R}^n_+ := \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : x_i \geq 0, i \in \{1, 2, ..., n\}\}.$$

The non-positive orthant can be defined similarly. Let $K \subseteq X$. The classical scalar equilibrium problem [44], described by a bifunction $\varphi : K \times K \longrightarrow \mathbb{R}$, consists in finding $x_0 \in K$ such that $\varphi(x_0, y) \geq 0,$ for all $y \in K$.

In this section, we extend Theorem 4.1.3 to the vector case. Moreover, the conditions will be assumed not on the whole set K, but rather on a self-segment-dense subset of K. In what follows we introduce the vector equilibrium problems described by a vector bifunction, and the suitable semicontinuity and convexity properties of a vector valued function, in order to rewrite the conditions $(i) - (iii)$ of Theorem 4.1.3 to the vector case.

Let Z be another locally convex Hausdorff topological vector space, let $K \subseteq X$ be a nonempty subset and let $C \subseteq Z$ be a convex and pointed cone. Assume that the interior of the cone C is nonempty and consider the mapping $f : K \times K \longrightarrow Z$. The vector equilibrium problem, introduced in [16], consist in finding $x_0 \in K$, such that

$$f(x_0, y) \notin -\operatorname{int} C, \forall y \in K. \tag{4. 5}$$

Recall that this problem is called weak vector equilibrium problem [99, 101].

The following equilibrium problem, called strong vector equilibrium problem, see [99, 101], is also a valid extension of the scalar equilibrium problem to vector valued case. We would like to point out, that in the formulation of the strong vector equilibrium problem we do not assume that the cone C has nonempty interior.

The strong vector equilibrium problem consists in finding $x_0 \in K$, such that

$$f(x_0, y) \notin -C \setminus \{0\}, \forall y \in K. \tag{4. 6}$$

It can easily be observed, that for $Z = \mathbb{R}$ and $C = \mathbb{R}_+ = [0, \infty[$, the previous problems reduce to the classical scalar equilibrium problem. Note that, if $\operatorname{int} C \neq \emptyset$ and $x_0 \in K$ is a solution of (4. 6), then x_0 is also a solution of (4. 5).

A map $f : K \longrightarrow Z$ is said to be C-upper semicontinuous at $x \in K$, iff for any neighborhood V of $f(x)$ there exists a neighborhood U of x such that $f(u) \in V - C$ for all $u \in U \cap K$. Obviously, if f is continuous at $x \in K$, then it is also C-upper semicontinuous at $x \in K$. Assume that C has nonempty interior. According to [203], f is C-upper semicontinuous at $x \in K$, if and only if, for any $k \in \operatorname{int} C$, there exists a neighborhood U of x, such that $f(u) \in f(x) + k - \operatorname{int} C$ for all $u \in U \cap K$.

Similarly, even if $\operatorname{int} C = \emptyset$, one can introduce the so called strongly C-upper semicontinuity of f as follows: f is strongly C-upper semicontinuous at $x \in K$, if and only if, for any $k \in C \setminus \{0\}$, there exists a neighborhood U of x, such that $f(u) \in f(x) + k - C \setminus \{0\}$

for all $u \in U \cap K$. The map $f : K \longrightarrow Z$ is said to be C-lower semicontinuous, (strongly C-lower semicontinuous) at $x \in K$, iff the map $-f$ is C-upper semicontinuous, (strongly C-upper semicontinuous) at $x \in K$. We say that f is C-upper semicontinuous, (strongly C-upper semicontinuous, C-lower semicontinuous, strongly C-lower semicontinuous) on K, if f is C-upper semicontinuous, (strongly C-upper semicontinuous, C-lower semicontinuous, strongly C-lower semicontinuous) at every $x \in K$.

Definition 4.2.1. The mapping $f : K \longrightarrow Z$ is called a C-function on K, iff for every finite subset $\{x_1, x_2, \ldots, x_n\} \subseteq K$, $n \in \mathbb{N}$ and every $\lambda_i \geq 0$, $i \in \{1, 2, \ldots, n\}$, such that $\sum_{i=1}^n \lambda_i = 1$ and $\sum_{i=1}^n \lambda_i x_i \in K$, one has

$$\sum_{i=1}^n \lambda_i f(x_i) - f\left(\sum_{i=1}^n \lambda_i x_i\right) \in C.$$

Remark 4.2.1. Note that in the previous definition we did not assume the convexity of the set K. It is obvious, that when K is convex, the condition $\sum_{i=1}^n \lambda_i x_i \in K$, is automatically satisfied for every finite subset $\{x_1, x_2, \ldots, x_n\} \subseteq K$, $n \in \mathbb{N}$ and every $\lambda_i \geq 0$, $i \in \{1, 2, \ldots, n\}$, such that $\sum_{i=1}^n \lambda_i = 1$. Moreover, in this case in the definition of a C-function it is enough to assume that for every $x_1, x_2 \in K$ and $t \in [0,1]$ one has: $tf(x_1) + (1-t)f(x_2) - f(tx_1 + (1-t)x_2) \in C$. Nevertheless, next example shows that the latter relation is not sufficient when K is not convex.

Example 4.2.1. Let $K \subseteq \mathbb{R}^2$ be the following set.

$$K := \{(-t, -t) \in \mathbb{R}^2 : t \in [0,1]\} \cup \{(t, -t) \in \mathbb{R}^2 : t \in [0,1]\} \cup \{(0, t) \in \mathbb{R}^2 : t \in [0,1]\}.$$

Obviously, K is the reunion of three closed line segments having $(0,0)$ as a common endpoint, hence it is not convex. Let $C = [0, \infty[\subseteq \mathbb{R}$. Then C is a closed, convex and pointed cone in $\mathbb{R}$. Consider the mapping $f : K \longrightarrow \mathbb{R}$, $f(x) = \begin{cases} 0, \text{ if } x \neq (0,0) \\ 1, \text{ if } x = (0,0). \end{cases}$

Then, it can easily be verified, that for every $x_1, x_2 \in K$ and $t \in [0,1]$, such that $tx_1 + (1-t)x_2 \in K$, one has $tf(x_1) + (1-t)f(x_2) - f(tx_1 + (1-t)x_2) \in C$. On the other hand, for $\lambda_1 = \lambda_2 = \frac{1}{4}$ and $\lambda_3 = \frac{1}{2}$, and $x_1 = (-1,-1), x_2 = (1,-1), x_3 = (0,1)$ one has $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = (0,0) \in K$.

At the same time, $\sum_{i=1}^3 \lambda_i f(x_i) - f\left(\sum_{i=1}^3 \lambda_i x_i\right) = -1 \notin C$.

Remark 4.2.2. Assume that K is convex. When $Z = \mathbb{R}^n$ and the cone C contains (or is equal, or is contained in) the non-negative orthant of $\mathbb{R}^n$, then a C-function is called C-convex (or convex, or strictly C-convex), when C contains (or is equal, or is contained in)

the non-positive orthant of $\mathbb{R}^n$, then a C-function is called C-concave (or concave, or strictly C-concave). It is well known, that a vector function f is convex, iff all its components are convex. In what follows, we provide an example of a C-function with non-convex components.

Example 4.2.2. Let $f : [0,\infty[\longrightarrow \mathbb{R}^2$, $f(x) = (x^2 + x, -x^2)$ and consider the closed, convex and pointed cone $C := \{(x,y) \in \mathbb{R}^2 : x \in [0,\infty[, -x \leq y \leq 0\}$.

Then, an easy computation shows, that for every $x_1, x_2 \in [0,\infty[$ and $t \in [0,1]$, one has $tf(x_1) + (1-t)f(x_2) - f(tx_1 + (1-t)x_2) \in C$, hence f is a C-function on $[0,\infty[$. Nevertheless, the second component of f is a concave function on $[0,\infty[$.

We will use the following notation for the closed line segments in X with the endpoints x and y

$$[x,y] := \{z \in X : z = x + t(y-x), t \in [0,1]\}.$$

The line segments $]x,y[$, $]x,y]$ and $[x,y[$ are defined as $[x,y] \setminus \{x,y\}$, $[x,y] \setminus \{x\}$ and $[x,y] \setminus \{y\}$, respectively. In [149], Definition 3.4, The Luc has introduced the notion of a so-called *segment-dense* set and, based on this concept, obtained some existence results of solutions for densely pseudomonotone variational inequalities. Let $V \subseteq X$ be a convex set. One says, that the set $U \subseteq V$ is segment-dense in V, iff for each $x \in V$, there can be found $y \in U$, such that x is a cluster point of the set $[x,y] \cap U$. The notion of a segment-dense set has successfully been used in the context of weighted quasi-variational inequalities [17], and densely relative pseudomonotone variational inequalities [18]. In this section, we use a related denseness notion, in order to obtain solution existence of some vector equilibrium problems. However, we would like to emphasize that our results are some natural extensions of the original result of Ky Fan (Theorem 4.1.3), and we do not assume any monotonicity property of the vector bifunctions involved.

In what follows, we present the concept of a self-segment-dense set (cf. [143,144]), which is in some sense compatible with the convexity property of sets.

Consider the sets $U \subseteq V \subseteq X$ and assume that V is convex. We say that U is self-segment-dense in V, iff U is dense in V and

$$\forall x, y \in U, \text{ the set } [x,y] \cap U \text{ is dense in } [x,y].$$

Remark 4.2.3. Obviously in one dimension the concept of a self-segment-dense set, (or the concept of a segment-dense set), is equivalent to the concept of a dense set.

The proof of the results obtained in the next sections are based on Lemma 4.1.2.

Let us emphasize that Lemma 4.1.2 does not remain valid in case we replace the self-segment-denseness of U in V, by its denseness in V, as the next example shows.

Example 4.2.3. Let V be the closed unit ball of an infinite dimensional Banach space X, and let $x, y \in V$, $x \neq y$. Moreover, consider $u, v \in]x, y[$, such that $u = x + t_1(y - x)$, $v = x + t_2(y - x)$, with $t_1, t_2 \in]0, 1[$, $t_1 < t_2$. Then obviously $U = V \setminus [u, v]$ is dense in V, (and it is also segment-dense), but not self-segment-dense, since for $x, y \in U$ the set $[x, y] \cap U = [x, u[\cup]v, y]$ is not dense in $[x, y]$.

This also shows, that $\mathrm{cl}(\mathrm{co}\{x, y\} \cap U) \neq \mathrm{co}\{x, y\}$.

4.2.2 Self-Segment-Dense Sets and the Weak Vector Equilibrium Problem

In this paragraph, we obtain some sufficient conditions that ensure the existence of a solution for the weak vector equilibrium problem (4. 5). The conditions, that we consider, are imposed not on the whole domain of the vector bifunction f, but rather on a self-segment-dense subset of it. We also show, that the self-segment-dense property of this set is essential in obtaining our results, and cannot be replaced by the usually denseness property. Lemma 4.1.2 plays an important role in the proofs of our results. We treat both the cases when the set K, the domain of the vector bifunction f, is compact, respectively closed. We pay a special attention to the case when the set K is a closed subset of reflexive Banach space X.

The main result of this section is the following.

Theorem 4.2.1. *Let X and Z be Hausdorff, locally convex topological vector spaces, let $C \subseteq Z$ be a convex and pointed cone with nonempty interior and let K be a nonempty, convex and compact subset of X. Let $D \subseteq K$ be a self-segment-dense set and consider the mapping $f : K \times K \longrightarrow Z$ satisfying*

(i) $\forall y \in D$, *the mapping* $x \longrightarrow f(x, y)$ *is C-upper semicontinuous on* K,

(ii) $\forall x \in K$, *the mapping* $y \longrightarrow f(x, y)$ *is C-upper semicontinuous on* $K \setminus D$,

(iii) $\forall x \in D$, *the mapping* $y \longrightarrow f(x, y)$ *is a C-function on* D,

(iv) $\forall x \in D$, $f(x, x) \notin -\mathrm{int}\, C$.

Then, there exists an element $x_0 \in K$ such that

$$f(x_0, y) \notin -\mathrm{int}\, C, \forall y \in K.$$

Proof. We give two different proofs.

I. Assume the contrary, that is, for every $x \in K$ there exists $y \in K$ such that $f(x,y) \in -\operatorname{int}C$. Then, for every $y \in K$ consider $V_y = \{x \in K : f(x,y) \in -\operatorname{int}C\}$. It is obvious that $K \subseteq \cup_{y\in K} V_y$. We show that $(V_y)_{y\in D}$ is an open cover of K. First of all observe that for all $y \in D$, one has $V_y = K \setminus G(y)$, where $G(y)$ is the set $\{x \in K : f(x,y) \notin -\operatorname{int}C\}$. We show that $G(y)$ is closed for all $y \in D$. Indeed, for fixed $y_0 \in D$ consider the net $(x_\alpha) \subseteq G(y_0)$ and let $\lim x_\alpha = x_0$. Assume that $x_0 \notin G(y_0)$. Then $f(x_0,y_0) \in -\operatorname{int}C$. According to (i) the function $x \longrightarrow f(x,y_0)$ is C-upper semicontinuous at x_0, hence for every $k \in \operatorname{int}C$ there exists U, a neighborhood of x_0, such that $f(x,y_0) \in f(x_0,y_0) + k - \operatorname{int}C$ for all $x \in U$. But then for $k = -f(x_0,y_0) \in \operatorname{int}C$ one obtains that there exits α_0, such that $f(x_\alpha,y_0) \in -\operatorname{int}C$, for $\alpha \geq \alpha_0$, which contradicts the fact that $(x_\alpha) \subseteq G(y_0)$. Consequently, V_y is open for every $y \in D$.

Assume now that there exists $x_0 \in K$ such that $x_0 \notin \cup_{y\in D} V_y$. Then, we have that $f(x_0,y) \notin -\operatorname{int}C$, for all $y \in D$. We show that $f(x_0,y) \notin -\operatorname{int}C$, for all $y \in K$. Indeed, for $y_0 \in K \setminus D$, by the denseness of D in K, we have that for every neighborhood U of y_0 there exists a $u_0 \in U \cap D$. At this point, the assumption (ii), the upper semicontinuity of $f(x_0,y)$ on $K \setminus D$, assures that for all $k \in \operatorname{int}C$ there exists a neighborhood U of y_0, such that $f(x_0,u) \in f(x_0,y_0) + k - \operatorname{int}C$. Assume that $f(x_0,y_0) \in -\operatorname{int}C$. Then let $k = -f(x_0,y_0) \in \operatorname{int}C$. Thus, we have that there exists a neighborhood U of y_0, such that $f(x_0,u) \in -\operatorname{int}C$. But, by choosing $u_0 \in U \cap D$ we get that $f(x_0,u_0) \in -\operatorname{int}C$, contradiction. Hence, $f(x_0,y) \notin -\operatorname{int}C$, for all $y \in K$, which contradicts our assumption, that for every $x \in K$ there exists $y \in K$, such that $f(x,y) \in -\operatorname{int}C$. Consequently, $(V_y)_{y\in D}$ is an open cover of the compact set K, in conclusion it contains a finite subcover. In other words, there exists $y_1, y_2, ..., y_n \in D$ such that $K \subseteq \cup_{i=1}^n V_{y_i}$. Consider $\left(p_i\right)_{1\leq i\leq n}$ a continuous partition of unity associated to the open cover $\left(V_{y_i}\right)_{1\leq i\leq n}$. Then $p_i : K \longrightarrow [0,1]$ is continuous and $\operatorname{supp}(p_i) = \operatorname{cl}\{x \in K : p_i(x) \neq 0\} \subseteq V_{y_i}$ for all $i \in \{1,2,...,n\}$, moreover $\sum_{i=1}^n p_i(x) = 1$, for all $x \in K$.

Consider the mapping

$$\varphi : \operatorname{co}\{y_1,y_2,...,y_n\} \longrightarrow \operatorname{co}\{y_1,y_2,...,y_n\},\ \varphi(x) = \sum_{i=1}^n p_i(x)y_i.$$

Obviously φ is continuous, and $\operatorname{co}\{y_1,y_2,...,y_n\}$ is a compact and convex subset of the finite dimensional space $\operatorname{span}\{y_1,y_2,...,y_n\}$. Hence, by the Brouwer fixed point theorem, there exists $x_0 \in \operatorname{co}\{y_1,y_2,...,y_n\}$ such that $\varphi(x_0) = x_0$.

Let $J = \{i \in \{1,2,...,n\} : p_i(x_0) > 0\}$. Obviously the set J is nonempty, since $\sum_{i\in J} p_i(x_0) = 1$, and $\varphi(x_0) = \sum_{i=1}^n p_i(x_0)y_i = \sum_{i\in J} p_i(x_0)y_i = x_0$. The latter equality shows, that $x_0 \in \operatorname{co}\{y_i :$

$i \in J\}$. On the other hand, from $p_i(x_0) > 0$ for all $i \in J$, we obtain that $x_0 \in \cap_{i\in J} V_{y_i}$. Since $\cap_{i\in J} V_{y_i}$ is open, we obtain that there exists U, an open and convex neighbourhood of x_0, such that $U \subseteq \cap_{i\in J} V_{y_i}$. Here we find very useful the conclusion of Lemma 4.1.2. Indeed, since $\mathrm{co}\{y_i : i \in J\} \cap U \neq \emptyset$, according to Lemma 4.1.2, we have that there exists $y_0 \in \mathrm{co}\{y_i : i \in J\} \cap U \cap D$. Hence, we have $y_0 = \sum_{i\in J} \lambda_i y_i \in \mathrm{co}\{y_i : i \in J\} \cap U \cap D$, where $\lambda_i \geq 0$ for all $i \in J$ and $\sum_{i\in J} \lambda_i = 1$. By (iv), in the hypothesis of the theorem, one gets $f(y_0, y_0) \notin -\mathrm{int}\, C$. On the other hand, by using (iii) we get $f(y_0, y_0) = f(y_0, \sum_{i\in J} \lambda_i y_i) \leq \sum_{i\in J} \lambda_i f(y_0, y_i)$, which shows that $\sum_{i\in J} \lambda_i f(y_0, y_i) - f(y_0, y_0) \in C$. But, $y_0 \in U$, thus $f(y_0, y_i) \in -\mathrm{int}\, C$, for all $i \in J$. Hence, $\sum_{i\in J} \lambda_i f(y_0, y_i) \in -\mathrm{int}\, C$, which leads to

$$-f(y_0, y_0) \in C - \sum_{i\in J} \lambda_i f(y_0, y_i) \subseteq \mathrm{int}\, C,$$

contradiction. □

II. The second proof is based on Ky Fan's Lemma. We consider the set-valued map $G : D \rightrightarrows K$, $G(y) = \{x \in K : f(x, y) \notin -\mathrm{int}\, C\}$. We have shown in the first part of the proof, that $G(y)$ is closed for all $y \in D$. Since K is compact, we have that $G(y) \subseteq K$, is compact for all $y \in D$. We show next, that G is a KKM mapping. We claim that for all $y_1, y_2, ..., y_n \in D$ one has $\mathrm{co}\{y_1, y_2, ..., y_n\} \cap D \subseteq \bigcup_{i=1}^n G(y_i)$. Indeed, assume that there exist $y_1, y_2, ..., y_n \in D$ and $y \in \mathrm{co}\{y_1, y_2, ..., y_n\} \cap D$, such that $y \notin \bigcup_{i=1}^n G(y_i)$.

Hence, there exist $\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$, satisfying $\sum_{i=1}^n \lambda_i = 1$ such that $\sum_{i=1}^n \lambda_i y_i \in D$ and $\sum_{i=1}^n \lambda_i y_i \notin \bigcup_{i=1}^n G(y_i)$. This is equivalent to $f(\sum_{i=1}^n \lambda_i y_i, y_i) \in -\mathrm{int}\, C$, for all $i \in \{1, 2, \ldots, n\}$, and hence, by the convexity of $-\mathrm{int}\, C$, we have that $\sum_{i=1}^n \lambda_i f\left(\sum_{i=1}^n \lambda_i y_i, y_i\right) \in -\mathrm{int}\, C$. From assumption (iii), we have that $\sum_{i=1}^n \lambda_i f\left(\sum_{i=1}^n \lambda_i y_i, y_i\right) - f\left(\sum_{i=1}^n \lambda_i y_i, \sum_{i=1}^n \lambda_i y_i\right) \in C$, or equivalently, $f\left(\sum_{i=1}^n \lambda_i y_i, \sum_{i=1}^n \lambda_i y_i\right) \in \sum_{i=1}^n \lambda_i f\left(\sum_{i=1}^n \lambda_i y_i, y_i\right) - C \subseteq -\mathrm{int}\, C$, which contradicts (iv). Consequently, $\mathrm{co}\{y_1, y_2, \ldots, y_n\} \cap D \subseteq \bigcup_{i=1}^n G(y_i)$, holds true, and leads to

$$\mathrm{cl}(\mathrm{co}\{y_1, y_2, \ldots, y_n\} \cap D) \subseteq \mathrm{cl}\left(\bigcup_{i=1}^n G(y_i)\right).$$

Furthermore, since $G(y_i)$ is closed for all $i \in \{1, 2, \ldots, n\}$, we have $\mathrm{cl}\left(\bigcup_{i=1}^n G(y_i)\right) = \bigcup_{i=1}^n G(y_i)$. On the other hand, according to Lemma 4.1.2 we have

$$\mathrm{cl}(\mathrm{co}\{y_1, y_2, \ldots, y_n\} \cap D) = \mathrm{co}\{y_1, y_2, \ldots, y_n\},$$

hence $\mathrm{co}\{y_1, y_2, \ldots, y_n\} \subseteq \bigcup_{i=1}^n G(y_i)$. In conclusion G is a KKM map.

Thus, according to Ky Fan's Lemma, $\bigcap_{y\in D} G(y) \neq \emptyset$. In other words, there exists $x_0 \in K$,

such that $f(x_0,y) \notin -\operatorname{int}C$ for all $y \in D$.

Finally, if $y_0 \in K \setminus D$, by the denseness of D in K, we obtain that for every neighborhood U of y_0, there exists a $u_0 \in U \cap D$. At this point, the assumption (ii), the C-upper semicontinuity of $f(x_0,y)$ on $K \setminus D$, assures that for all $k \in \operatorname{int}C$ there exists a neighborhood U of y_0 such that $f(x_0,u) \in f(x_0,y_0) + k - \operatorname{int}C$. Assume that $f(x_0,y_0) \in -\operatorname{int}C$. Then, let $k = -f(x_0,y_0) \in \operatorname{int}C$. Thus, we have that there exists a neighborhood U of y_0, such that $f(x_0,u) \in -\operatorname{int}C$. But, by choosing $u_0 \in U \cap D$ we get that $f(x_0,u_0) \in -\operatorname{int}C$, contradiction. □

Remark 4.2.4. The approach, based on Ky Fan's Lemma, in the proof of Theorem 4.2.1, is well known in the literature, see, for instance, [17, 18, 137, 149]. Let us notice, that the compactness assumption of the set K in the hypothesis of the previous theorem is rather a strong condition. Fortunately, the compactness condition can be removed by assuming only the closedness of K, but also a so-called coercivity condition. This can be done in the virtue of Fan's Lemma, which do not require the compactness of the set $G(y)$ for every $y \in K$, but in only one point. Therefore, the following result holds.

Theorem 4.2.2. *Let X and Z be Hausdorff, locally convex topological vector spaces, let $C \subseteq Z$ be a convex and pointed cone with nonempty interior, and let K be a nonempty, convex and closed subset of X. Let $D \subseteq K$ be a self-segment-dense set, and consider the mapping $f : K \times K \longrightarrow Z$ satisfying*

(i) $\forall y \in D$, *the mapping* $x \longrightarrow f(x,y)$ *is C-upper semicontinuous on* K,

(ii) $\forall x \in K$, *the mapping* $y \longrightarrow f(x,y)$ *is C-upper semicontinuous on* $K \setminus D$,

(iii) $\forall x \in D$, *the mapping* $y \longrightarrow f(x,y)$ *is a C-function on* D,

(iv) $\forall x \in D$, $f(x,x) \notin -\operatorname{int}C$,

(v) $\exists K_0 \subseteq X$ *compact and* $y_0 \in D$, *such that* $f(x,y_0) \in -\operatorname{int}C$, *for all* $x \in K \setminus K_0$.

Then, there exists an element $x_0 \in K$ such that

$$f(x_0,y) \notin -\operatorname{int}C, \forall y \in K.$$

Proof. Consider the set-valued map $G : D \rightrightarrows K$, $G(y) = \{x \in K : f(x,y) \notin -\operatorname{int}C\}$. According to the proof of Theorem 4.2.1 G is a KKM map, and $G(y)$ is closed for all $y \in D$. We show that $G(y_0)$ is compact. For this is enough to show that $G(y_0) \subseteq K_0$. Assume the contrary, that is $G(y_0) \not\subseteq K_0$. Then, there exits $z \in G(y_0) \setminus K_0$. This implies that $z \in K \setminus K_0$, and according

to (v) $f(z,y_0) \in -\operatorname{int} C$, which contradicts the fact that $z \in G(y_0)$. Hence, $G(y_0)$ is a closed subset of the compact set K_0 which shows that $G(y_0)$ is compact. The rest of the proof is similar to the proof of Theorem 4.2.1, therefore we omit it. □

Remark 4.2.5. The condition (v) in the hypothesis of Theorem 4.2.2 is hard to be verified. However, it is well known that in a reflexive Banach space X, the closed ball, with radius $r > 0$, $\overline{B}_r := \{x \in X : \|x\| \leq r\}$, is weakly compact. Therefore, if we endow the reflexive Banach space X with the weak topology, we can take $K_0 = \overline{B}_r \cap K$, hence, condition (v) in Theorem 4.2.2 becomes : there exists $r > 0$ and $y_0 \in D$, such that for all $x \in K$ satisfying $\|x\| > r$, one has $f(x,y_0) \in -\operatorname{int} C$.

Furthermore, in this setting, condition (v) in the hypothesis of Theorem 4.2.2 can be weakened by assuming that there exists $r > 0$, such that for all $x \in K$ satisfying $\|x\| > r$, there exists some $y_0 \in K$ with $\|y_0\| < \|x\|$, and for which $f(x,y_0) \in -\operatorname{int} C$ holds. More precisely, we have the following result.

Theorem 4.2.3. *Let X be a reflexive Banach space and let Z be a Hausdorff, locally convex topological vector space. Let $C \subseteq Z$ be a convex and pointed cone with non-empty interior, and let K be a nonempty, convex and closed subset of X. Let $D \subseteq K$ be a self-segment-dense set with respect to the weak topology of X, and consider the mapping $f : K \times K \longrightarrow Z$ satisfying the following conditions.*

(i) $\forall y \in D$, *the mapping $x \longrightarrow f(x,y)$ is C-upper semicontinuous on K, in the weak topology of X,*

(ii) $\forall x \in K$, *the mapping $y \longrightarrow f(x,y)$ is C-upper semicontinuous on $K \setminus D$, in the weak topology of X,*

(iii) $\forall x \in K$, *the mapping $y \longrightarrow f(x,y)$ is a C-function on K,*

(iv) $\forall x \in K$, $f(x,x) = 0$,

(v) $\exists r > 0$, *such that, for all $x \in K$, $\|x\| > r$, there exists $y_0 \in K$ with $\|y_0\| < \|x\|$, such that $f(x,y_0) \in -\operatorname{int} C \cup \{0\}$.*

Then, there exists an element $x_0 \in K$ such that

$$f(x_0,y) \notin -\operatorname{int} C, \forall y \in K.$$

Proof. Let $r > 0$, such that (v) holds, and let $r_1 > r$. Consider $K_0 = K \cap \overline{B}_{r_1}$. Obviously, K_0 is weakly compact, hence, according to Theorem 4.2.1, there exists $x_0 \in K_0$ such that $f(x_0, y) \notin -\operatorname{int} C, \forall y \in K_0$. We show that $f(x_0, y) \notin -\operatorname{int} C, \forall y \in K$.

First, we show that there exists $z_0 \in K_0$, $\|z_0\| < r_1$ such that $f(x_0, z_0) = 0$. Indeed, if $\|x_0\| < r_1$ then let $z_0 = x_0$ and the conclusion follows from (iv).

Assume now, that $\|x_0\| = r_1 > r$. Then, according to (v), we have that there exists $z_0 \in K$, $\|z_0\| < \|x_0\| = r_1$ such that $f(x_0, z_0) \in -\operatorname{int} C \cup \{0\}$. On the other hand one has $z_0 \in K_0$, hence $f(x_0, z_0) \notin -\operatorname{int} C$, which leads to $f(x_0, z_0) = 0$.

Let $y \in K$. Then, there exists $\lambda \in [0,1]$ such that $\lambda z_0 + (1-\lambda)y \in K_0$, consequently $f(x_0, \lambda z_0 + (1-\lambda)y) \notin -\operatorname{int} C$.

From (iii), we have $\lambda f(x_0, z_0) + (1-\lambda) f(x_0, y) - f(x_0, \lambda z_0 + (1-\lambda)y) \in C$ or, equivalently, $(1-\lambda) f(x_0, y) - f(x_0, \lambda z_0 + (1-\lambda)y) \in C$. Assume now, that we have $f(x_0, y) \in -\operatorname{int} C$. Then, $-f(x_0, \lambda z_0 + (1-\lambda)y) \in -(1-\lambda) f(x_0, y) + C \subseteq \operatorname{int} C$. In other words, $f(x_0, \lambda z_0 + (1-\lambda)y) \in -\operatorname{int} C$, contradiction. □

Remark 4.2.6. If one compares the hypotheses of Theorem 4.2.3 and Theorem 4.2.1, or Theorem 4.2.2, observes that the conditions (iii) and (iv) have considerably been changed. This is due the fact that condition (v) in Theorem 4.2.3 with the assumptions (iii) and (iv) of Theorem 4.2.1 or Theorem 4.2.2 does not assure the existence of a solution for the weak vector equilibrium problem, when K is closed but not compact. Our purpose is to overcome this situation by replacing (v) with a condition that assures the existence of a solution under the original assumptions (iii) and (iv).

In fact, we show that, if for all $x \in K$ the mapping $y \longrightarrow f(x, y)$ is a C-function on D, and for all $x \in D$ one has $f(x, x) \notin -\operatorname{int} C$, then we can replace (v) by:

$\exists r > 0$, such that, for all $x \in K$ satisfying $\|x\| \leq r$, there exists $y_0 \in D$ with $\|y_0\| < r$, such that $f(x, y_0) \in -\operatorname{int} C \cup \{0\}$.

The following result holds.

Theorem 4.2.4. *Let X be a reflexive Banach space and let Z be a Hausdorff, locally convex topological vector space. Let $C \subseteq Z$ be a convex and pointed cone with non-empty interior, and let K be a nonempty, convex and closed subset of X. Let $D \subseteq K$ be a self-segment-dense set with respect to the weak topology of X, and consider the mapping $f : K \times K \longrightarrow Z$ satisfying the following conditions.*

(i) $\forall y \in D$, the mapping $x \longrightarrow f(x, y)$ is C-upper semicontinuous on K, in the weak topology of X,

(ii) $\forall x \in K$, *the mapping* $y \longrightarrow f(x,y)$ *is C-upper semicontinuous on* $K \setminus D$, *in the weak topology of* X,

(iii) $\forall x \in K$, *the mapping* $y \longrightarrow f(x,y)$ *is a C-function on* D,

(iv) $\forall x \in D$, $f(x,x) \notin -\operatorname{int} C$,

(v) $\exists r > 0$, *such that, for all* $x \in K$, $\|x\| \leq r$, *there exists* $y_0 \in D$ *with* $\|y_0\| < r$, *such that* $f(x,y_0) \in -\operatorname{int} C \cup \{0\}$.

Then, there exists an element $x_0 \in K$ *such that*

$$f(x_0,y) \notin -\operatorname{int} C, \forall y \in K.$$

Proof. Let $r > 0$ such that (v) holds, and consider $K_0 = K \cap \overline{B}_r$. Obviously, K_0 is weakly compact, hence, according to Theorem 4.2.1 there exists $x_0 \in K_0$ such that $f(x_0,y) \notin -\operatorname{int} C$, $\forall y \in K_0$. We show, that $f(x_0,y) \notin -\operatorname{int} C$, $\forall y \in K$. According to (v) there exists $z_0 \in D$ with $\|z_0\| < r$, such that $f(x,z_0) \in -\operatorname{int} C \cup \{0\}$. On the other hand, $z_0 \in K_0$, hence $f(x_0,z_0) \notin -\operatorname{int} C$. Consequently $f(x_0,z_0) = 0$. Let $y \in D \setminus K_0$. Then, in virtue of self-segment-denseness of D in K, there exists $\lambda \in [0,1]$ such that $\lambda z_0 + (1-\lambda)y \in D \cap K_0$, consequently $f(x_0, \lambda z_0 + (1-\lambda)y) \notin -\operatorname{int} C$.

From (iii), we have $\lambda f(x_0,z_0) + (1-\lambda) f(x_0,y) - f(x_0, \lambda z_0 + (1-\lambda)y) \in C$ or, equivalently $(1-\lambda) f(x_0,y) - f(x_0, \lambda z_0 + (1-\lambda)y) \in C$. Assume now, that we have $f(x_0,y) \in -\operatorname{int} C$. Then, $-f(x_0, \lambda z_0 + (1-\lambda)y) \in -(1-\lambda) f(x_0,y) + C \subseteq \operatorname{int} C$, or, in other words $f(x_0, \lambda z_0 + (1-\lambda)y) \in -\operatorname{int} C$, contradiction.

Hence, $f(x_0,y) \notin -\operatorname{int} C$, for all $y \in D$. Finally, if $y \in K \setminus D$ by the denseness of D in K for every neighborhood U of y there exists a $u_0 \in U \cap D$.

At this point, the assumption (ii), the C-upper semicontinuity of $y \longrightarrow f(x_0,y)$ on $K \setminus D$, assures that for all $k \in \operatorname{int} C$, there exists a neighborhood U of y, such that $f(x_0,u) \in f(x_0,y) + k - \operatorname{int} C$. Assume that $f(x_0,y) \in -\operatorname{int} C$. Then, consider $k = -f(x_0,y) \in \operatorname{int} C$. Thus, we have that there exists a neighborhood U of y, such that $f(x_0,u) \in -\operatorname{int} C$. But, by choosing $u_0 \in U \cap D$ we get that $f(x_0,u_0) \in -\operatorname{int} C$, contradiction. □

In what follows, we show that the assumption, that D is self-segment-dense, in the hypotheses of the previous theorems is essential, and it cannot be replaced by the denseness of D. Indeed, let us consider the Hilbert space of square-summable sequences l_2, and let $K = \{x \in l_2 : \|x\| \leq 1\}$ be its closed unit ball. Further, consider the set $D = \{x \in l_2 : \|x\| = 1\}$

the unit sphere of l_2. It is well known that l_2, endowed with the weak topology, is a Hausdorff, locally convex topological vector space, and by reflexivity of l_2, K is compact in this topology. Furthermore, we have seen in Example 4.1.2 that D is dense, but not self-segment-dense in K. In this setting we define the vector-valued map $f : K \times K \to \mathbb{R}^2$, $f(x,y) := (\langle x,y\rangle - 1, \langle x,y\rangle - 1)$, which has the following properties:

(a) for all $y \in K$, $x \longrightarrow f(x,y)$ is continuous on K,

(b) for all $x \in K$, $y \longrightarrow f(x,y)$ is continuous on K,

(c) for all $x \in K$, $y \longrightarrow f(x,y)$ is affine, hence convex and also concave on K,

(d) $f(x,x) = (0,0)$ for all $x \in D$.

Further, consider $\mathbb{R}^2_+ = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}$, the nonnegative orthant of $\mathbb{R}^2$, which is obviously a convex and pointed cone. An easy verification shows, that f satisfies all the assumptions of Theorem 4.2.1, except the assumption that D is self-segment-dense (here D is only dense) and consequently the conclusion of the above mentioned theorem fails, since for $y = 0 \in K$ and for all $x \in K$, one has

$$f(x,y) = (-1,-1) \in -\operatorname{int}\mathbb{R}^2_+.$$

4.2.3 Self-Segment-Dense Sets and the Strong Vector Equilibrium Problem

Solution existence for the strong vector equilibrium problem (4. 6), can be provided under some similar conditions, as have been obtained in the previous section, for the weak vector equilibrium problem. However, note that the strong C-upper semicontinuity of a map, differs significantly from the C-upper semicontinuity property, as will be shown in Remark 4.2.10. Therefore, despite of similar statements to those presented in the previous section, the results of this section can be considered original and new. For the sake of completeness we give some full proofs.

Theorem 4.2.5. *Let X and Z be Hausdorff, locally convex topological vector spaces, let $C \subseteq Z$ be a convex and pointed cone and let K be a nonempty, convex and compact subset of X. Let $D \subseteq K$ be a self-segment-dense set in K and consider the mapping $f : K \times K \longrightarrow Z$ satisfying*

(i) $\forall y \in D$, the mapping $x \longrightarrow f(x,y)$ is strongly C-upper semicontinuous on K,

(ii) $\forall x \in K$, *the mapping* $y \longrightarrow f(x,y)$ *is strongly C-upper semicontinuous on* $K \setminus D$,

(iii) $\forall x \in D$, *the mapping* $y \longrightarrow f(x,y)$ *is a C-function on* D,

(iv) $\forall x \in D$, $f(x,x) \notin -C \setminus \{0\}$.

Then, there exists an element $x_0 \in K$ *such that*

$$f(x_0,y) \notin -C \setminus \{0\}, \forall y \in K.$$

Proof. Assume the contrary, that is, for every $x \in K$ there exists $y \in K$ such that $f(x,y) \in -C \setminus \{0\}$. Then, for every $y \in K$ consider $V_y = \{x \in K : f(x,y) \in -C \setminus \{0\}\}$. Obviously $K \subseteq \cup_{y \in K} V_y$. We show that $(V_y)_{y \in D}$ is an open cover of K. First of all, observe that for all $y \in D$, one has $V_y = K \setminus G(y)$, where $G(y)$ stands for the set $\{x \in K : f(x,y) \notin -C \setminus \{0\}\}$. We show that $G(y)$ is closed for all $y \in D$. Indeed, for fixed $y_0 \in D$ consider the net $(x_\alpha) \subseteq G(y_0)$ and let $\lim x_\alpha = x_0$. Assume that $x_0 \notin G(y_0)$. Then $f(x_0,y_0) \in -C \setminus \{0\}$. According to (i) the function $x \longrightarrow f(x,y_0)$ is strongly C-upper semicontinuous at x_0, hence for any $k \in C \setminus \{0\}$, there exists a neighborhood U of x_0 such that $f(u,y_0) \in f(x_0,y_0) + k - C \setminus \{0\}$ for all $u \in U$. But then, for $k = -f(x_0,y_0) \in C \setminus \{0\}$, one obtains that there exits α_0, such that $f(x_\alpha,y_0) \in -C \setminus \{0\}$, for $\alpha \geq \alpha_0$, which contradicts the fact that $(x_\alpha) \subseteq G(y_0)$. Consequently, V_y is open for every $y \in D$.

Assume now, that there exists $x_0 \in K$, such that $x_0 \notin \cup_{y \in D} V_y$. Then, one has that $f(x_0,y) \notin -C \setminus \{0\}$, for all $y \in D$. We show that $f(x_0,y) \notin -C \setminus \{0\}$, for all $y \in K$. Indeed, for $y_0 \in K \setminus D$, by the denseness of D in K, we have that for every neighborhood U of y_0, there exists a $u_0 \in U \cap D$. At this point, the assumption (ii), the strongly C-upper semicontinuity of $f(x_0,y)$ on $K \setminus D$, assures that for all $k \in C \setminus \{0\}$ there exists a neighborhood U of y_0 such that $f(x_0,u) \in f(x_0,y_0) + k - C \setminus \{0\}$. Assume that $f(x_0,y_0) \in -C \setminus \{0\}$. Then, let $k = -f(x_0,y_0) \in C \setminus \{0\}$. Thus, we have that there exists a neighborhood U of y_0, such that $f(x_0,U) \subseteq -C \setminus \{0\}$. But, by choosing $u_0 \in U \cap D$ we get that $f(x_0,u_0) \in -C \setminus \{0\}$, contradiction. Hence, $f(x_0,y) \notin -C \setminus \{0\}$, for all $y \in K$, which contradicts our assumption, that for every $x \in K$ there exists $y \in K$ such that $f(x,y) \in -C \setminus \{0\}$. Consequently, $(V_y)_{y \in D}$ is an open cover of the compact set K, in conclusion it contains a finite subcover. In other words, there exists $y_1, y_2, ..., y_n \in D$, such that $K \subseteq \cup_{i=1}^n V_{y_i}$. Consider $\left(p_i\right)_{1 \leq i \leq n}$ a continuous partition of unity associated to the open cover $\left(V_{y_i}\right)_{1 \leq i \leq n}$. Then, for all $i \in \{1,2,...,n\}$, the functions $p_i : K \longrightarrow [0,1]$ are continuous and it holds that their support $\text{supp}(p_i) = \text{cl}\{x \in K : p_i(x) \neq 0\} \subseteq V_{y_i}$. Moreover,

$\sum_{i=1}^{n} p_i(x) = 1$, for all $x \in K$. Consider the mapping $\varphi : \mathrm{co}\{y_1, y_2, ..., y_n\} \longrightarrow \mathrm{co}\{y_1, y_2, ..., y_n\}$

$$\varphi(x) = \sum_{i=1}^{n} p_i(x) y_i.$$

Obviously φ is continuous, and $\mathrm{co}\{y_1, y_2, ..., y_n\}$ is a compact and convex subset of the finite dimensional space $\mathrm{span}\{y_1, y_2, ..., y_n\}$. Hence, by the Brouwer fixed point theorem, there exists $x_0 \in \mathrm{co}\{y_1, y_2, ..., y_n\}$ such that $\varphi(x_0) = x_0$.

Let $J = \{i \in \{1, 2, ..., n\} : p_i(x_0) > 0\}$. Obviously the set J is nonempty, since $\sum_{i \in J} p_i(x_0) = 1$, and $\varphi(x_0) = \sum_{i=1}^{n} p_i(x_0) y_i = \sum_{i \in J} p_i(x_0) y_i = x_0$. The latter equality shows, that $x_0 \in \mathrm{co}\{y_i : i \in J\}$. On the other hand, from $p_i(x_0) > 0$ for all $i \in J$, we obtain that $x_0 \in \cap_{i \in J} V_{y_i}$. Since $\cap_{i \in J} V_{y_i}$ is open, we obtain that there exists U, an open and convex neighbourhood of x_0, such that $U \subseteq \cap_{i \in J} V_{y_i}$.

Since the set $\mathrm{co}\{y_i : i \in J\} \cap U$ is nonempty, from Lemma 4.1.2, we have that there exists $y_0 \in \mathrm{co}\{y_i : i \in J\} \cap U \cap D$.

Hence, we have $y_0 = \sum_{i \in J} \lambda_i y_i \in \mathrm{co}\{y_i : i \in J\} \cap U \cap D$ for some $\lambda_i \geq 0$, $i \in J$, satisfying $\sum_{i \in J} \lambda_i = 1$. From (iv) one gets $f(y_0, y_0) \notin -C \setminus \{0\}$. On the other hand, by using (iii), we get $f(y_0, y_0) = f(y_0, \sum_{i \in J} \lambda_i y_i) \leq \sum_{i \in J} \lambda_i f(y_0, y_i)$, which shows that $\sum_{i \in J} \lambda_i f(y_0, y_i) - f(y_0, y_0) \in C$. But, $y_0 \in U$, which implies that for all $i \in J$, one has $f(y_0, y_i) \in -C \setminus \{0\}$.

Hence, $\sum_{i \in J} \lambda_i f(y_0, y_i) \in -C \setminus \{0\}$, which leads to

$$-f(y_0, y_0) \in C - \sum_{i \in J} \lambda_i f(y_0, y_i) \subseteq C \setminus \{0\},$$

contradiction. □

Remark 4.2.7. Obviously, one can give a proof of the previous theorem based on Ky Fan's Lemma, analogously to the proof of Theorem 4.2.1. Therefore, the rigid assumption of compactness of the set K in the hypothesis of the previous theorem can be replaced by its closedness and a coercivity condition. This can be done in the virtue of Fan's Lemma, which do not require the compactness of $G(y)$ for every $y \in K$, but in only one point. In conclusion, the following result holds.

Theorem 4.2.6. *Let X and Z be Hausdorff, locally convex topological vector spaces, let $C \subseteq Z$ be a convex and pointed cone and let K be a nonempty, convex and closed subset of X. Let $D \subseteq K$ be a self-segment-dense set in K and consider the mapping $f : K \times K \longrightarrow Z$ satisfying*

(i) $\forall y \in D$, the mapping $x \longrightarrow f(x, y)$ is strongly C-upper semicontinuous on K,

(ii) $\forall x \in K$, *the mapping* $y \longrightarrow f(x,y)$ *is strongly C-upper semicontinuous on* $K \setminus D$,

(iii) $\forall x \in D$, *the mapping* $y \longrightarrow f(x,y)$ *is a C-function on* D,

(iv) $\forall x \in D$, $f(x,x) \notin -C \setminus \{0\}$,

(v) $\exists K_0 \subseteq X$ *compact and* $y_0 \in D$, *such that* $f(x,y_0) \in -C \setminus \{0\}$, *for all* $x \in K \setminus K_0$.

Then, there exists an element $x_0 \in K$ *such that*

$$f(x_0,y) \notin -C \setminus \{0\}, \forall y \in K.$$

Proof. The proof is similar to the proof of Theorem 4.2.2 therefore we omit it. □

Remark 4.2.8. If X is a reflexive Banach space, endowed with the weak topology, and $\overline{B}_r := \{x \in X : \|x\| \leq r\} \subseteq X$, is a closed ball with radius $r > 0$, then one can take $K_0 = K \cap \overline{B}_r$. Therefore, condition (v) in Theorem 4.2.6 becomes: there exists $r > 0$ and $y_0 \in D$, such that, for all $x \in K$ satisfying $\|x\| > r$, one has $f(x,y_0) \in -C \setminus \{0\}$.

Furthermore, in this setting, condition (v) in the hypothesis of Theorem 4.2.6 can be weakened by assuming that there exists $r > 0$ such that, for all $x \in K$ satisfying $\|x\| > r$, there exists some $y_0 \in K$ with $\|y_0\| < \|x\|$, and for which $f(x,y_0) \in -C \setminus \{0\}$ holds.

More precisely, we have the following result.

Theorem 4.2.7. *Let* X *be a reflexive Banach space and let* Z *be a Hausdorff, locally convex topological vector space, let* $C \subseteq Z$ *be a convex and pointed cone and let* K *be a nonempty, convex and closed subset of* X. *Let* $D \subseteq K$ *be a self-segment-dense set in the weak topology of* X, *and consider the mapping* $f : K \times K \longrightarrow Z$ *satisfying*

(i) $\forall y \in D$, *the mapping* $x \longrightarrow f(x,y)$ *is strongly C-upper semicontinuous on* K, *in the weak topology of* X,

(ii) $\forall x \in K$, *the mapping* $y \longrightarrow f(x,y)$ *is strongly C-upper semicontinuous on* $K \setminus D$, *in the weak topology of* X,

(iii) $\forall x \in K$, *the mapping* $y \longrightarrow f(x,y)$ *is a C-function on* K,

(iv) $\forall x \in K$, $f(x,x) = 0$,

(v) $\exists r > 0$, *such that, for all* $x \in K$, $\|x\| > r$, *there exists* $y_0 \in K$ *with* $\|y_0\| < \|x\|$, *such that* $f(x,y_0) \in -C$.

Then, there exists an element $x_0 \in K$ such that

$$f(x_0, y) \notin -C \setminus \{0\}, \forall y \in K.$$

Proof. Let $r > 0$ such that (v) holds, and let $r_1 > r$. Consider $K_0 = K \cap \overline{B}_{r_1}$. Obviously, K_0 is weakly compact, hence, according to Theorem 4.2.5 there exists $x_0 \in K_0$ such that $f(x_0, y) \notin -C \setminus \{0\}, \forall y \in K_0$. We claim that $f(x_0, y) \notin -C \setminus \{0\}, \forall y \in K$.

We show, as in the proof of Theorem 4.2.3, that there exists $z_0 \in K_0$, $\|z_0\| < r_1$ such that $f(x_0, z_0) = 0$. As in the proof of Theorem 4.2.3, we show that for $y \in K$, there exists $\lambda \in [0,1]$, such that $\lambda z_0 + (1-\lambda)y \in K_0$, $f(x_0, \lambda z_0 + (1-\lambda)y) \notin -C \setminus \{0\}$, and the relation $(1-\lambda)f(x_0, y) - f(x_0, \lambda z_0 + (1-\lambda)y) \in C$ holds. Assume that we have $f(x_0, y) \in -C \setminus \{0\}$. Then, $-f(x_0, \lambda z_0 + (1-\lambda)y) \in -(1-\lambda)f(x_0, y) + C \subseteq C \setminus \{0\}$, or, in other words $f(x_0, \lambda z_0 + (1-\lambda)y) \in -C \setminus \{0\}$, contradiction. □

Remark 4.2.9. One can observe, that the conditions (iii) and (iv) in the hypothesis of Theorem 4.2.7, considerably differ from the assumptions used in the hypothesis of Theorem 4.2.5. This is due the fact, that condition (v) in Theorem 4.2.7 with the assumptions (iii) and (iv) of Theorem 4.2.5, does not assure the existence of a solution for the strong vector equilibrium problem, when K is closed but not compact.

Next, we obtain the existence of a solution for strong vector equilibrium problem under the original assumptions (iii) and (iv), by replacing (v) with a more suitable condition. In fact, we show that, if for all $x \in K$, the mapping $y \longrightarrow f(x, y)$ is a C-function on D, and for all $x \in D$, $f(x, x) \notin -C \setminus \{0\}$, then we can replace (v) by:

$\exists r > 0$, such that, for all $x \in K$ satisfying $\|x\| \leq r$, there exists $y_0 \in D$ with $\|y_0\| < r$, such that $f(x, y_0) \in -C$.

The following result holds.

Theorem 4.2.8. *Let X be a reflexive Banach space and let Z be a Hausdorff, locally convex topological vector space, let $C \subseteq Z$ be a convex and pointed cone and let K be a nonempty, convex and closed subset of X. Let $D \subseteq K$ be a self-segment-dense set in the weak topology of X, and consider the mapping $f : K \times K \longrightarrow Z$ satisfying*

(i) $\forall y \in D$, the mapping $x \longrightarrow f(x, y)$ is strongly C-upper semicontinuous on K, in the weak topology of X,

(ii) $\forall x \in K$, the mapping $y \longrightarrow f(x, y)$ is strongly C-upper semicontinuous on $K \setminus D$, in the weak topology of X,

(iii) $\forall x \in K$, *the mapping* $y \longrightarrow f(x,y)$ *is a C-function on* D,

(iv) $\forall x \in D$, $f(x,x) \notin -C \setminus \{0\}$,

(v) $\exists r > 0$, *such that, for all* $x \in K$, $\|x\| \leq r$, *there exists* $y_0 \in D$ *with* $\|y_0\| < r$, *such that* $f(x,y_0) \in -C$.

Then, there exists an element $x_0 \in K$ *such that*

$$f(x_0,y) \notin -C \setminus \{0\}, \forall y \in K.$$

Proof. For the sake of completeness we give a full proof. Let $r > 0$ such that (v) holds, and consider $K_0 = K \cap \overline{B}_r$. Obviously, K_0 is weakly compact, hence, according to Theorem 4.2.5 there exists $x_0 \in K_0$ such that $f(x_0,y) \notin -C \setminus \{0\}, \forall y \in K_0$. We show, that $f(x_0,y) \notin -C \setminus \{0\} \forall y \in K$. According to (v), there exists $z_0 \in D$ with $\|z_0\| < r$, such that $f(x_0,z_0) \in -C$. On the other hand, $z_0 \in K_0$, hence $f(x_0,z_0) \notin -C \setminus \{0\}$. Consequently, $f(x_0,z_0) = 0$. Let $y \in D \setminus K_0$. Then, in virtue of self-segment-denseness of D in K, there exists $\lambda \in [0,1]$, such that $\lambda z_0 + (1-\lambda)y \in D \cap K_0$, consequently $f(x_0, \lambda z_0 + (1-\lambda)y) \notin -C \setminus \{0\}$.

From (iii), we have $\lambda f(x_0,z_0) + (1-\lambda) f(x_0,y) - f(x_0, \lambda z_0 + (1-\lambda)y) \in C$ or, equivalently $(1-\lambda) f(x_0,y) - f(x_0, \lambda z_0 + (1-\lambda)y) \in C$. Assume now, that we have $f(x_0,y) \in -C \setminus \{0\}$. Then, $-f(x_0, \lambda z_0 + (1-\lambda)y) \in -(1-\lambda) f(x_0,y) + C \subseteq C \setminus \{0\}$, or, in other words $f(x_0, \lambda z_0 + (1-\lambda)y) \in -C \setminus \{0\}$, contradiction.

Hence, $f(x_0,y) \notin -C \setminus \{0\}$, for all $y \in D$. Finally, if $y \in K \setminus D$, since D is dense in K, we have that for every neighborhood U of y, there exists a $u_0 \in U \cap D$. At this point, the assumption (ii), the proper C-upper semicontinuity of the mapping $y \longrightarrow f(x_0,y)$ on $K \setminus D$, assures that for all $k \in C \setminus \{0\}$, there exists a neighborhood U of y, such that $f(x_0,u) \in f(x_0,y) + k - C \setminus \{0\}$. Assume now that $f(x_0,y) \in -C \setminus \{0\}$ and consider $k = -f(x_0,y) \in C \setminus \{0\}$. Then, we have that there exists a neighborhood U of y, such that $f(x_0,u) \in -C \setminus \{0\}$. But, by choosing $u_0 \in U \cap D$ we obtain that $f(x_0,u_0) \in -C \setminus \{0\}$, contradiction. $\square$

In what follows, we show that the assumption that D is self-segment-dense in K, in the hypothesis of Theorem 4.2.5, is essential and it cannot be replaced by the usual denseness of D. Indeed, let us consider the Hilbert space of square-summable sequences l_2, and let K be its unit ball, while D be its unit sphere. It is well known that l_2, endowed with the weak topology, is a Hausdorff locally convex topological vector space, and by the reflexivity of l_2, K is compact in this topology. Furthermore, we have seen in Example 4.1.2 that D is dense, but not self-segment-dense in K. In this setting we define the vector-valued map

$f : K \times K \longrightarrow \mathbb{R}^2$, $f(x,y) := (\langle x,y\rangle - 1, 0)$. Further, consider the set $C = \mathbb{R}_+ \times \{0\} = \{(x,0) : x \in \mathbb{R}, x \geq 0\}$, which is obviously a convex and pointed cone. It can easily be verified that all the assumptions of Theorem 4.2.5 are satified, except the assumption that D is self-segment-dense (here D is only dense) and also that the conclusions of the above mentioned theorem fails, since for $y = 0 \in K$ and for all $x \in K$, one has

$$f(x,y) = (-1,0) \in -C \setminus \{(0,0)\}.$$

4.2.4 Applications to Vector Optimization

Let X and Z be Hausdorff, locally convex topological vector spaces, let $C \subseteq Z$ be a convex and pointed cone and let K be a nonempty subset of X. Let $F : K \longrightarrow Z$ be a vector function. Recall that a point $x_0 \in K$ is called a weakly efficient point of F, iff $F(y) - F(x_0) \notin -\operatorname{int} C$ for all $y \in K$. Similarly, $x_0 \in K$ is called an efficient point of F, iff $F(y) - F(x_0) \notin -C \setminus \{0\}$ for all $y \in K$; see [100]. Based on the results obtained in the previous sections, we can give some conditions that ensure the existence of a weakly efficient point, (efficient point), of a vector function F. Indeed, consider the mapping $f : K \times K \longrightarrow Z$, $f(x,y) = F(y) - F(x)$. It is obvious that $x_0 \in K$ is a weak efficient point, (efficient point), of F, if and only if, x_0 is a solution of the weak vector equilibrium problem $f(x_0,y) \notin -\operatorname{int} C$, $\forall y \in K$, (x_0 is a solution of the strong vector equilibrium problem $f(x_0,y) \notin -C \setminus \{0\}$, $\forall y \in K$).

Assume now, that K is a nonempty, convex and compact subset of X. Let $D \subseteq K$ be a self-segment-dense set. It can easily be verified that in this case, the assumptions (i) in the hypothesis of Theorem 4.2.1, becomes F is C-lower semicontinuous on K. Then, since K is compact, one can conclude, that there exists a weak efficient point of F; see [150]. Note that the condition (iv) in the hypothesis of Theorem 4.2.1 is automatically satisfied. Moreover, it is obvious that the condition (iii) in the hypothesis of Theorem 4.2.1, which becomes that F is a C-function on D, is superfluous. Nevertheless, this is not the case when K is not a compact set, but a closed subset of a reflexive Banach space X. In conclusion, according to the Theorem 4.2.4, in reflexive Banach spaces the following result holds.

Theorem 4.2.9. *Let X be a reflexive Banach space and let Z be a Hausdorff, locally convex topological vector space, let $C \subseteq Z$ be a convex and pointed cone with nonempty interior and let K be a nonempty, convex and closed subset of X. Let $D \subseteq K$ be a self-segment-dense set in the weak topology of X, and consider the mapping $F : K \longrightarrow Z$ satisfying*

(i) F is C-lower semicontinuous on K,

(ii) *F is C-upper semicontinuous on $K \setminus D$,*

(iii) *F is a C-function on D,*

(iv) *$\exists r > 0$, such that, for all $x \in K$, $\|x\| \leq r$, there exists $y_0 \in D$ with $\|y_0\| < r$, such that $F(y_0) - F(x) \in -\operatorname{int} C \cup \{0\}$.*

Then, there exists a weak efficient point of F.

Proof. Consider the mapping $f : K \times K \longrightarrow Z$, $f(x,y) = F(y) - F(x)$. Then, one can apply Theorem 4.2.4, and the conclusion follows. □

We show next, that the assumption (iii) in the hypothesis of the previous theorem, is essential.

Example 4.2.4. Let $F : [0,\infty[\longrightarrow \mathbb{R}$, $F(x) = \begin{cases} 0, \text{ if } x \leq 1 \\ -x+1, \text{ if } x > 1. \end{cases}$ Consider $C = [0,\infty[$, a convex and pointed cone, with nonempty interior. Let D be the set of non-negative rational numbers. Then obviously, the conditions (i) and (ii) in the hypothesis of Theorem 4.2.9, are satisfied automatically and (iv) is satisfied with $r = 1$. It is also obvious that F is not a C-function on D. Hence, (iii) fails, and also the conclusion of the previous theorem, since F has no minima.

Similarly, Theorem 4.2.8 applied to the vector valued bifunction $f : K \times K \longrightarrow Z$, $f(x,y) = F(y) - F(x)$, assures the existence of an efficient point, of the vector function $F : K \longrightarrow Z$, in reflexive Banach spaces. More precisely, the following result holds.

Theorem 4.2.10. *Let X be a reflexive Banach space and let Z be a Hausdorff, locally convex topological vector space, let $C \subseteq Z$ be a convex and pointed cone and let K be a nonempty, convex and closed subset of X. Let $D \subseteq K$ be a self-segment-dense set in the weak topology of X, and consider the mapping $F : K \longrightarrow Z$ satisfying*

(i) *F is strongly C-lower semicontinuous on K,*

(ii) *F is strongly C-upper semicontinuous on $K \setminus D$,*

(iii) *F is a C-function on D,*

(iv) *$\exists r > 0$, such that, for all $x \in K$, $\|x\| \leq r$, there exists $y_0 \in D$ with $\|y_0\| < r$, such that $F(y_0) - F(x) \in -C$.*

Then, there exists an efficient point of F.

4.2.5 Applications to Minty Type Vector Variational Inequalities

In this paragraph, we give some new existence results for weak, respectively strong vector variational inequalities of Minty type. Let X and Z be Hausdorff, locally convex topological vector spaces, let $C \subseteq Z$ be a convex and pointed cone and let K be a nonempty subset of X. Let us denote $L(X,Z)$ the set of all linear and continuous operators from X to Z. Let $F : X \longrightarrow L(X,Z)$. For $x^* \in L(X,Z)$ and $x \in X$, we denote by $\langle x^*, x\rangle$ the vector $x^*(x) \in Z$. The weak vector variational inequality of Minty type consists in finding $x_0 \in K$, such that $\langle F(y), y - x_0\rangle \notin -\operatorname{int} C$, for all $y \in K$. Analogously, the strong vector variational inequality of Minty type consists in finding $x_0 \in K$, such that $\langle F(y), y - x_0\rangle \notin -C \setminus \{0\}$, for all $y \in K$.

Note that, if we take $f : K \times K \longrightarrow Z$, $f(x,y) = \langle F(y), y - x\rangle$, then the weak, (strong), vector variational inequality problem of Minty type becomes the appropriate vector equilibrium problem. In conclusion, the following results hold.

Theorem 4.2.11. *Let X and Z be Hausdorff, locally convex topological vector spaces, let $C \subseteq Z$ be a convex and pointed cone with nonempty interior and let K be a nonempty, convex and compact subset of X. Let $D \subseteq K$ be a self-segment-dense set and consider the mapping $F : K \longrightarrow L(X,Z)$ satisfying*

(i) $\forall x \in K$, the mapping $y \longrightarrow \langle F(y), y - x\rangle$ is C-upper semicontinuous on $K \setminus D$,

(ii) $\forall x \in D$, the mapping $y \longrightarrow \langle F(y), y - x\rangle$ is a C-function on D,

Then, there exists an element $x_0 \in K$ such that

$$\langle F(y), y - x_0\rangle \notin -\operatorname{int} C, \forall y \in K.$$

Proof. Consider the mapping $f : K \times K \longrightarrow Z$, $f(x,y) = \langle F(y), y - x\rangle$. Then, one can apply Theorem 4.2.1, and the conclusion follows. □

Remark 4.2.10. Note that the condition (i) of Theorem 4.2.1, that is, $x \longrightarrow \langle F(y), y - x\rangle$ is C-upper semicontinuous on K for every $y \in D$ is automatically satisfied. However this is not the case for strongly C-upper semicontinuity. Indeed, let $X = Z = \mathbb{R}^2$ and

$$F : X \to L(X,Z), F(u,v) = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}.$$

Then, $\langle F(u,v), (u,v) - (x,y)\rangle = (u(u-x), v(v-y))$. Let C be the non-negative orthant of $\mathbb{R}^2$, i.e. $C = \mathbb{R}^2_+$. Let $K = [-1,1] \times [-1,1]$ and $D = \mathbb{Q}^2 \cap K$, where $\mathbb{Q}$ is the set of rational numbers. According to Example 4.1.1, D is self-segment-dense in K. We show that for

$(u,v) = (-1,-1) \in D$ the map $(x,y) \longrightarrow (u(u-x), v(v-y))$ is not strongly C-upper semicontinuous at $(x,y) = (0,1)$. Indeed, assume the contrary, that is, for all $(k,h) \in C \setminus \{(0,0)\}$ there exits U a neighbourhood of $(0,1)$, such that for all $(s,t) \in U$ one has $(s+1,t+1) \in (1,2) + (k,h) - \mathbb{R}^2_+\{(0,0)\}$. Obviously, one can take $U = (-\varepsilon,\varepsilon) \times (1-\varepsilon, 1+\varepsilon)$, for some $\varepsilon > 0$. Let $(k,h) = (0,1)$. Then, one must have $(s,t) \in (0,2) - \mathbb{R}^2_+ \setminus \{(0,0)\}$ for all $(s,t) \in U$, which leads to contradiction if one takes $s > 0$.

In conclusion, as an application of Theorem 4.2.5, for strong vector variational inequalities the following result holds.

Theorem 4.2.12. *Let X and Z be Hausdorff, locally convex topological vector spaces, let $C \subseteq Z$ be a convex and pointed cone and let K be a nonempty, convex and compact subset of X. Let $D \subseteq K$ be a self-segment-dense subset of K, and consider the mapping $F : K \longrightarrow L(X,Z)$ satisfying*

(i) $\forall y \in D$, the mapping $x \longrightarrow \langle F(y), y-x\rangle$ is strongly C-upper semicontinuous on K,

(ii) $\forall x \in K$, the mapping $y \longrightarrow \langle F(y), y-x\rangle$ is strongly C-upper semicontinuous on $K \setminus D$,

(iii) $\forall x \in D$, the mapping $y \longrightarrow \langle F(y), y-x\rangle$ is a C-function on D

Then, there exists an element $x_0 \in K$ such that

$$\langle F(y), y-x_0\rangle \notin -C \setminus \{0\}, \forall y \in K.$$

Remark 4.2.11. One can use Theorem 4.2.4, (Theorem 4.2.8), to obtain some sufficient conditions that ensure the solution existence of weak, (strong), vector variational inequalities of Minty type, in reflexive Banach spaces.

4.3 Minimax results on dense sets and dense families of functionals

Let X and Y be two arbitrary sets and let $f : X \times Y \longrightarrow \mathbb{R}$ be a bifunction. Recall that a minimax theorem deals with sufficient conditions under which the equality

$$\inf_{x\in X} \sup_{y\in Y} f(x,y) = \sup_{y\in Y} \inf_{x\in X} f(x,y)$$

holds. Among the most general minimax results are the ones obtained by Fan [87] and Sion [197], and both assume the compactness of X. As a matter of fact, minimax results on dense

sets, (that is, X is dense in a subset of a topological space), are absent in the literature. The reason for this lack is the fact that, according to Example 4.3.8, the general minimax results of Fan and Sion cannot be extended on usual dense sets. Nevertheless, in this section we obtain some new minimax results on a special type of dense set that we call self-segment-dense [139, 143, 144]. Moreover, under the strong assumption of equicontinuity of the function family $\{f(\cdot,y)\}_{y\in Y}$ we are able to obtain some minimax results on general dense sets. Our approach is based on some results that ensures that the infimum of a convex function on a dense set coincides with its global infimum. In first step we provide conditions that assure the infimum of a convex function on a dense set of its domain coincide with the infimum taken over the convex hull of that dense set. As a second step we give conditions that assure the infimum on this convex hull is equal to the global infimum of the function. In the same manner, we provide some conditions that ensure the coincidence of two convex function that are equal on a dense subset of their domain. Then, we apply these results in order to obtain some minimax results on dense sets. Several examples and counterexamples circumscribe our research and motivates our approach considering special type of dense sets.

Let us mention that the results from this section were partially published in [140]:[S. László, *Minimax results on dense sets and dense families of functionals*, Siam Journal on Optimization, (accepted 2016)].

4.3.1 Preliminaries

In what follows, for the convenience of the reader, we recall Fan's minimax result (see [51, 87]), and Sion's minimax result (see [197]), respectively.

Theorem 4.3.1. *Suppose that X and Y are non-empty sets and let $f : X \times Y \to \mathbb{R}$ be a function convexlike on X and concavelike on Y. Suppose that X is compact and $x \longrightarrow f(x,y)$ is lower semicontinuous on X for each $y \in Y$. Then,*

$$\min_{x\in X} \sup_{y\in Y} f(x,y) = \sup_{y\in Y} \min_{x\in X} f(x,y).$$

Sion's minimax results holds under different assumptions.

Theorem 4.3.2. *Let X be a compact and convex subset of topological vector space and let Y be a convex subset of a topological vector space. Let $f : X \times Y \to \mathbb{R}$ be a function and assume that $x \longrightarrow f(x,y)$ is lower semicontinuous and quasi-convex on X for each $y \in Y$, and $y \longrightarrow f(x,y)$ is upper semicontinuous and quasi-concave on Y for each $x \in X$. Then,*

$$\min_{x\in X} \sup_{y\in Y} f(x,y) = \sup_{y\in Y} \min_{x\in X} f(x,y).$$

In this section we obtain some Fan type minimax result, where the set X is a (special) dense subset of a convex set. However, as it is emphasized in Remark 4.3.14, our results also hold under the appropriate Sion type conditions. We also show that our minimax results fail in general, it is not enough to assume only that X is a dense subset of a compact and convex set.

For a comprehensive survey in the field of minimax theorems we refer to [194, 195].

4.3.2 Convexity notions for real valued functions

Let X be a real Hausdorff, locally convex topological vector space. For a non-empty set $D \subseteq X$, we denote by $\operatorname{int}(D)$ its interior, by $\operatorname{cl}(D)$ its closure, by $\operatorname{co}(D)$ its convex hull, by $\overline{\operatorname{co}}(D) = \operatorname{cl}(\operatorname{co}(D))$ its closed convex hull and by $\operatorname{lin}(D)$ the subspace of X generated by D. We say that $P \subseteq D$ is dense in D iff $D \subseteq \operatorname{cl}(P)$, and that $P \subseteq X$ is relatively compact iff $\operatorname{cl}(P)$ is compact.

We say that the function $f : X \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is convex if

$$\forall x, y \in X,\ \forall t \in [0,1]\ : f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

with the conventions $(+\infty) + (-\infty) = +\infty$, $0 \cdot (+\infty) = +\infty$ and $0 \cdot (-\infty) = 0$. We consider $\operatorname{dom} f = \{x \in X : f(x) < +\infty\}$ the *domain* of f and $\operatorname{epi} f = \{(x,r) \in X \times \mathbb{R} : f(x) \leq r\}$ its *epigraph*. We call f *proper* if $\operatorname{dom} f \neq \emptyset$ and $f(x) > -\infty$ for all $x \in X$. By $\overline{f}$ we denote the *lower semicontinuous hull* of f, namely the function whose epigraph is the closure of $\operatorname{epi} f$ in $X \times \mathbb{R}$, that is $\operatorname{epi}(\overline{f}) = \operatorname{cl}(\operatorname{epi} f)$.

We say that the set $U \subseteq \operatorname{dom} f$ is graphically dense in $\operatorname{dom} f$, (see [39, 184]), if for all $x \in \operatorname{dom} f$ there exists a net $(u_i) \subseteq U$ such that $u_i \longrightarrow x$ and $f(u_i) \longrightarrow f(x)$.

Let $f : X \to \overline{\mathbb{R}}$ be a proper function and consider the set $U \subseteq \operatorname{dom} f$. We define the set $\operatorname{epi} f_U$ as

$$\operatorname{epi} f_U := \{(u,r) \in U \times \mathbb{R} : f(u) \leq r\},$$

and the function $\overline{f_U} : X \longrightarrow \overline{\mathbb{R}}$ by

$$\operatorname{epi} \overline{f_U} = \operatorname{cl}(\operatorname{epi} f_U).$$

We say that f is convex on U, iff for all $u, v \in U$, $t \in [0,1]$ such that $(1-t)u + tv \in U$ one has

$$f((1-t)u + tv) \leq (1-t)f(u) + tf(v).$$

Note that we did not assume the convexity of U in the previous definition.

We say that f is lower semicontinuous at $x \in X$ if for every net $(x^i) \subseteq X$ converging to x one has $\liminf_{x^i \longrightarrow x} f(x^i) \geq f(x)$. f is upper semicontinuous at x iff $-f$ is lower semicontinuous at x. f is lower semicontinuous on X if is lower semicontinuous at every point of X. Note that f is lower semicontinuous on X iff $\operatorname{epi} f$ is closed, that is $f = \overline{f}$.

We say that f is quasiconvex, if its domain is convex and for all $x, y \in \operatorname{dom} f$ and $t \in [0,1]$ one has $f((1-t)x + ty) \leq \max\{f(x), f(y)\}$. f is quasiconcave iff $-f$ is quasiconvex.

For two arbitrary sets X and Y, and a bifunction $f : X \times Y \longrightarrow \mathbb{R}$, Ky Fan [87] introduced the following notions.

f is convexlike on X, iff for all $x_1, x_2 \in X$, $t \in [0,1]$ there exists $x_3 \in X$ such that

$$f(x_3, y) \leq (1-t) f(x_1, y) + t f(x_2, y) \, \forall y \in Y.$$

f is concavelike on Y iff for all $y_1, y_2 \in Y$, $t \in [0,1]$ there exists $y_3 \in Y$ such that

$$f(x, y_3) \geq (1-t) f(x, y_1) + t f(x, y_2) \, \forall x \in X.$$

Note that in these definitions no algebraic structure on X and Y are assumed. Obviously, if the mapping $x \longrightarrow f(x,y)$ is convex for every $y \in Y$ then f is convexlike in its first variable.

4.3.3 On some remarkable properties of self-segment-dense sets

Let X be a real Hausdorff, locally convex topological vector space. We will use the following notations for the open, respectively closed, line segments in X with the endpoints x and y

$$\begin{aligned}]x,y[\quad &:= \quad \big\{z \in X : z = x + t(y-x), \, t \in]0,1[\big\}, \\ [x,y] \quad &:= \quad \big\{z \in X : z = x + t(y-x), \, t \in [0,1]\big\}. \end{aligned}$$

The line segments $]x,y]$, respectively $[x,y[$ are defined similarly. In [149], Definition 3.4, The Luc has introduced the notion of a so-called *segment-dense* set. Let $V \subseteq X$ be a convex set. One says that the set $U \subseteq V$ is segment-dense in V if for each $x \in V$ there can be found $y \in U$ such that x is a cluster point of the set $[x,y] \cap U$.

In what follows we present a denseness notion (cf. [143, 144]) which is slightly different from the concept of The Luc presented above, but which is in some sense compatible with the convexity property of sets.

Definition 4.3.1. Consider the sets $U \subseteq V \subseteq X$ and assume that V is convex. We say that U is self-segment-dense in V if U is dense in V and

$$\forall x, y \in U, \text{ the set } [x,y] \cap U \text{ is dense in } [x,y].$$

Remark 4.3.1. Obviously in one dimension the concepts of a segment-dense set respectively a self-segment-dense set are equivalent to the concept of a dense set.

In what follows we provide an essential example of a self-segment-dense set.

Example 4.3.1. [see also [143], Example 2.1] Let V be the real Hilbert space of square summable sequences l_2 and define U to be the set

$$U := \{(x) = (x_1, ..., x_n, ...) \in l_2 : x_i \in \mathbb{Q}, \text{ for all } i \in \mathbb{N}\},$$

where $\mathbb{Q}$ denotes the set of all rational numbers. Then, it is clear that U is dense in l_2. On the other hand U is not segment-dense in l_2, since for $(x) = \left(\sqrt{2}, \frac{1}{2}, ..., \frac{1}{n}, ...\right) \in l_2$ and for every $(y) = (y_1, y_2, ..., y_n, ...) \in U$, one has $[(x),(y)] \cap U = \{(y)\}$.

It can easily be observed that U is self-segment-dense in l_2, since for every $(x),(y) \in U$, $(x) = (x_1, ..., x_n, ...)$, $(y) = (y_1, ..., y_n, ...)$ we have $[(x),(y)] \cap U = \{(x_1 + t(y_1 - x_1), ..., x_n + t(y_n - x_n), ...) : t \in [0,1] \cap \mathbb{Q}\}$, which is obviously dense in $[(x),(y)]$.

Another very interesting example comes from the general infinite dimensional setting. Let X be a nonreflexive Banach space and let B be the closed unit ball of X. Let X^{**} be the bidual space of X and let B^{**} be its closed unit ball. According to Goldstine Theorem [86] in this case B is dense in B^{**} in the weak* topology of X^{**}. Moreover, in virtue of convexity of B, we have that B is actually self-segment-dense in B^{**} with respect to the weak* topology of X^{**}.

To further circumscribe the notion of a self-segment-dense set we provide an example of a subset that is dense but not self-segment-dense.

Example 4.3.2. Let X be an infinite dimensional real Hilbert space. It is known that the unit sphere $U = \{x \in X : \|x\| = 1\}$, is dense with respect to the weak topology in the unit ball $B = \{x \in X : \|x\| \leq 1\}$, but it is obviously not self-segment-dense since any segment with endpoints on the sphere does not intersect the sphere in any other points. Moreover, U is not segment-dense in the sense of The Luc either, because for every $x \in U$ one has $[0,x] \cap U = \{x\}$. Note that the same argument is also valid if X is a normed space. In this case one can take two points x and $-x$ belonging to the unit sphere, and show that the intersection

of the segment $[-x,x]$ with the unit sphere is $\{-x,x\}$. Moreover, if X is not strictly convex then the unit sphere contains segments.

In what follows we provide an example of a dense set in the strong topology of a Hilbert space, which is also segment-dense but is not self-segment-dense (see also Proposition 3.4, [39]).

Example 4.3.3. Let $V = X = l_2$ be the real Hilbert space of square summable sequences and define U to be the set

$$U := \{(x) = (x_1, ..., x_n, ...) \in l_2 : |x_1| < \sup_{n\geq 2} n|x_n|\}.$$

Then U is open, dense and segment-dense in l_2, but is not self-segment-dense.

Proof. First of all observe that the complement of U is closed. Indeed, for a sequence $(x^i) = (x_1^i, ..., x_n^i, ...) \in l_2 \setminus U$ converging to $(x) = (x_1, ..., x_n, ...) \in l_2$, one has $|x_1^i| \geq n|x_n^i|$ for all $n \in \mathbb{N}$. Since $x_n^i \longrightarrow x_n$, $i \longrightarrow \infty$ for all $n \in \mathbb{N}$, obviously $|x_1| \geq n|x_n|$ for all $n \in \mathbb{N}$. Hence, $l_2 \setminus U$ is closed, which shows that U is open. Consider now $(x) = (x_1, ..., x_n, ...) \in l_2 \setminus U$. We show that for every $\varepsilon > 0$ there exists $(y) \in U$ such that $\|(x) - (y)\| < \varepsilon$. Indeed, let $\varepsilon > 0$ and consider $n_0 \in \mathbb{N}$ such that $n_0 > \frac{3|x_1|}{\varepsilon}$. Let $(y) = (y_1, y_2, ..., y_n, ...) \in l_2, y_n = x_n$ for all $n \neq n_0$ and $y_{n_0} = |x_{n_0}| + \frac{\varepsilon}{3}$. Then $n_0|y_{n_0}| > |y_1|$, hence $(y) \in U$.

On the other hand

$$\|(y) - (x)\| = |\frac{\varepsilon}{3} + |x_{n_0}| - x_{n_0}| \leq 2|x_{n_0}| + \frac{\varepsilon}{3} \leq \frac{2|x_1|}{n_0} + \frac{\varepsilon}{3} < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Next, we show that U is not self-segment-dense in l_2.

Indeed, consider $(x) = (1, 1, \frac{1}{9}, ..., \frac{1}{n^2}, ...)$ and $(y) = (1, -1, \frac{1}{18}, ..., \frac{1}{2n^2}, ...) \in U$. We show that $(1-t)(x) + t(y) \cap U = \emptyset$ for all $t \in \left[\frac{1}{4}, \frac{3}{4}\right]$. We have

$$(1-t)(x) + t(y) = \left(1, 1-2t, \frac{2(1-t)+t}{18}, ..., \frac{2(1-t)+t}{2n^2}, ...\right),$$

hence $(1-t)(x) + t(y) \in U$, if and only if $1 < 2|1-2t|$. But then $t \notin \left[\frac{1}{4}, \frac{3}{4}\right]$. Consequently $\mathrm{cl}([(x),(y)] \cap U) \neq [(x),(y)]$, which shows that U is not self-segment-dense in l_2.

It remained to show, that U is segment-dense in l_2. Actually we will show something more, that is, for every $(y) \in l_2$ there exists $(x) \in U$ such that $[(x),(y)[\subseteq U$. When $(y) \in U$ the statement follows from the fact that U is open. Let now $(y) = (y_1, y_2, ..., y_n, ..) \in l_2 \setminus U$. Then $|y_1| \geq n|y_n|$ for all $n \in \mathbb{N}$, hence one has $-\frac{|y_1|}{n} \leq y_n \leq \frac{|y_1|}{n}$ for all $n \in \mathbb{N}$. Consider $(x) =$

$\left(y_1, \frac{\sqrt{\ln 2}}{2}, ..., \frac{\sqrt{\ln n}}{n}, ...\right)$. By using the integral test, one easily can show that $(x) \in l_2$. Since $\sup_{n\geq 2} n|x_n| = \infty > |y_1|$ we get that $(x) \in U$. For $t \in]0,1]$ we show that $(1-t)(y)+t(x) \in U$, that is $[x,y[\subseteq U$. Indeed, one has

$$(1-t)(y)+t(x) = \left(y_1, (1-t)y_2 + t\frac{\sqrt{\ln 2}}{2}, ..., (1-t)y_n + t\frac{\sqrt{\ln n}}{n}, ...\right).$$

Since for $n \in \mathbb{N}$ big enough and for $t > 0$ fixed, one has

$$n\left|(1-t)y_n + t\frac{\sqrt{\ln n}}{n}\right| \geq n\left|-(1-t)\frac{|y_1|}{n} + t\frac{\sqrt{\ln n}}{n}\right| \geq t\sqrt{\ln n} - (1-t)|y_1|,$$

we get that

$$\sup_{n\in\mathbb{N}} n\left|(1-t)y_n + t\frac{\sqrt{\ln n}}{n}\right| = \infty.$$

Hence, $(1-t)(y)+t(x) \in U$ for all $t \in]0,1]$. □

Remark 4.3.2. Note that every convex subset of a topological vector space is self-segment-dense in its closure. In particular dense subspaces and dense affine subsets are self-segment-dense. Therefore if $U \subseteq V$ is dense in V and V is convex, then $\mathrm{co}(U)$ is self-segment-dense in V.

Next we provide some remarkable properties of a self-segment-dense set. We also show that these results do not hold if we replace the self-segment-dense property of the set involved by its denseness.

The following lemma (see [144], [5]) gives an interesting characterization of self-segment-dense sets and will be used in the sequel.

Lemma 4.3.1. *[Lemma 2.1, [144]] Let X be a Hausdorff locally convex topological vector space, let $V \subseteq X$ be a convex set and let $U \subseteq V$ a self-segment-dense set in V. Then, for all finite subset $\{u_1, u_2, \ldots, u_n\} \subseteq U$ one has*

$$\mathrm{cl}(\mathrm{co}\{u_1, u_2, \ldots, u_n\} \cap U) = \mathrm{co}\{u_1, u_2, \ldots, u_n\}.$$

Remark 4.3.3. Observe that under the hypothesis of Lemma 4.3.1, one has that the intersection $\mathrm{co}\{u_1, u_2, \ldots, u_n\} \cap U$ is self-segment-dense in $\mathrm{co}\{u_1, u_2, \ldots, u_n\}$.

Let us emphasize that this result does not remain valid in case we replace the self-segment-denseness of U in V, by its denseness in V, as the next example shows.

Example 4.3.4. [Example 2.3, [139]] Let V be the closed unit ball of an infinite dimensional Banach space X, and let $x, y \in V, x \neq y$. Moreover, consider $u, v \in]x, y[$, $u = x + t_1(y - x)$, $v = x + t_2(y - x)$, with $t_1, t_2 \in]0, 1[$, $t_1 < t_2$. Then obviously $U = V \setminus [u, v]$ is dense in V, but not self-segment-dense, since for $x, y \in U$ the set $[x, y] \cap U = [x, u[\cup]v, y]$ is not dense in $[x, y]$. This also shows, that $\mathrm{cl}(\mathrm{co}\{x, y\} \cap U) \neq \mathrm{co}\{x, y\}$.

An easy consequence of Lemma 4.3.1 is the following more general result.

Lemma 4.3.2. *Let X be a Hausdorff locally convex topological vector space, let $V \subseteq X$ be a convex set and let $U \subseteq V$ a self-segment-dense set in V. Then, for every subset $\mathcal{S} \subseteq U$ one has*

$$\mathrm{cl}(\mathrm{co}(\mathcal{S}) \cap U) = \overline{\mathrm{co}}(\mathcal{S}).$$

In other words, $\mathrm{co}(\mathcal{S}) \cap U$ *is self-segment-dense in* $\mathrm{co}(\mathcal{S})$.

Proof. Let $x \in \mathrm{co}(\mathcal{S})$. We show that for every neighbourhood G of x one has $G \cap U \neq \emptyset$. Indeed, $x = \sum_{i=1}^{n} \lambda_i u_i$ for some $u_i \in S \subseteq U, \lambda_i \geq 0, i \in \{1, 2, \ldots, n\}, n \in \mathbb{N}, \sum_{i=1}^{n} \lambda_i = 1$. Hence, $x \in \mathrm{co}\{u_1, u_2, \ldots, u_n\}$ and according to Lemma 4.3.1, $\mathrm{co}\{u_1, u_2, \ldots, u_n\} \cap U = \mathrm{co}\{u_1, u_2, \ldots, u_n\} \cap S$ is dense in $\mathrm{co}\{u_1, u_2, \ldots, u_n\}$. Consequently $G \cap U \neq \emptyset$ which shows that $\mathrm{co}(\mathcal{S}) \cap U$ is dense in $\mathrm{co}\,\mathcal{S}$. It is also self-segment-dense since for $u_1, u_2 \in \mathrm{co}(\mathcal{S}) \cap U$ one has

$$\mathrm{cl}([u_1, u_2] \cap \mathrm{co}(\mathcal{S}) \cap U) = \mathrm{cl}([u_1, u_2] \cap U) = [u_1, u_2].$$

□

Remark 4.3.4. Obviously, the precedent lemma also ensures that $\mathrm{co}(\mathcal{S}) \cap U$ is self-segment-dense in $\overline{\mathrm{co}}(\mathcal{S})$. A particular instance is, that $\mathrm{cl}(U) = \overline{\mathrm{co}}(U)$.

In what follows we present a simple but very useful result concerning on self-segment-dense sets, by showing that the self-segment-dense property of a subset of some base set implies the convexity of the base set.

Lemma 4.3.3. *Let $V \subseteq X$ be closed and let $U \subseteq V$ be dense in V, with the property that for all $u, v \in U$ one has that $[u, v] \cap U$ is dense in $[u, v]$. Then V is convex, hence U is actually self-segment-dense in V.*

Proof. Observe that for all $u, v \in U$ one has $[u, v] \subseteq V$ since

$$[u, v] = \mathrm{cl}([u, v] \cap U) \subseteq \mathrm{cl}(U) = V.$$

Assume now that V is not convex, i.e., there exist $x, y \in V$, and $t_0 \in (0,1)$ such that $(1-t_0)x + t_0 y = z_0 \notin V$. Since U is dense in V, there exists the nets $(x_i), (y_i) \subseteq U$ such that $x_i \longrightarrow x$, $y_i \longrightarrow y$. But then $[x_i, y_i] \subseteq V$ for all i, hence $(1-t_0)x_i + t_0 y_i \in V$ for all i. Since V is closed we obtain that $\lim((1-t_0)x_i + t_0 y_i) = z_0 \in V$, contradiction. □

Remark 4.3.5. Note that the closedness of V in the hypothesis of Lemma 4.3.3 is essential. Indeed, let $A \subseteq \mathbb{R}^2$ be the square with vertices $(-1,-1), (-1,1), (1,1)$ and $(1,-1)$, let V be $A \backslash](-1,-1),(1,-1)[$ and let U be the interior of A. Here, the interior of the square is meant relative to the plane of the square. Then obviously U has the property, that for all $u, v \in U$ one has that $[u,v] \cap U$ is dense in $[u,v]$. Moreover, U is dense in V. Observe that V is not convex, because $(-1,-1),(1,-1) \in V$ but $[(-1,-1),(1,-1)] \not\subseteq V$. This is due to the fact that V is not closed, hence Lemma 4.3.3 cannot be applied.

4.3.4 Convex functions and dense sets

In what follows we provide some results concerning convex functions on dense sets. Our results are based on the concepts of a self-segment-dense and a segment-dense set, respectively. We obtain conditions that ensure the coincidence of two proper, convex and lower semicontinuous function that are equal on a dense subset of their domain. Further, we analyze the situation when the infimum of a convex function is equal to the infimum of that function taken over a dense subset of its domain. An example shows that the use of the concepts of a self-segment-dense set and segment-dense set are essential in order to obtain these results.

Theorem 4.3.3. *Let $f : X \longrightarrow \overline{\mathbb{R}}$ be a proper function and let $U \subseteq \operatorname{dom} f$ be a dense set.*

(a) Then, $\overline{f_U}$ is lower semicontinuous, U is graphically dense in $\operatorname{dom} \overline{f_U}$ and $\inf_{x \in U} f(x) = \inf_{x \in X} \overline{f_U}(x)$.

(b) Assume that f is lower semicontinuous on U. Then, $f(u) = \overline{f_U}(u)$ for all $u \in U$. Moreover, if f is lower semicontinuous on X then $\operatorname{dom} \overline{f_U} \subseteq \operatorname{dom} f$ and $\overline{f_U}(x) \geq f(x)$ for all $x \in X$. Further, in this case, $\overline{f_U} = f$ on X, if and only if, U is graphically dense in $\operatorname{dom} f$.

(c) Assume that $\operatorname{dom} f$ is convex, U is self-segment-dense in $\operatorname{dom} f$ and f is convex on U. Then, $\overline{f_U}$ is also convex and and $\operatorname{epi} f_U$ is self-segment-dense in $\operatorname{epi} \overline{f_U}$.

Proof. (a) The lower semicontinuity of $\overline{f_U}$ follows from the closedness of $\operatorname{epi} \overline{f_U}$. We show that U is graphically dense in $\operatorname{dom} \overline{f_U}$. Let $(u_i, r_i) \subseteq \operatorname{epi} f_U$, a net converging to $(x, \overline{f_U}(x)) \in$

$\operatorname{epi}\overline{f_U}$. Then, $(u_i, r_i) \subseteq \operatorname{epi}\overline{f_U}$, hence $r_i \geq \overline{f_U}(u_i)$. Consequently

$$\overline{f_U}(x) = \lim r_i \geq \liminf \overline{f_U}(u_i) \geq \overline{f_U}(x),$$

which shows that $\liminf \overline{f_U}(u_i) = \overline{f_U}(x)$.

(Note that we have also shown that $\liminf f(u_i) = \overline{f_U}(x)$.) Hence, there exists a subnet of the net $(\overline{f_U}(u_i))$, say $(\overline{f_U}(u_j))$ which converges to $\overline{f_U}(x)$. Since (u_i) converges to x, obviously $\lim u_j = x$, which completes the proof. (One may similarly show, that there exists a net $(u_j) \subseteq U$ such that $\lim u_j = x$ and $\lim f(u_j) = \overline{f_U}(x)$.)

We show next, that $\inf_{x\in U} f(x) = \inf_{x\in X} \overline{f_U}(x)$. Note at first that $\inf_{x\in U} f(x) \geq \inf_{x\in X} \overline{f_U}(x)$. Indeed, for every $x \in U$ one has $(x, f(x)) \in \operatorname{epi} f_U \subseteq \operatorname{epi}\overline{f_U}$, hence $\overline{f_U}(x) \leq f(x)$ for all $x \in U$. Consequently,

$$\inf_{x\in X} \overline{f_U}(x) \leq \inf_{x\in U} \overline{f_U}(x) \leq \inf_{x\in U} f(x).$$

If $\inf_{x\in U} f(x) = -\infty$ then the statements is obvious. Assume now, that $\inf_{x\in U} f(x) = \alpha > -\infty$. Then, $f(x) \geq \alpha$ for all $x \in U$. Let $x \in \operatorname{dom}\overline{f_U}$. Then, according to the previous part of proof, there exists a net $(u_i) \subseteq U$, such that $\overline{f_U}(x) = \lim f(u_i) \geq \alpha$, hence $\inf_{x\in X} \overline{f_U}(x) \geq \alpha$. This also shows that, in this case, $\overline{f_U}$ is proper.

(b) We show that $\overline{f_U}(u) = f(u)$ for all $u \in U$. Let $u \in U$. Since $(u, f(u)) \in \operatorname{epi} f_U \subseteq \operatorname{epi}\overline{f_U}$ one has

$$\overline{f_U}(u) \leq f(u).$$

Let $r \in \mathbb{R}$ such that $(u, r) \in \operatorname{epi}\overline{f_U}$. Then, there exists a net $((u_i, r_i)) \subseteq \operatorname{epi} f_U$ such that $(u_i, r_i) \longrightarrow (u, r)$. Obviously $r_i \geq f(u_i)$ and in virtue of lower semicontinuity of f on U we have

$$r = \lim r_i = \liminf r_i \geq \liminf f(u_i) \geq f(u).$$

Sice r is a arbitrary, provided $r \geq \overline{f_U}(u)$, one has

$$\overline{f_U}(u) \geq f(u).$$

Assume now that f is lower semicontinuous on X. Obviously, $\operatorname{epi} f$ is closed, hence $\operatorname{epi}\overline{f_U} = \operatorname{cl}(\operatorname{epi} f_U) \subseteq \operatorname{epi} f$. The latter relation leads to

$$\operatorname{dom}\overline{f_U} = \operatorname{pr}_X(\operatorname{epi}\overline{f_U}) \subseteq \operatorname{pr}_X(\operatorname{epi} f) = \operatorname{dom} f.$$

Consequently, for all $x \in \operatorname{dom}\overline{f_U}$ one has $(x, \overline{f_U}(x)) \in \operatorname{epi}\overline{f_U} \subseteq \operatorname{epi} f$ which leads to $f(x) \leq$

$\overline{f_U}(x)$, for all $x \in \operatorname{dom} \overline{f_U}$. Since $\overline{f_U}(x) = +\infty$ for all $x \in X \setminus \operatorname{dom} \overline{f_U}$ we get that $f(x) \leq \overline{f_U}(x)$, for all $x \in X$. This also shows that, in this case, $\overline{f_U}$ is proper.

It remained to show, that $\overline{f_U} = f$ on X, if and only if U is graphically dense in $\operatorname{dom} f$. Assume that $\overline{f_U} = f$ on X and let $x \in \operatorname{dom} f$. Then, according to (a), there exists a net $(u_j) \subseteq U$ such that $\lim u_j = x$ and $\lim \overline{f_U}(u_j) = \overline{f_U}(x)$. But $\overline{f_U}(u_j) = f(u_j)$ and $\overline{f_U}(x) = f(x)$. Conversely, consider a net $(u_j) \subseteq U$ such that $\lim u_j = x$ and $\lim f(u_j) = f(x)$. Note that $\overline{f_U}(x) \leq \lim f(u_j) = f(x)$. On the other hand, $\overline{f_U}(x) \geq f(x)$, which completes the proof.

(c) Assume now that $\operatorname{dom} f$ is convex, U is self-segment-dense in $\operatorname{dom} f$ and that f is convex on U. Observe that by definition we have that $\operatorname{epi} f_U$ is dense in $\operatorname{epi} \overline{f_U}$ and $\operatorname{epi} \overline{f_U}$ is closed.

Note that $\overline{f_U}$ is convex, if and only if $\operatorname{epi} \overline{f_U}$ is convex. For showing the convexity of $\operatorname{epi} \overline{f_U}$ we use Lemma 4.3.3. To this end, we show that for all $(u,r),(v,s) \in \operatorname{epi} f_U$ we have that

$$\operatorname{cl}([(u,r),(v,s)] \cap \operatorname{epi} f_U) = [(u,r),(v,s)].$$

Indeed, assume the contrary, i.e., there exist $(u,r),(v,s) \in \operatorname{epi} f_U$ such that $\operatorname{cl}([(u,r),(v,s)] \cap \operatorname{epi} f_U) \neq [(u,r),(v,s)]$. Then, there exist two distinct points $(u_1,r_1),(v_1,s_1) \in [(u,r),(v,s)]$ such that

$$](u_1,r_1),(v_1,s_1)[\cap \operatorname{epi} f_U = \emptyset.$$

Since $u_1, v_1 \in [u,v]$ and U is self-segment-dense in $\operatorname{dom} f$, one has that $]u_1,v_1[\cap U \neq \emptyset$, hence there exists $t_0 \in]0,1[$ such that

$$(1-t_0)u_1 + t_0 v_1 = u_0 \in]u_1,v_1[\cap U.$$

By the convexity of f on U we have

$$f(u_0) \leq (1-t_0)f(u_1) + t_0 f(v_1) \leq (1-t_0)r + t_0 s,$$

which leads to

$$(u_0, (1-t_0)r + t_0 s) \in \operatorname{epi} f_U.$$

The latter relation shows that $(1-t_0)(u_1,r) + t_0(v_1,s) \in \operatorname{epi} f_U$ which contradicts the assumption that $](u_1,r_1),(v_1,s_1)[\cap \operatorname{epi} f_U = \emptyset$. Thus, according to Lemma 4.3.3 $\operatorname{epi} \overline{f_U}$ is convex and $\operatorname{epi} f_U$ is self-segment-dense in $\operatorname{epi} \overline{f_U}$. □

Remark 4.3.6. Note that $\inf_{x \in U} f(x) = \inf_{x \in X} \overline{f_U}(x)$ implies that $\overline{f_U}$ is proper, provided $\inf_{x \in U} f(x) \neq -\infty$. The convexity of $\overline{f_U}$ is not guaranteed if in the hypothesis of Theorem

4.3.3 (c) we assume only that U is dense in $\operatorname{dom} f$, as the next example shows.

Example 4.3.5. Let X be an infinite dimensional real Hilbert space. According to Example 4.3.2, the union of the unit sphere with $\{0\}$, is dense with respect to the weak topology in the unit ball $B = \{x \in X : \|x\| \leq 1\}$ of X, but this set is not self-segment-dense in B. Hence, let $U = \{x \in X : \|x\| = 1\} \cup \{0\}$. Let $y_0 \in U$, $y_0 \neq 0$, and consider the function

$$f : X \longrightarrow \overline{\mathbb{R}},\ f(x) = \begin{cases} \langle y_0, x\rangle^3, \text{ if } x \in B \\ +\infty, \text{ otherwise.} \end{cases}$$

Then, trivially, f is convex on U, since for $x, 0, -x \in U$ one has $f(0) = \frac{1}{2}f(-x) + \frac{1}{2}f(x)$, but $\overline{f_U}$ is not convex on X. Indeed, it can easily be verified that $\overline{f_U}(y_0) = 1$, $\overline{f_U}(-y_0) = -1$ and $\overline{f_U}\left(-\frac{1}{2}y_0\right) = -\frac{1}{8}$. Obviously $-\frac{1}{2}y_0 = \left(1 - \frac{1}{4}\right)(-y_0) + \frac{1}{4}y_0 \in [-y_0, y_0]$, but

$$-\frac{1}{8} = \overline{f_U}\left(\left(1 - \frac{1}{4}\right)(-y_0) + \frac{1}{4}y_0\right) > \left(1 - \frac{1}{4}\right)\overline{f_U}(-y_0) + \frac{1}{4}\overline{f_U}(y_0) = -\frac{1}{2}.$$

Corollary 4.3.1. *Let $f : X \to \overline{\mathbb{R}}$ be a proper and continuous function and let $U \subseteq \operatorname{dom} f$ be a dense set. Then,* $\inf_{x\in U} f(x) = \inf_{x\in X} f(x)$.

Proof. According to Theorem 4.3.3 (a), $\inf_{x\in U} f(x) = \inf_{x\in X} \overline{f_U}(x)$. Since f is continuous, obviously it is lower semicontinuous on X and U is graphically dense in $\operatorname{dom} f$. Hence, by Theorem 4.3.3 (b), one has $\overline{f_U} = f$ on X. □

However, Example 4.3.7 below, shows that the continuity assumption of f in the hypothesis of Corollary 4.3.1 is essential and cannot be replaced by lower semicontinuity.

Remark 4.3.7. According to Theorem 4.3.3, for any proper, lower semicontinuous and convex function f, and any dense set $U \subseteq \operatorname{dom} f$ there exists a proper and lower semicontinuous function $\overline{f_U}$ which coincides with f on U, dominates f on X and is not equal to f, provided U is not graphically dense in $\operatorname{dom} f$. Moreover, if U is also self-segment-dense, then $\overline{f_U}$ is convex. The general problem, that if two proper, convex and lower semicontinuous functions are equal on a dense set whether they coincide, has been investigated by Benoist and Daniilidis in [39], in a Banach space context. According to Proposition 3.4 [39], in infinite dimensions the answer is negative, which also follows from our argument above. Nevertheless, in finite dimension the answer is affirmative, as follows from Corollary 3.7 [39]. However, there are special type of dense sets in infinite dimensional Hausdorff topological vector spaces, where the coincidence result holds. We show next, that if two proper, convex and lower semicon-

tinuous functions are equal on a segment-dense set of their common domain, then they are equal everywhere.

Proposition 4.3.1. *Let X be a Hausdorff locally convex topological vector space. Let $f,g : X \to \overline{\mathbb{R}}$ be two proper, convex and lower semicontinuous functions. Let $U \subseteq \operatorname{dom} f \cap \operatorname{dom} g$ be a segment-dense set such that $f(u) = g(u)$ for all $u \in U$. Then, $f = g$ on $\operatorname{dom} f \cap \operatorname{dom} g$.*

Proof. Let $x_0 \in \operatorname{dom} f \cap \operatorname{dom} g$. Since U is segment-dense in $\operatorname{dom} f \cap \operatorname{dom} g$, there exists $u_0 \in U$ such that x_0 is a cluster point of the set $[u_0, x_0] \cap U$. We show that $f(x_0) = g(x_0)$. Indeed, by the convexity and lower semicontinuity of f and g, (in virtue Proposition 1.3.4, [165]) one has that for every sequence $(x_n) \subseteq [u_0, x_0]$ converging to x_0, it holds

$$f(x_0) = \lim_{x_n \longrightarrow x_0} f(x_n) \text{ and } g(x_0) = \lim_{x_n \longrightarrow x_0} g(x_n).$$

But, then for $(u_n) \subseteq [u_0, x_0] \cap U$ converging to x_0, we get

$$f(x_0) = \lim_{u_n \longrightarrow x_0} f(u_n) = \lim_{u_n \longrightarrow x_0} g(u_n) = g(x_0).$$

□

Corollary 4.3.2. *Let X be a Hausdorff locally convex topological vector space, let $f : X \to \overline{\mathbb{R}}$ be a proper, convex and lower semicontinuous function and let $U \subseteq \operatorname{dom} f$ be a segment-dense set in $\operatorname{dom} f$. Then, $f = \overline{f_U}$ on X and $\inf_{x \in U} f(x) = \inf_{x \in X} f(x)$.*

Proof. Using the same arguments as in the proof of Proposition 4.3.1, one can easily show that U is graphically dense in $\operatorname{dom} f$. Hence, according to Theorem 4.3.3 (b), $f = \overline{f_U}$ on X. But Theorem 4.3.3 also shows that $\inf_{x \in U} f(x) = \inf_{x \in X} \overline{f_U}(x)$. □

Remark 4.3.8. Note that the conclusion of Corollary 4.3.2 fails if we assume that $f : X \to \overline{\mathbb{R}}$ is convex only on U. Moreover $\operatorname{dom} \overline{f_U}$ might not be closed even when $\operatorname{dom} f$ is compact and f is lower semicontinuous as the following simple example shows.

Example 4.3.6. Consider the function

$$f : \mathbb{R} \longrightarrow \overline{\mathbb{R}},\ f(x) = \begin{cases} \frac{1}{x}, \text{ if } x \in]0,1] \\ 0, \text{ if } x = 0 \\ +\infty, \text{ otherwise.} \end{cases}$$

Then, $\operatorname{dom} f = [0,1]$ is compact and f is convex on $U =]0,1]$ and lower semicontinuous on $\operatorname{dom} f$. Further U is segment-dense (and also self-segment-dense), in $\operatorname{dom} f$. But, $1 =$

$\inf_{x\in U} f(x) \neq \inf_{x\in X} f(x) = 0$. On the other hand, $\overline{f_U}(x) = \begin{cases} \frac{1}{x}, \text{ if } x \in]0,1] \\ +\infty, \text{ otherwise} \end{cases}$, hence $f \neq \overline{f_U}$ on $\mathbb{R}$.

It can easily be observed that $\operatorname{dom} \overline{f_U} =]0,1]$ which is not closed.

Next we provide a general coincidence result, involving self-segment-dense sets, in infinite dimension.

Proposition 4.3.2. *Let X be a Hausdorff locally convex topological vector space. Let $f : X \to \overline{\mathbb{R}}$ be a proper, convex and lower semicontinuous function. Let $U \subseteq \operatorname{dom} f$ be a self-segment-dense set. Then, $f = \overline{f_U}$ on $\operatorname{co}(U)$ and $\overline{f_U} = \overline{f_{\operatorname{co}(U)}}$ on X.*

Proof. Obviously $\operatorname{co}(U) \subseteq \operatorname{dom} \overline{f_U}$. Let $x_0 \in \operatorname{co}(U)$. Then $x_0 = \sum_{i=1}^n \lambda_i u_i$ for some $u_1, ..., u_n \in U$ and $\lambda_1, ..., \lambda_n \in [0,1]$ with $\sum_{i=1}^n \lambda_i = 1$. Hence, $x_0 \in \operatorname{co}\{u_1, ..., u_n\}$. Let $Y = \operatorname{lin}\{u_1, ..., u_n\}$ and consider the function

$$\tilde{f} : Y \longrightarrow \overline{\mathbb{R}}, \tilde{f}(x) = \begin{cases} f(x), \text{ if } x \in \operatorname{co}\{u_1, ..., u_n\} \\ +\infty, \text{ otherwise.} \end{cases}$$

Then, $\tilde{f}$ is proper convex and lower semicontinuous, and in virtue of Theorem 4.3.3, $\tilde{f}(u) = f(u) = \overline{f_U}(u)$ for all $u \in \operatorname{co}\{u_1, ..., u_n\} \cap U$. Obviously Y is finite dimensional and according to Lemma 4.3.1, $\operatorname{co}\{u_1, ..., u_n\} \cap U$ is dense, (actually is self-segment-dense), in $\operatorname{co}\{u_1, ..., u_n\}$, hence according to Corollary 3.7 [39], $\tilde{f} = \overline{f_U}$ on $\operatorname{co}\{u_1, ..., u_n\}$. In particular $f(x_0) = \overline{f_U}(x_0)$. Since $x_0 \in \operatorname{co}(U)$ was arbitrary chosen, it follows that $f = \overline{f_U}$ on $\operatorname{co}(U)$.

Let us denote $g = \overline{f_U}$. According to Theorem 4.3.3 (a), U is graphically dense in $\operatorname{dom} g$, hence $\operatorname{co}(U)$ is also graphically dense in $\operatorname{dom} g$. By Theorem 4.3.3 (b), one has $\overline{g_{\operatorname{co}(U)}} = g$ on X. But $g_{\operatorname{co}(U)} = f_{\operatorname{co}(U)}$ and the conclusion follows. □

Remark 4.3.9. According to Proposition 4.3.2, if two proper convex and lower semicontinuous functions f and g coincide on a self-segment-dense set U of their domain, then they also coincide on $\operatorname{co}(U)$. Indeed, in this case $\overline{f_U} = \overline{g_U}$, hence $f = \overline{f_U} = \overline{g_U} = g$ on $\operatorname{co}(U)$.

Remark 4.3.10. Note that under the hypothesis of Proposition 4.3.2, one has $\inf_{x\in U} f(x) = \inf_{x\in \operatorname{co}(U)} f(x)$. Indeed, according to Proposition 4.3.2 one has $f = \overline{f_U}$ on $\operatorname{co}(U)$. On the other hand, according to Theorem 4.3.3 one has $\inf_{x\in U} f(x) = \inf_{x\in X} \overline{f_U}(x)$. Therefore, the following inequalities

$$\inf_{x\in U} f(x) \geq \inf_{x\in \operatorname{co}(U)} f(x) = \inf_{x\in \operatorname{co}(U)} \overline{f_U}(x) \geq \inf_{x\in X} \overline{f_U}(x) = \inf_{x\in U} f(x)$$

must hold with equality everywhere.

Moreover, if $\mathrm{co}(U)$ is closed then $f = \overline{f_U}$ on X, because

$$U \subseteq \mathrm{co}(U) \subseteq \mathrm{dom}\,\overline{f_U} \subseteq \mathrm{dom}\, f \subseteq \mathrm{cl}(U) \subseteq \overline{\mathrm{co}}(U) = \mathrm{co}(U).$$

Hence, in this case,

$$\inf_{x\in U} f(x) = \inf_{x\in X} f(x). \quad (*)$$

The next example shows that $(*)$ fails even if $\mathrm{co}(U)$ is compact, when we only assume that U is dense in $\mathrm{dom}\, f$. It also shows, that the continuity assumption on f in Corollary 4.3.1, the segment-dense property of U in Corolary 4.3.2 and the self-segment-denseness of U in Proposition 4.3.2 are essential.

Example 4.3.7. Let X be an infinite dimensional real Hilbert space. Let $K = \{x \in X : \|x\| \leq 1\}$ be the unit ball of X and let $U = \{x \in X : \|x\| = 1\}$. Then according to Example 4.3.2, U is dense in K with respect to the weak topology of X, but is neither segment-dense not self-segment-dense in K. Obviously $\mathrm{co}(U) = K$, which according to Banach-Alaoglu Theorem [86] is weakly compact. Consider the function

$$f : X \longrightarrow \mathbb{R},\ f(x) = \begin{cases} \|x\|, \text{ if } x \in K \\ +\infty, \text{ otherwise.} \end{cases}$$

Then, obviously f is proper, convex and weakly lower semicontinuous and $\mathrm{dom}\, f = K$. Nevertheless

$$\inf_{x\in U} f(x) = 1$$

and

$$\min_{x\in X} f(x) = 0.$$

It is also obvious the fact, that in this case $f \neq \overline{f_U}$ on $\mathrm{co}(U)$, since

$$\overline{f_U}(x) = \begin{cases} 1, \text{ if } x \in K \\ +\infty, \text{ otherwise.} \end{cases}$$

Next we present some other interesting conditions that ensure the validity of $(*)$.

Proposition 4.3.3. *Let X be a Hausdorff locally convex topological vector space. Let $f : X \longrightarrow \overline{\mathbb{R}}$ be a proper, convex and lower semicontinuous function, and let $U \subseteq \mathrm{dom}\, f$ be self-segment-dense in $\mathrm{dom}\, f$. Assume further that $\mathrm{co}(U)$ is segment-dense in $\mathrm{dom}\, f$. Then, $\overline{f_U} = \overline{f_{\mathrm{co}(U)}} = f$ on X and $\inf_{x\in U} f(x) = \inf_{x\in X} f(x)$.*

Proof. According to Proposition 4.3.2, $\overline{f_U} = \overline{f_{\mathrm{co}(U)}}$ on X. According to Corollary 4.3.2, $\overline{f_{\mathrm{co}(U)}} = f$ on X. Hence, $\overline{f_U} = \overline{f_{\mathrm{co}(U)}} = f$ on X. From Theorem 4.3.3 (a), one has $\inf_{x\in U} f(x) = \inf_{x\in X} \overline{f_U}(x)$, consequently $\inf_{x\in U} f(x) = \inf_{x\in X} f(x)$. □

Remark 4.3.11. Note that each of the following conditions ensure that $\mathrm{co}(U)$ is segment-dense in $\mathrm{dom}\, f$.

(a) For all $x \in \mathrm{dom}\, f$ there exists $y \in \mathrm{co}(U)$, such that $[y,x) \subseteq \mathrm{co}(U)$.

(b) The interior of $\mathrm{co}(U)$ is non-empty. In this case it is known that the following relation, sometimes called line segment principle [156, 189], holds

$$\lambda\, \mathrm{int}(\mathrm{co}(U)) + (1-\lambda)\overline{\mathrm{co}}(U) \subseteq \mathrm{int}(\mathrm{co}(U)), \forall \lambda \in (0,1]. \tag{4. 7}$$

(c) X is finite dimensional. In this case $\mathrm{co}(U)$ has nonempty relative interior, and a similar relation to (4. 7) holds.

Remark 4.3.12. If X is a Banach space then, according to Mazur Theorem [86], the weak and strong closure of a convex set coincides. Therefore, if we endow X with the weak topology, in (4. 7) it is enough to assume that the strong interior of $\mathrm{co}(U)$ is nonempty.

Remark 4.3.13. Example 4.3.7 shows that also in hypothesis of Proposition 4.3.3 the self-segment-dense assumption on U cannot be replaced by the usual denseness assumption. Indeed, for the sets X, K, U and the function f considered Example 4.3.7 all the assumptions in the hypothesis of Proposition 4.3.3 are fulfilled excepting U is self-segment-dense. Here U is only dense. Since $\mathrm{co}(U) = K = \mathrm{dom}\, f$, obviously $\mathrm{co}(U)$ is segment-dense in $\mathrm{dom}\, f$. Nevertheless

$$1 = \inf_{x\in U} f(x) > 0 = \min_{x\in X} f(x).$$

4.3.5 Minimax results on dense sets

In this pragraph we prove several minimax theorems on dense sets. We obtain some results where the conditions imposed to the bifunction that describes the minimax problem are considered relative to a dense set. Several examples and counterexamples circumscribe the results of this section. In particular we show that the general minimax results of Fan and Sion cannot be extended to usual dense sets.

Based on the results previously stated, we are able to give a proof for the following minimax theorem, an extension of Fan's minimax result on self-segment-dense sets.

Theorem 4.3.4. *Let K be a nonempty, compact and convex subset of the Hausdorff locally convex topological vector space X and let Y be an arbitrary nonempty set. Let $U \subseteq K$ be a self-segment-dense set in K and assume that $\mathrm{co}(U)$ is segment-dense in K. Consider further the mapping $f : K \times Y \longrightarrow \mathbb{R}$, and assume that the following assumptions are fulfilled.*

(i) The map $x \longrightarrow f(x,y)$ is proper, convex and lower semicontinuous on K for all $y \in Y$.

(ii) The map $y \longrightarrow f(x,y)$ is concavelike, for all $x \in K$.

Then,

$$\inf_{x\in U} \sup_{y\in Y} f(x,y) = \sup_{y\in Y} \inf_{x\in U} f(x,y).$$

Proof. In virtue of Proposition 4.3.3, one has

$$\sup_{y\in Y} \inf_{x\in U} f(x,y) = \sup_{y\in Y} \min_{x\in K} f(x,y).$$

Theorem 4.3.1 assures that

$$\sup_{y\in Y} \min_{x\in K} f(x,y) = \min_{x\in K} \sup_{y\in Y} f(x,y).$$

On the other hand, the function $g : K \longrightarrow \overline{\mathbb{R}}$, $g(x) = \sup_{y\in Y} f(x,y)$ is proper, convex and lower semicontinuous as a pointwise supremum of a family of proper, convex and lower semicontinuous functions, hence by Proposition 4.3.3 one obtains

$$\min_{x\in K} \sup_{y\in Y} f(x,y) = \inf_{x\in U} \sup_{y\in Y} f(x,y).$$

Thus, we have

$$\sup_{y\in Y} \inf_{x\in U} f(x,y) = \sup_{y\in Y} \min_{x\in K} f(x,y) = \min_{x\in K} \sup_{y\in Y} f(x,y) = \inf_{x\in U} \sup_{y\in Y} f(x,y),$$

and the conclusion follows. □

Remark 4.3.14. Note that, in case Y is a convex subset of a topological vector space, then (ii) can be replaced by the following: the map $y \longrightarrow f(x,y)$ is upper semicontinuous and quasiconcave, for all $x \in K$. Then, the conclusion of Theorem 4.3.4 follows by using Theorem 4.3.2 instead of Theorem 4.3.1 in its proof. The same argument is valid in every result presented bellow.

Remark 4.3.15. For every fixed $y \in Y$, we denote by $\overline{f_U}(\cdot, y)$, the function defined as

$$\operatorname{epi} \overline{f_U}(\cdot, y) = \operatorname{cl}(\operatorname*{epi}_U f(\cdot, y)).$$

If one obtains a condition that assures the domain of $\overline{f_U}(\cdot, y)$ being the same closed subset of the compact set K for every $y \in Y$, in particular $\operatorname{dom} \overline{f_U}(\cdot, y) = K$ for every $y \in Y$, then one can assume that the conditions (i) and (ii) in Theorem 4.3.4 are fulfilled only on U. More precisely the following result holds.

Theorem 4.3.5. *Let K be a nonempty, compact and convex subset of the Hausdorff locally convex topological vector space X and let Y be an arbitrary nonempty set. Let $U \subseteq K$ be a self-segment-dense set in K and assume that $\operatorname{co}(U)$ is segment-dense in K. Consider further the mapping $f : K \times Y \longrightarrow \mathbb{R}$, and assume that the following assumptions are fulfilled.*

(i) The map $x \longrightarrow f(x, y)$ is proper, convex and lower semicontinuous on U for all $y \in Y$.

(ii) The map $y \longrightarrow f(x, y)$ is concavelike, for all $x \in U$.

(iii) For every $y \in Y$, $\operatorname{dom} \overline{f_U}(\cdot, y) = K$.

Then,

$$\inf_{x \in U} \sup_{y \in Y} f(x, y) = \sup_{y \in Y} \inf_{x \in U} f(x, y).$$

Proof. In virtue of Theorem 4.3.3, one has

$$\sup_{y \in Y} \inf_{x \in U} f(x, y) = \sup_{y \in Y} \min_{x \in K} \overline{f_U}(x, y).$$

According to Theorem 4.3.3, the function $x \longrightarrow \overline{f_U}(x, y)$ is convex, proper and lower semicontinuous for all $y \in Y$.

We show that $\overline{f_U}$ is concavelike in its second variable, that is, for every $y_1, y_2 \in Y$ and $t \in [0, 1]$ there exists $y_3 \in Y$ such that

$$(1 - t)\overline{f_U}(x, y_1) + t\overline{f_U}(x, y_2) \leq \overline{f_U}(x, y_3) \text{ for all } x \in \operatorname{co}(U).$$

From the hypothesis of the theorem, we have that for every $y_1, y_2 \in Y$ and $t \in [0, 1]$ there exists $y_3 \in Y$ such that $(1 - t)f(u, y_1) + tf(u, y_2) \leq f(u, y_3)$ for all $u \in U$. From Theorem 4.3.3 we get that $f(u, y_j) = \overline{f_U}(u, y_j)$ for all $u \in U$, $j = 1, 2, 3$. Thus, $(1 - t)\overline{f_U}(u, y_1) + t\overline{f_U}(u, y_2) \leq \overline{f_U}(u, y_3)$ for all $u \in U$. Let $x \in \operatorname{co}(U)$. By the construction of $\overline{f_U}(\cdot, y_3)$ we obtain that there

exists a net $((u^i, r_i)) \subseteq \operatorname{epi} f_U(\cdot, y_3)$, such that $(u^i, r_i) \longrightarrow (x, \overline{f_U}(x, y_3))$. We have

$$r_i \geq f(u^i, y_3) = \overline{f_U}(u^i, y_3), \text{ and } r_i \longrightarrow \overline{f_U}(x, y_3),$$

consequently

$$\overline{f_U}(x, y_3) = \liminf r_i \geq \liminf \overline{f_U}(u^i, y_3).$$

Since $\overline{f_U}(\cdot, y_j)$, $j = 1, 2, 3$ is lower semicontinuous we have, that

$$(1-t)\overline{f_U}(x, y_1) + t\overline{f_U}(x, y_2) \leq$$

$$\leq (1-t)\liminf \overline{f_U}(u^i, y_1) + t\liminf \overline{f_U}(u^i, y_2) \leq$$

$$\leq \liminf((1-t)\overline{f_U}(u^i, y_1) + t\overline{f_U}(u^i, y_2)) \leq$$

$$\leq \liminf \overline{f_U}(u^i, y_3) = \overline{f_U}(x, y_3).$$

Hence,

$$(1-t)\overline{f_U}(x, y_1) + t\overline{f_U}(x, y_2) \leq \overline{f_U}(x, y_3) \text{ for all } x \in \operatorname{co}(U).$$

Now, Theorem 4.3.1 assures that

$$\sup_{y \in Y} \min_{x \in K} \overline{f_U}(x, y) = \min_{x \in K} \sup_{y \in Y} \overline{f_U}(x, y).$$

On the other hand, the function $g : K \longrightarrow \overline{\mathbb{R}}$, $g(x) = \sup_{y \in Y} \overline{f_U}(x, y)$ is proper, convex and lower semicontinuous as a pointwise supremum of a family of proper, convex and lower semicontinuous functions, hence according to Proposition 4.3.3 one has

$$\min_{x \in K} \sup_{y \in Y} \overline{f_U}(x, y) = \inf_{x \in U} \sup_{y \in Y} \overline{f_U}(x, y).$$

But, according to Theorem 4.3.3, we have $f(x, y) = \overline{f_U}(x, y)$ for all $x \in U$. Hence,

$$\inf_{x \in U} \sup_{y \in Y} \overline{f_U}(x, y) = \inf_{x \in U} \sup_{y \in Y} f(x, y),$$

and the conclusion follows. □

First of all we would like to emphasize that the conclusions of Theorem 4.3.4 and Theorem 4.3.5 do not remain valid if in their hypotheses we assume only that the set U is dense in K as the next example shows. Moreover, the assumptions imposed on the bifunction f in

Theorem 4.3.1 and Theorem 4.3.2 are also satisfied, however their conclusions fail. This fact shows that the general results of Fan and Sion cannot be extended on general dense sets.

Example 4.3.8. Let X be an infinite dimensional real Hilbert space. Let

$$K = Y = \{x \in X : \|x\| \leq 1\}$$

be the unit ball of X and let $U = \{x \in X : \|x\| = 1\}$. Then according to Example 4.3.2, U is dense in K with respect to the weak topology of X, but is not self-segment-dense in K. Obviously $\mathrm{co}(U) = K$ which according to Banach-Alaoglu Theorem is weakly compact. Obviously, in this case $\mathrm{co}(U)$ is segment-dense in K. Consider the function

$$f : K \times Y \longrightarrow \mathbb{R},\ f(x,y) = \langle x,y \rangle.$$

Then, it can easily be verified that the conditions in the hypotheses of Theorem 4.3.1, Theorem 4.3.2, Theorem 4.3.4 and Theorem 4.3.5 are fulfilled. Nevertheless

$$\inf_{x \in U} \sup_{y \in Y} f(x,y) = 1$$

and

$$\sup_{y \in Y} \inf_{x \in U} f(x,y) = 0.$$

As immediate consequences of Theorem 4.3.5 we have the following results.

Corollary 4.3.3. *Let K be a nonempty, compact and convex subset of the Hausdorff locally convex topological vector space X and let Y be an arbitrary nonempty set. Let $U \subseteq K$ be a self-segment-dense set. Consider the mapping $f : K \times Y \longrightarrow \mathbb{R}$, and assume that the following assumptions are fulfilled.*

(i) The map $x \longrightarrow f(x,y)$ is convex and lower semicontinuous on U for all $y \in Y$.

(ii) The map $y \longrightarrow f(x,y)$ is concavelike, for all $x \in U$.

(iii) For every $y \in Y$, $\mathrm{dom}\, \overline{f_U}(\cdot, y) = K$.

Assume further that one of the following conditions hold.

(a) $\mathrm{co}(U) = K$.

(b) For all $x \in K$ there exists $y \in \mathrm{co}(U)$, such that $[y,x) \subseteq \mathrm{co}(U)$.

(c) The interior of $\mathrm{co}(U)$ *is non-empty.*

(d) X is finite dimensional.

Then,

$$\inf_{x\in U}\sup_{y\in Y} f(x,y) = \sup_{y\in Y}\inf_{x\in U} f(x,y).$$

Proof. Observe that according to Remark 4.3.11, each of the conditions (a)-(d) assures that $\mathrm{co}(U)$ is segment-dense in K. The conclusion follows from Theorem 4.3.5. □

Corollary 4.3.4. *Let K be a nonempty and convex subset of the Hausdorff locally convex space X and let Y be an arbitrary nonempty set. Let $U \subseteq K$ be a self-segment-dense set, let $\mathcal{S} \subseteq U$ be a subset of U and assume that* $\mathrm{co}(\mathcal{S})$ *is compact. Consider further the mapping $f : K \times Y \longrightarrow \mathbb{R}$, and assume that the following assumptions are fulfilled.*

(i) The map $x \longrightarrow f(x,y)$ is convex and lower semicontinuous on $\mathrm{co}(\mathcal{S}) \cap U$ *for all $y \in Y$.*

(ii) The map $y \longrightarrow f(x,y)$ is concavelike, for all $x \in \mathrm{co}(\mathcal{S}) \cap U$.

Then,

$$\inf_{x\in \mathrm{co}(\mathcal{S})\cap U}\sup_{y\in Y} f(x,y) = \sup_{y\in Y}\inf_{x\in \mathrm{co}(\mathcal{S})\cap U} f(x,y).$$

Proof. Note that according to Lemma 4.3.2, $\mathrm{co}(\mathcal{S}) \cap U$ is self-segment-dense in $\mathrm{co}(\mathcal{S})$. Obviously $\mathrm{co}(\mathrm{co}(\mathcal{S}) \cap U) = \mathrm{co}(\mathcal{S})$, which shows that $\mathrm{co}(\mathrm{co}(\mathcal{S}) \cap U)$ is segment-dense in $\mathrm{co}(\mathcal{S})$. Hence, Theorem 4.3.5 can be applied for the function

$$\tilde{f} : \mathrm{co}(S) \times Y \longrightarrow \mathbb{R},\ \tilde{f}(x,y) = f(x,y).$$

□

We would like to emphasize that the conclusion of Corollary 4.3.4 fails even in finite dimension if in its hypothesis we replace the condition U is self-segment-dense in K by the condition that U is dense, or segment-dense in K.

Example 4.3.9. Let $K = Y = \{(x^1,x^2) \in \mathbb{R}^2 : (x^1)^2 + (x^2)^2 \leq 1\}$ be the closed unit ball of $\mathbb{R}^2$ and let $U = K \setminus \left[\left(-\frac{1}{2},0\right), \left(\frac{1}{2},0\right)\right]$. Obviously U is dense in K, and also segment-dense in the sense of The Luc, but not self-segment-dense, since for $u_1 = (-1,0), u_2 = (1,0) \in U$ one has

$$\mathrm{cl}([u_1,u_2] \cap U) = \left[(-1,0), \left(-\frac{1}{2},0\right)\left[\cup\right]\left(\frac{1}{2},0\right), (1,0)\right] \neq [u_1,u_2].$$

Consider the bifunction $f : K \times K \longrightarrow \mathbb{R}$, $f(x,y) = f((x^1,x^2),(y^1,y^2)) = x^1y^1 + x^2y^2$. Then, it is straightforward that for every subset $\mathcal{S} \subseteq U$ the conditions (i) and (ii) of Corollary 4.3.4 are satisfied. Nevertheless its conclusion fails as we will show in what follows. This is due to the fact that U is not self-segment-dense in K.

Indeed, let $\mathcal{S} = \{u_1, u_2\}$. Then,

$$\inf_{x\in \mathrm{co}(\mathcal{S})\cap U} \sup_{y\in Y} f(x,y) = \inf_{x^1\in[-1,-\frac{1}{2}[\cup]\frac{1}{2},1]} \sup_{y\in K} x^1y^1 = \inf_{x^1\in[-1,-\frac{1}{2}[\cup]\frac{1}{2},1]} |x^1| = \frac{1}{2}.$$

On the other hand,

$$\sup_{y\in Y} \inf_{x\in \mathrm{co}(\mathcal{S})\cap U} f(x,y) = \sup_{y\in K} \inf_{x^1\in[-1,-\frac{1}{2}[\cup]\frac{1}{2},1]} x^1y^1 = \sup_{y\in K} -|y^1| = 0.$$

In the next Corollary we assume that Y is finite.

Corollary 4.3.5. *Let K be a nonempty and convex subset of the Hausdorff locally convex topological vector space X and let Y be an arbitrary nonempty and finite set. Let $U \subseteq K$ be a self-segment-dense set in K. Consider further the mapping $f : K \times Y \longrightarrow \mathbb{R}$, and assume that the following assumptions are fulfilled.*

(i) The map $x \longrightarrow f(x,y)$ is convex and lower semicontinuous on U for all $y \in Y$.

(ii) The map $y \longrightarrow f(x,y)$ is concavelike, for all $x \in U$.

(iii) For every $y \in Y$, $\inf_{x\in U} f(x,y)$ is attained.

Then,

$$\inf_{x\in U} \sup_{y\in Y} f(x,y) = \sup_{y\in Y} \min_{x\in U} f(x,y).$$

Proof. The result is a consequence of Theorem 4.3.5. Let $S = \cup_{y\in Y}\{u_y\}$, where $u_y \in S_y = \{u \in U : \min_{x\in U} f(x,y) = f(u,y)\}$. Then $\mathrm{co}(S) \subseteq \mathrm{dom}\, f(\cdot,y) = K$ for all $y \in Y$. Obviously $\mathrm{co}(S)$ is compact as a convex hull of a finite set, further according to Lemma 4.3.1, $\mathrm{co}(S)\cap U$ is self-segment-dense in $\mathrm{co}(S)$. Let $\tilde{f}$ be the restriction of f on $\mathrm{co}(S)\times Y$, that is

$$\tilde{f} : \mathrm{co}(S)\times Y \longrightarrow \mathbb{R},\ \tilde{f}(x,y) = f(x,y).$$

Then, Theorem 4.3.5 can be applied, hence

$$\inf_{x\in \mathrm{co}(S)\cap U} \sup_{y\in Y} \tilde{f}(x,y) = \sup_{y\in Y} \inf_{x\in \mathrm{co}(S)\cap U} \tilde{f}(x,y).$$

In other words

$$\inf_{x\in \mathrm{co}(S)\cap U}\sup_{y\in Y} f(x,y) = \sup_{y\in Y}\inf_{x\in \mathrm{co}(S)\cap U} f(x,y).$$

But, by construction of S one has that for every $y \in Y$, $\min_{x\in U} f(x,y) = \inf_{x\in \mathrm{co}(S)\cap U} f(x,y)$, hence

$$\sup_{y\in Y}\min_{x\in U} f(x,y) = \sup_{y\in Y}\inf_{x\in \mathrm{co}(S)\cap U} f(x,y) = \inf_{x\in \mathrm{co}(S)\cap U}\sup_{y\in Y} f(x,y) \geq \inf_{x\in U}\sup_{y\in Y} f(x,y).$$

Since $\inf_{x\in U}\sup_{y\in Y} f(x,y) \geq \sup_{y\in Y}\inf_{x\in U} f(x,y)$ always holds the conclusion follows. □

Observe that in the previous result we assumed that Y is finite. This fact assured that the set $\mathrm{co}(S)$ is compact. Note that Theorem 4.3.5 is a great theoretical result, nevertheless the segment-dense requirement of $\mathrm{co}(U)$ in some cases might be restrictive. In what follows we present a minimax result on general dense sets, where we do not assume that $\mathrm{co}(U)$ is segment-dense. However, we have to consider some quite strong conditions imposed to the bifunction that describes the minimax problem. Further, observe that if we replace the lower semicontinuity assumption of the maps $x \longrightarrow f(x,y)$ on U for all $y \in Y$ in the hypothesis of Theorem 4.3.5 by their continuity assumption on K, then Corollary 4.3.1 assures that $\inf_{x\in U} f(x,y) = \inf_{x\in K} f(x,y)$ for all $y \in Y$. Therefore, one can renounce to the assumption U is self-segment-dense and $\mathrm{co}(U)$ is segment-dense in K, in the hypothesis of Theorem 4.3.5, provided the condition (i) is replaced by the condition: the map $x \longrightarrow f(x,y)$ is convex and continuous on K for all $y \in Y$. In this case due to the continuity of f in its first variable, the condition (ii) becomes: the map $y \longrightarrow f(x,y)$ is concavelike, for all $x \in K$. Further, we will need an extra assumption in order to assure the continuity of the function $g(x) = \sup_{y\in Y} f(x,y)$.

For $K \subseteq X$ let us denote by $C(K)$ the space of continuous real valued functions, that is $C(K) = \{f : K \to \mathbb{R} : f \text{ continuous }\}$. A subset $S \subseteq C(K)$ is said to be equicontinuous if for every $x \in K$ and every $\varepsilon > 0$, x has a neighborhood U_x such that $\forall y \in U_x \cap K, \forall f \in S, |f(y) - f(x)| < \varepsilon$. A set $S \subseteq C(K)$ is said to be pointwise bounded if for every $x \in K$, $\sup_{f\in S} |f(x)| < \infty$. If K is also compact, we can endow $C(K)$ with the uniform norm, $\|f\| = \sup_{x\in K} |f(x)|$. In case K is compact, the Arzelà-Ascoli theorem [85], affirms that a subset S of $C(K)$ is relatively compact in the topology induced by the uniform norm of $C(K)$, if and only if it is equicontinuous and pointwise bounded. These concepts allow us to obtain a result in which we do not assume the self-segment-denseness of U or the segment-denseness of $\mathrm{co}(U)$.

Theorem 4.3.6. *Let K be a nonempty, compact and convex subset of the Hausdorff locally convex topological vector space X and let Y be an arbitrary nonempty set. Let $U \subseteq K$ be a dense set in K. Consider further the mapping $f : K \times Y \longrightarrow \mathbb{R}$, and assume that the following assumptions are fulfilled.*

(i) The map $x \longrightarrow f(x,y)$ is convex on K for all $y \in Y$.

(ii) The map $y \longrightarrow f(x,y)$ is concavelike, for all $x \in K$.

(iii) The family $(f(\cdot,y))_{y\in Y}$ is an equicontinuous family of $C(K)$.

(iv) For all $x \in K$ one has $\sup_{y\in Y} f(x,y) < \infty$.

Then,

$$\inf_{x\in U} \sup_{y\in Y} f(x,y) = \sup_{y\in Y} \inf_{x\in U} f(x,y).$$

Proof. Note that condition (iii) ensures that $x \longrightarrow f(x,y)$ is continuous on K for all $y \in Y$. In virtue of Corollary 4.3.1, one has

$$\sup_{y\in Y} \inf_{x\in U} f(x,y) = \sup_{y\in Y} \min_{x\in K} f(x,y). \tag{4. 8}$$

Theorem 4.3.1 assures that

$$\sup_{y\in Y} \min_{x\in K} f(x,y) = \min_{x\in K} \sup_{y\in Y} f(x,y). \tag{4. 9}$$

On the other hand, the function $g : K \longrightarrow \overline{\mathbb{R}}$, $g(x) = \sup_{y\in Y} f(x,y)$ is proper, convex and lower semicontinuous as a pointwise supremum of a family of proper, convex and lower semicontinuous functions. From (iv) we have that g is not extended valued, that is $g(x) \in \mathbb{R}$ for all $x \in K$. We show that g is continuous on K. Indeed, let $x_0 \in K$. Since the family $(f(\cdot,y))_{y\in Y}$ is equicontinuous one has that for every $\varepsilon > 0$ there exists U_0 a neighbourhood of x_0, such that, for all $x \in U_0 \cap K$ and for all $y \in Y$ one has

$$|f(x,y) - f(x_0,y)| < \frac{\varepsilon}{2}. \tag{4. 10}$$

Let us fix $\varepsilon > 0$ and let $x_1 \in U_0 \cap K$. Then by the definition of supremum, there exist $y_0, y_1 \in Y$ such that $f(x_0,y_0) + \frac{\varepsilon}{2} > g(x_0)$ and $f(x_1,y_1) + \frac{\varepsilon}{2} > g(x_1)$. Obviously, $g(x_0) \geq f(x_0,y_1)$ and $g(x_1) \geq f(x_1,y_0)$. Hence,

$$f(x_1,y_0) - f(x_0,y_0) - \frac{\varepsilon}{2} \leq g(x_1) - g(x_0) \leq f(x_1,y_1) - f(x_0,y_1) + \frac{\varepsilon}{2},$$

and using (4. 10), one gets

$$|g(x_1) - g(x_0)| \leq \varepsilon.$$

According to Corollary 4.3.1, one has $\inf_{x\in U} g(x) = \min_{x\in K} g(x)$. In other words,

$$\min_{x\in K} \sup_{y\in Y} f(x,y) = \inf_{x\in U} \sup_{y\in Y} f(x,y). \tag{4. 11}$$

The conclusion of theorem follows from (4. 8), (4. 9) and (4. 11). □

Remark 4.3.16. Example 4.3.8 becomes again a counterexample for Theorem 4.3.6. However, note that in this case the contradiction is not provided by the fact that the set U considered is not self-segment-dense, but by the fact that the corresponding family of functions, $(f(\cdot,y))_{y\in Y} = (\langle\cdot,y\rangle)_{y\in B}$ is not an equicontinuous family in the weak topology of X.

4.3.6 Dense families of functionals

In this paragraph we apply our minimax results in order to prove the denseness of some family of functionals in $C(Y)$ and $B(Y)$, where $C(Y)$ is the space of continuous functions on Y endowed with the uniform norm and $B(Y)$ is the space of bounded functions on Y endowed with the topology of uniform norm. We show that also here the use of the concept of a self-segment-dense set is essential. Finally, we show that James' famous reflexivity result can be treated as a density problem in a parameterized family of functionals.

Remark 4.3.17. Consider the bounded mapping $f : K \times Y \longrightarrow \mathbb{R}$.

Note that in case $\{f(\cdot,y)\}_{y\in Y}$ is an equicontinuous family of $C(K)$, then for any dense set $U \subseteq K$ one has that

$$\forall k \in K, \forall \varepsilon > 0, \exists u \in U \text{ such that } \sup_{y\in Y} |f(k,y) - f(u,y)| < \varepsilon.$$

In other words, $\|f(u,\cdot) - f(k,\cdot)\| < \varepsilon$ in the uniform norm of $B(Y)$. Consequently, the family of functionals $\{f(u,\cdot)\}_{u\in U}$ is dense in $\{f(x,\cdot)\}_{x\in K}$.

The next abstract theorem provides the denseness of the set of functions $\{f(u,\cdot)\}_{u\in U}$ in $\{f(x,\cdot)\}_{x\in K}$ in $B(Y)$, without the assumption of equicontinuity of the family $\{f(\cdot,y)\}_{y\in Y}$.

Theorem 4.3.7. *Let K be a nonempty subset of the Hausdorff locally convex topological vector space X and let Y be a nonempty subset of a locally convex topological vector space. Let $U \subseteq K$ be a dense set in K. Consider further the bounded mapping $f : K \times Y \longrightarrow \mathbb{R}$, and assume that the following assumptions are fulfilled.*

(i) The map $x \longrightarrow f(x,y)$ is continuous on K for all $y \in Y$.

(ii) The map $y \longrightarrow f(x,y)$ is continuous, for all $x \in K$.

(iii) For every $k \in K$, $\sup_{y\in Y} \inf_{x\in U} |f(x,y) - f(k,y)| = \inf_{x\in U} \sup_{y\in Y} |f(x,y) - f(k,y)|$.

Then, $\{f(u,\cdot)\}_{u\in U}$ is dense in $\{f(x,\cdot)\}_{x\in K} \subseteq B(Y)$.

Proof. Let $k \in K$. We show that for all $\varepsilon > 0$ there exists $u \in U$ such that

$$\sup_{y\in Y} |f(u,y) - f(k,y)| < \varepsilon.$$

In other words, $\|f(u,\cdot) - f(k,\cdot)\| < \varepsilon$ in the uniform norm of $B(Y)$.

This means that $\{f(u,\cdot)\}_{u\in U}$ is dense in $\{f(x,\cdot)\}_{x\in K}$.

Let $\varepsilon > 0$ and $y_0 \in Y$ be fixed. Since the mapping $x \longrightarrow f(x,y_0)$ is continuous there exists an open neighbourhood V of k such that for all $x \in V$ one has

$$|f(x,y_0) - f(k,y_0)| < \frac{\varepsilon}{8}.$$

Since U is dense in K, there exists $u_0 \in V \cap U$ such that

$$|f(u_0,y_0) - f(k,y_0)| < \frac{\varepsilon}{8}. \tag{4. 12}$$

On the other hand the mapping $y \longrightarrow f(u_0,y)$ is continuous at y_0, hence there exists V_0 an open convex neighbourhood of y_0 such that for all $y \in V_0$ one has

$$|f(u_0,y) - f(u_0,y_0)| < \frac{\varepsilon}{8}. \tag{4. 13}$$

It can easily be observed, that (4.12) and (4.13) lead to

$$|f(u_0,y) - f(k,y_0)| < \frac{\varepsilon}{4}, \forall y \in V_0. \tag{4. 14}$$

Finally, the mapping $y \longrightarrow f(k,y)$ is continuous at y_0, hence there exists V_0' an open convex neighbourhood of y_0 such that for all $y \in V_0'$ one has

$$|f(k,y) - f(k,y_0)| < \frac{\varepsilon}{4}. \tag{4. 15}$$

From (4.14) and (4.15) we obtain, that for all $y \in V(y_0) = V_0 \cap V_0'$ one has

$$|f(u_0,y) - f(k,y)| < \frac{\varepsilon}{2}. \tag{4. 16}$$

In other words, for every $y_0 \in Y$ there exist $V(y_0)$, an open and convex neighbourhood of y_0, and $u_0 \in U$ such that

$$\sup_{y \in V(y_0)} |f(u_0, y) - f(k, y)| \leq \frac{\varepsilon}{2}.$$

Obviously $\cup_{y_0 \in Y} V(y_0)$ is an open cover of the set Y.

Note that for every $y_0 \in Y$ there exists $V(y_0)$ such that $y_0 \in V(y_0)$, hence

$$\inf_{u \in U} |f(u, y_0) - f(k, y_0)| \leq |f(u_0, y_0) - f(k, y_0)| \leq \frac{\varepsilon}{2}.$$

Thus,

$$\sup_{y \in Y} \inf_{u \in U} |f(u, y) - f(k, y)| \leq \frac{\varepsilon}{2} < \varepsilon.$$

The latter relation combined with (iii) leads to

$$\inf_{u \in U} \sup_{y \in Y} |f(u, y) - f(k, y)| < \varepsilon.$$

Consequently, there exists $u^* \in U$ such that

$$\sup_{y \in Y} |f(u^*, y) - f(k, y)| < \varepsilon.$$

□

When Y is compact the following density result holds in $C(Y)$.

Theorem 4.3.8. *Let K be a nonempty convex subset of the Hausdorff locally convex topological vector space X and let Y be a compact and symmetric subset of a locally convex topological vector space. Let $U \subseteq K$ be a self-segment-dense set in K. Consider further the mapping $f : K \times Y \longrightarrow \mathbb{R}$, and assume that the following assumptions are fulfilled.*

(i) The map $x \longrightarrow f(x, y)$ is convex on U and continuous on K for all $y \in Y$.

(ii) The map $y \longrightarrow f(x, y)$ is affine and continuous, for all $x \in K$.

Then, $\{f(u, \cdot)\}_{u \in U}$ is dense in $\{f(x, \cdot)\}_{x \in K} \subseteq C(Y)$.

Proof. Let $k \in K$ and let $\varepsilon > 0$. As in the proof of Theorem 4.3.7 one can show that for every $y_0 \in Y$ there exist $V(y_0)$, an open and convex neighbourhood of y_0, and $u_0 \in U$ such that

$$\sup_{y \in V(y_0)} |f(u_0, y) - f(k, y)| \leq \frac{\varepsilon}{2}.$$

Obviously $\cup_{y_0\in Y}V(y_0)$ is an open cover of the compact set Y, hence it contains a finite subcover. In other words, there exist $y_1,...,y_n \in Y$ and $u_1,...,u_n \in U$ such that $Y = \cup_{i=1}^n V(y_i)$ and

$$\sup_{y\in V(y_i)} |f(u_i,y) - f(k,y)| \leq \frac{\varepsilon}{2} \text{ for all } i \in \{1,2,...,n\}.$$

Note that for every $y_0 \in Y$ there exists $V(y_j)$ such that $y_0 \in V(y_j)$. Further, it is obvious that

$$\inf_{u\in\{u_1,...,u_n\}} |f(u,y_0) - f(k,y_0)| \leq |f(u_j,y_0) - f(k,y_0)| \leq \frac{\varepsilon}{2}$$

hence,

$$\sup_{y\in Y}\inf_{u\in\{u_1,...,u_n\}} |f(u,y) - f(k,y)| \leq \frac{\varepsilon}{2}.$$

The latter relation leads to

$$\sup_{y\in Y}\inf_{u\in \mathrm{co}\{u_1,...,u_n\}\cap U} (f(u,y) - f(k,y)) \leq \frac{\varepsilon}{2}.$$

Obviously $\mathrm{co}\{u_1,...,u_n\}$ is compact and according to Lemma 4.3.1, the intersection

$$\mathrm{co}\{u_1,...,u_n\} \cap U$$

is self-segment-dense in $\mathrm{co}\{u_1,...,u_n\}$. We show that Corollary 4.3.1 can be applied to the function $g : K \times Y \longrightarrow \mathbb{R}$, $g(x,y) = f(x,y) - f(k,y)$. Indeed, the mapping $x \longrightarrow g(x,y)$ is convex on U and continuous on K for all $y \in Y$ hence it is also convex on $\mathrm{co}\{u_1,...,u_n\} \cap U$, (since U is self-segment-dense in K), and lower semicontinuous on $\mathrm{co}\{u_1,...,u_n\}$ for all $y \in Y$.

We show that the map $y \longrightarrow g(x,y)$ is concave (hence also concavelike) for all $x \in \mathrm{co}\{u_1,...,u_n\} \cap U$.

Indeed, let $x \in \mathrm{co}\{u_1,...,u_n\} \cap U$. According to the hypothesis of the theorem, the mapping $y \longrightarrow f(x,y)$ is affine, hence for every $y_1, y_2 \in Y$ and $t \in [0,1]$ one has

$$g(x,(1-t)y_1 + ty_2) = f(x,(1-t)y_1 + ty_2) - f(k,(1-t)y_1 + ty_2) =$$

$$(1-t)(f(x,y_1) - f(k,y_1)) + t(f(x,y_2) - f(k,y_2)) = (1-t)g(x,y_1) + tg(x,y_2).$$

By applying Corollary 4.3.4, we obtain

$$\inf_{u\in \mathrm{co}\{u_1,...,u_n\}\cap U}\sup_{y\in Y}(f(u,y) - f(k,y)) = \sup_{y\in Y}\inf_{u\in \mathrm{co}\{u_1,...,u_n\}\cap U}(f(u,y) - f(k,y)) \leq$$

$$\leq \frac{\varepsilon}{2} < \varepsilon.$$

Hence, there exists $u^* \in \mathrm{co}\{u_1, ..., u_n\} \cap U$ such that

$$\sup_{y \in Y}(f(u^*, y) - f(k, y)) < \varepsilon.$$

Conversely, since Y is symmetric and f is affine in the second variable we have

$$\sup_{y \in Y}(-f(u^*, y) + f(k, y)) = \sup_{y \in Y}(f(u^*, -y) - f(k, -y))$$

and

$$\sup_{y \in Y}(f(u^*, -y) - f(k, -y)) = \sup_{y \in Y}(f(u^*, y) - f(k, y)) < \varepsilon.$$

Hence,

$$\sup_{y \in Y}|f(u^*, y) - f(k, y)| < \varepsilon.$$

□

In the next result we drop the compactness assumption on Y, but we assume instead some minimax results.

Theorem 4.3.9. *Let K be a nonempty convex subset of the Hausdorff locally convex topological vector space X and let Y be a closed convex bounded and symmetric subset of a locally convex topological vector space. Let $U \subseteq K$ be a self-segment-dense set in K. Consider further the bounded mapping $f : K \times Y \longrightarrow \mathbb{R}$, and assume that the following assumptions are fulfilled.*

(i) The map $x \longrightarrow f(x, y)$ is convex on U and continuous on K for all $y \in Y$.

(ii) The map $y \longrightarrow f(x, y)$ is affine and continuous on Y for all $x \in K$.

(iii) $\inf_{x \in U} \sup_{y \in Y_0} f(x, y) = \sup_{y \in Y_0} \inf_{x \in U} f(x, y)$, for every closed convex subset $Y_0 \subseteq Y$.

Then, $\{f(u, \cdot)\}_{u \in U}$ is dense in $\{f(x, \cdot)\}_{x \in K} \subseteq B(Y)$.

Proof. Let $k \in K$. Note at first that $\inf_{u \in U} f(u, y) \leq f(k, y)$ for all $y \in Y$. Indeed, since U is dense in K for $y \in Y$ fixed, consider a net (u_i) converging to k. Then, from (i) we obtain $\lim f(u_i, y) \to f(k, y)$, hence $\inf_{u \in U} f(u, y) \leq \lim f(u_i, y) \leq f(k, y)$.

Let $\varepsilon > 0$ be fixed. Assume that $\sup_{y\in Y} f(k,y) = M$ and $\inf_{y\in Y} f(k,y) = m$ and consider $n \in \mathbb{N}$ such that $\varepsilon > \frac{M-m}{n}$. For every $i \in \{1,2,...,n\}$ consider the set

$$Y_i = f^{-1}\left(k, \left[m + (i-1)\frac{M-m}{n}, m + i\frac{M-m}{n}\right]\right).$$

Then from (ii) the set Y_i is closed and convex for all $i \in \{1,2,...,n\}$ and obviously $\cup_{i=1}^{n} Y_i = Y$.

We apply (iii) for the sets Y_i, $i \in \{1,2,...,n\}$. Hence,

$$\inf_{x\in U} \sup_{y\in Y_i} f(x,y) = \sup_{y\in Y_i} \inf_{x\in U} f(x,y), \text{ for all } i \in \{1,2,...,n\}.$$

We have $\sup_{y\in Y_i} \inf_{u\in U} f(u,y) \leq \sup_{y\in Y_i} f(k,y) < \varepsilon + \inf_{y\in Y_i} f(k,y)$, for all $i \in \{1,2,...,n\}$, hence

$$\inf_{x\in U} \sup_{y\in Y_i} f(x,y) < \varepsilon + \inf_{y\in Y_i} f(k,y), \text{ for all } i \in \{1,2,...,n\}.$$

But then, for all $i \in \{1,2,...,n\}$ there exists $u_i \in U$ such that

$$\sup_{y\in Y_i} f(u_i,y) < \varepsilon + \inf_{y\in Y_i} f(k,y).$$

From the latter relation we get

$$\sup_{y\in Y_i}(f(u_i,y) - f(k,y)) \leq \sup_{y\in Y_i} f(u_i,y) - \inf_{y\in Y_i} f(k,y) < \varepsilon, \text{ for all } i \in \{1,2,...,n\}.$$

Obviously for every $y \in Y$ there exists $i \in \{1,2,...,n\}$ such that $y \in Y_i$, hence

$$\sup_{y\in Y} \inf_{u\in\{u_1,u_2,...,u_n\}} (f(u,y) - f(k,y)) < \varepsilon.$$

On the other hand

$$\inf_{u\in\{u_1,u_2,...,u_n\}} (f(u,y) - f(k,y)) \geq \inf_{u\in \mathrm{co}\{u_1,u_2,...,u_n\}\cap U} (f(u,y) - f(k,y)),$$

consequently

$$\sup_{y\in Y} \inf_{u\in \mathrm{co}\{u_1,u_2,...,u_n\}\cap U} (f(u,y) - f(k,y)) < \varepsilon.$$

We show as in the proof of Theorem 4.3.8, that Corollary 4.3.4 can be applied to the function $g : K \times Y \longrightarrow \mathbb{R}$, $g(x,y) = f(x,y) - f(k,y)$.

By applying Corollary 4.3.4, we obtain

$$\inf_{u\in \mathrm{co}\{u_1,\dots,u_n\}\cap U}\sup_{y\in Y}(f(u,y)-f(k,y)) = \sup_{y\in Y}\inf_{u\in \mathrm{co}\{u_1,\dots,u_n\}\cap U}(f(u,y)-f(k,y)) < \varepsilon.$$

Hence, there exists $u^* \in \mathrm{co}\{u_1,\dots,u_n\}\cap U$ such that

$$\sup_{y\in Y}(f(u^*,y)-f(k,y)) < \varepsilon.$$

Conversely, since Y is symmetric and f is affine in the second variable we have

$$\sup_{y\in Y}(-f(u^*,y)+f(k,y)) = \sup_{y\in Y}(f(u^*,-y)-f(k,-y))$$

and

$$\sup_{y\in Y}(f(u^*,-y)-f(k,-y)) = \sup_{y\in Y}(f(u^*,y)-f(k,y)) < \varepsilon.$$

Hence,

$$\sup_{y\in Y}|f(u^*,y)-f(k,y)| < \varepsilon.$$

□

Remark 4.3.18. We would like to emphasize that the self-segment-dense property of U in the hypotheses of Theorem 4.3.8 and Theorem 4.3.9 is essential and cannot be replaced by its denseness. Indeed, let X be an infinite dimensional real Hilbert space. Let $K = Y = \{x \in X : \|x\| \leq 1\}$ be the unit ball of X and let $U = \{x \in X : \|x\| = 1\}$. Then according to Example 4.3.2, U is dense in K with respect to the weak topology of X, but is not self-segment-dense in K. Obviously $K = Y$ is weakly compact. Consider the function

$$f : K \times Y \longrightarrow \mathbb{R},\ f(x,y) = \langle x,y\rangle.$$

Then, it can easily be verified that the conditions (i) and (ii) in the hypotheses of Theorem 4.3.8 and Theorem 4.3.9 are fulfilled. Observe further, that (iii) in the hypothesis of Theorem 4.3.9 also holds, since for every weakly closed convex subset $Y_0 \subseteq Y$ Theorem 4.3.1, can be applied for the function $f(x,y) = \langle x,y\rangle$.

Now, the family $\{f(u,\cdot)\}_{u\in U}$ is not dense in $\{f(x,\cdot)\}_{x\in K} \subseteq C(Y)$, since for $k = 0 \in K$ one has

$$\sup_{y\in Y}|f(u,y)-f(k,y)| = \sup_{y\in Y}|\langle u,y\rangle| = 1,\ \text{for all } u \in U.$$

Corollary 4.3.6. *Let K be a nonempty, compact and convex subset of the Hausdorff locally convex topological vector space X and let Y be a closed convex bounded and symmetric subset of a locally convex topological vector space. Let $U \subseteq K$ be a self-segment-dense set in K and suppose that $\mathrm{co}(U)$ is segment-dense in K. Consider further the bounded mapping $f : K \times Y \longrightarrow \mathbb{R}$, and assume that the following assumptions are fulfilled.*

(i) The map $x \longrightarrow f(x,y)$ is convex and continuous on K for all $y \in Y$.

(ii) The map $y \longrightarrow f(x,y)$ is affine and continuous on Y for all $x \in K$.

Then, $\{f(u,\cdot)\}_{u\in U}$ is dense in $\{f(x,\cdot)\}_{x\in K} \subseteq B(Y)$, where $B(Y)$ is the space of bounded functions on Y endowed with the topology of uniform norm.

Proof. The conclusion follows via Theorem 4.3.5, which assures that the condition (iii) in Theorem 4.3.9 is satisfied. □

Corollary 4.3.7. (James' Theorem) *Let X be a Banach space and let X^* be the dual of X. Then X is reflexive if and only if every $x^* \in X^*$ attains its norm on the close unit ball of X.*

Proof. The implication " $\Rightarrow$ " is straightforward. Indeed, if X is reflexive, then B is weakly compact, hence by Weierstrass theorem $\sup_{x\in B}\langle x^*,x\rangle$ is attained.

Moreover, for the converse implication it is enough to assume that for all $x^* \in B^* = \{x^* \in X^* : \|x^*\| \leq 1\}$, there exists $x_0 \in B$ such that $\|x^*\| = \langle x^*,x_0\rangle$, where B^* is the closed unit ball of X^*. Indeed, for $x^* \in X^* \setminus B^*$ one has $\frac{x^*}{\|x^*\|} \in B^*$, hence $\left\langle \frac{x^*}{\|x^*\|},x_0\right\rangle = \left\|\frac{x^*}{\|x^*\|}\right\| = 1$, for some $x_0 \in B$.

Let B^{**} be the unit ball of X^{**} and consider $U = \hat{B} \subseteq X^{**}$, where $\hat{B}$ is the canonical embedding of B in X^{**}. Then, by Goldstine theorem U is dense in B^{**} in the w^*-topology of X^{**} and since is convex, U is self-segment-dense in B^{**}. Consider the bifunction

$$f : B^{**} \times B^* \longrightarrow \mathbb{R},\ f(x^{**},x^*) = \langle x^{**},x^*\rangle.$$

From Simons' minimax theorem [193, 195] it follows that

$$\inf_{u\in U}\sup_{x^*\in B^*} f(u,x^*) = \sup_{x^*\in B^*}\inf_{u\in U} f(u,x^*).$$

Hence, by Theorem 4.3.9 one has that $\{f(u,\cdot)\}_{u\in U}$ is dense in $\{f(x^{**},\cdot)\}_{x^{**}\in B^{**}} \subseteq C(B^*)$, where $C(B^*)$ is endowed with the topology of uniform norm. In other words, for every $\varepsilon > 0$ and $x^{**} \in B^{**}$ there exists a $u \in U$ such that

$$\sup_{x^*\in B^*} |f(u,x^*) - f(x^{**},x^*)| < \varepsilon.$$

Equivalently,

$$\sup_{x^* \in B^*} |\langle u - x^{**}, x^* \rangle| = \|u - x^{**}\| < \varepsilon.$$

Hence U is dense B^{**} in the strong topology of X^{**}. But $\operatorname{cl} U = U$ in the strong topology, hence $U = B^{**}$. But this implies that $J(X) = X^{**}$, where $J(X)$ is the canonical embedding of X in X^{**}, that is, X is reflexive. $\square$

Index of notation

$\forall$ for all

$\exists$ there exists (at least one)

$\mathbb{N}$ the set of positive integers $\{1,2,...\}$

$\mathbb{R}$ the set of real numbers

(a,b), $]a,b[$ the open interval (open line segment) with the endpoints a and b, respectively

$[a,b)$, $[a,b[$ the left-closed and right-open interval with the endpoints a and b, respectively

$(a,b]$, $]a,b]$ the left-open and right-closed interval with the endpoints a and b, respectively

$[a,b]$ the closed interval (closed line segment) with the endpoints a and b, respectively

$\mathbb{R}_+$ the interval $(0,+\infty)$

$\overline{\mathbb{R}}$ the extended set of real numbers

$D(T)$ the domain of the operator T

$R(T)$ the range of the operator T

$G(T)$ the graph of the operator T

$\mathrm{co}(U)$ the convex hull of the set U

$\mathrm{cone}(U)$ the conic hull of the set U

$\mathrm{coneco}(U)$ the convex conic hull of the set U

$\mathrm{int}(U)$ the interior of the set U

$\mathrm{ri}(U)$ the relative interior of the set U

$\mathrm{core}(U)$ the algebraic interior of the set U

$\mathrm{cl}(U)$ the closure of the set U

$\mathrm{aff}(U)$ the affine hull of the set U

$\mathrm{lin}(U)$, $\mathrm{span}U$ the linear subspace generated by the set U

$\mathrm{dom}\, f$ the domain of the function f

$\mathrm{epi}\, f$ the epigraph of the function f

$\mathrm{co}\, f$ the convex hull of the function f

$\mathrm{cl}\, f$, $\overline{f}$ the lower semicontinuous hull of the function f

f^* the Fenchel-Moreau conjugate of the function f

$\langle \cdot, \cdot \rangle$ the bilinear pairing between two vector spaces which are in duality

$\| \cdot \|$ the norm of a normed space

$(dT)_x$ the Fréchet differential of the operator T in x

∇f the Fréchet differential (gradient operator) of the function f

∂f the (convex) subdifferential of the function f

δ_U the indicator function of the set U

σ_U the support function of the set U

$N_U(x)$ the normal cone to the set U at $x \in U$

X^* the topological dual space of the topological vector space X

X^{**} the topological bidual space of the topological vector space X

$w(X, X^*)$ the weak topology on X induced by X^*

$w^*(X^*, X)$ the weak* topology on X^* induced by X

$w^*(X^{**}, X^*)$ the weak* topology on X^{**} induced by X^*

$S : X \rightrightarrows Y$ a set valued operator from X to Y

φ_S the Fitzpatrick function of the monotone operator $S : X \rightrightarrows X^*$

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