TOPICS IN VARIATIONAL ANALYSIS VIA MONOTONICITY AND CONVEXITY

Summary

This thesis is built on some significant results achieved by the author after obtaining his Ph.D. degree in Mathematics at Babeş-Bolyai University, in 2011. The work contains four chapters consisting of twelve thematically linked sections. Every section is based on a published paper of the author. Therefore, for the readers convenience, every section can be read as it, without the previous read of any other section. We treat some of the most important problems of the variational analysis, that is, optimization, variational inequalities, equilibrium problems and minimax problems. Further, our results rely on the core concepts of variational analysis, namely, monotonicity and its generalizations and convexity and its extensions, respectively.

Recall that the concept of monotonicity for operators defined on a Banach space into its dual has been introduced some fifty years ago by the celebrated works of Browder and Minty. This notion have shown to be a cornerstone for the development of variational analysis, due to the fact that convexity of a proper, lower semicontinuous function can be characterized by the maximal monotonicity of its subdifferential.

During the last decades, the concept of classical monotonicity has imposed itself, due to its importance, and influenced some other branches of mathematics, such as differential equations or image processing, as well as economics, engineering, management science, probability theory and other applied sciences. Due to these interactions the concepts of monotonicity alongside with convexity were subjects of a dynamical evolution reflected in a number of new concepts - extensions of the classical assumption of monotonicity and convexity without the loss of valuable properties.

We use the concepts of monotonicity and convexity in order to obtain global injectivity results and to show solution existence of several variational inequalities and some scalar and vector equilibrium problems. Further, by using these concepts, we obtain some new minimax results on dense sets and via discrete and continuous dynamical systems we approximate the minimizer of an objective function having a complex structure.

Chapter 1 deals with local (generalized) monotonicities and local (generalized) convexities on special dense sets. Further some injectivity results are also obtained.

In the first section we continue the investigations started in G. Kassay, C. Pintea, S. László, *Monotone operators and closed countable sets*, Optimization **60**, 1059-1069 (2011), and improve the results obtained there. We manage to prove here that the local monotonicity of a single-valued operator assumed just on residual subsets of the source set is enough to guarantee the global monotonicity of that operator and the convexity of the inverse images therefore. As a consequence one can deduce that the local monotonicity on residual subsets of local homeomorphisms ensures their global injectivity. We pay some special attention to the residual sets arising as complements of σ-affine sets, σ-compact sets and σ-algebraic varieties. We also show that the obtained results cannot be extended to the set-valued monotone operators. However, in case of generalized monotonicities, much stronger results than those obtained in S. László, *Generalized monotone operators, generalized convex functions and closed countable sets*, J. Convex Anal. **18**, 1075-1091 (2011), can be provided even in the set-valued case. Indeed, in the second section we show that the local generalized monotonicity of a lower semicontinuous set-valued operator on some certain type of dense sets ensures the global generalized monotonicity of that operator. We achieve this goal gradually by showing at first that the lower semicontinuous set-valued functions of one real variable, which are locally generalized monotone on a dense subsets of their domain are globally generalized monotone. Then, these results are extended to the case of set-valued operators on arbitrary Banach spaces. We close this section with some results on the global generalized convexity of a real valued function, which is obtained out of its local counterpart on some dense sets. In the last section we provide sufficient conditions that ensure the convexity of the inverse images of an operator, monotone in some sense. Further, conditions that ensure the monotonicity and the local injectivity of an operator are also obtained. Combining the conditions that provide the local injectivity and the convexity of the inverse images of an operator, we are able to obtain some global injectivity results. As applications some new analytical conditions that assure the injectivity and univalency, respectively, of a complex function of one complex variable are obtained. We also show that some classical results, such as Alexander-Noshiro-Warschawski and Wolff theorem or Mocanu theorem are particular instances of our results.

In Chapter 2 we deal with the minimization problem of the sum of two functions, both in convex and nonconvex setting.

In the first section we propose a forward-backward proximal-type algorithm with inertial/memory effects for minimizing the sum of a nonsmooth function with a smooth one in the nonconvex setting. Every sequence of iterates generated by the algorithm converges to a critical point of the objective function provided an appropriate regularization of the objective satisfies the Kurdyka-Łojasiewicz inequality, which is for instance fulfilled for semi-algebraic functions. We illustrate the theoretical results by considering two numerical experiments: the first one concerns the ability of recovering the local optimal solutions of nonconvex optimization problems, while the second one refers to the restoration of a noisy blurred image. Further, in the second section, we consider a second order dynamical system of the from $\ddot{x}(t)+\gamma(t)\dot{x}(t)+x(t)-J_{\lambda(t)A}(x(t)-\lambda(t)D(x(t))-\lambda(t)\beta(t)B(x(t)))=0$, where $A:\mathcal{H}\rightrightarrows\mathcal{H}$ is a maximal monotone operator, $J_{\lambda(t)A}:\mathcal{H}\longrightarrow\mathcal{H}$ is the resolvent operator of $\lambda(t)A$, $D,B:\mathcal{H}\to\mathcal{H}$ are cocoercive operators defined on a real Hilbert space $\mathcal{H}$, $\lambda,\beta:[0,+\infty)\to[0,+\infty)$ are relaxation functions and $\gamma:[0,+\infty)\to[0,+\infty)$ a damping function, all depending on time. We show the existence and uniqueness of strong global solutions in the framework of the Cauchy-Lipschitz-Picard Theorem and prove ergodic asymptotic convergence for the generated trajectories to a zero of the operator $A+D+N_C$, where $C=\operatorname{zer}(B)$ and N_C is the normal cone operator, by using Lyapunov analysis combined with the celebrated Opial Lemma in its ergodic continuous version. Furthermore, we show the strong convergence of trajectories to the unique zero of $A+D+N_C$ in case A is a strongly monotone operator. The framework allows to address as particular case the minimization of the sum of a nonsmooth convex function with a smooth convex one and allows us to recover and improve several results from the literature.

Chapter 3 deals with the solution existence of variational inequalities. We treat both the cases where the operator involved is single valued and set valued. We apply these results in order to obtain new (unknown) coincidence point results for two operators and new fixed point results, respectively.

In the first section we introduce some new type of operators that generalize the notion of operator of type ql, (an extension of quasilinear functions and affine operators, respectively), that can be view as an extension of the monotonicity property of real valued functions. An example of operator belonging to this class, that is not of type ql, is also provided. Further, we give some sufficient conditions that ensure the existence of the solutions for an extended general variational inequality. We also show by an example that these results fail outside of the class of operators introduced in this section. Finally, as application, based on the existence results of the solutions for the extended general variational inequalities established before, we obtain a coincidence point result in Hilbert spaces. While existence results of the solution for the classical Stampacchia variational inequalities were abundant in the last years this is not the case of general variational inequality, respectively of multivalued variational inequality. Some variants of the general variational inequality problem, respectively the multivalued variational inequality problem have been studied in a Banach space context and several existence results of the solution for these problems were established in the case when one of the operators involved is of type ql and the operators involved possess some continuity properties. Moreover, it has been shown by examples that the existence results of solution for these problems, obtained in the papers mentioned above, fail outside of the class of ql type operators. The second and third section of this chapter are strongly connected. In the second section we obtain several existence results of the solution for general variational inequalities of Stampacchia type. These results will be used for providing some unknown coincidence point results in Hilbert spaces. Also here, as corollaries, several fixed point theorems are obtained. In the third section we obtain some existence results of the solution for general variational inequalities without assuming that the operators involved are of type ql. We do not assume any continuity property of the operators involved, instead we work with some sequential conditions imposed on these operators. We use these results to obtain some new coincidence point results in Hilbert spaces. The fourth section deals with several multivalued inequality problem, both of Stampachia and Minty type. First we state and prove a useful adaptation of KKM principle in Banach spaces. Further, we obtain some results concerning on existence of solution for these multivalued variational inequalities. By an example we show, that our results are the best possible in some sense, that is, if we drop the assumption that one of the operators involved is of type ql, in the hypothesis of our main theorems, then their conclusion fail. Finally, as applications of the results obtained, we provide some coincidence point results in Hilbert spaces.

Chapter 4 deals with scalar and vector equilibrium problems on dense sets. Further, several new minimax results on dense sets are provided.

In the first section, we deal with set-valued equilibrium problems, for which we provide sufficient conditions for the existence of a solution. The conditions, that we consider, are imposed not on the whole domain, but rather on a self segment-dense subset of it, a special type of dense subset. As an application, we obtain a generalized Debreu-Gale-Nikaïdo-type theorem, with a considerably weakened Walras law in its hypothesis. Furthermore, we consider a non-cooperative n-person game and prove the existence of a Nash equilibrium, under assumptions that are less restrictive than the classical ones. Further, in section two, we provide sufficient conditions, that ensure the existence of the solution of some vector equilibrium problems in Hausdorff topological vector spaces ordered by a cone. Also here, the conditions, that we consider, are imposed not on the whole domain of the operators involved, but rather on a self-segment-dense subset of it, a special type of dense subset. We apply the obtained results to vector optimization and vector variational inequalities. In the last section we provide conditions that assure the infimum of a proper, lower semicontinuous and convex function on a dense subset of its domain is equal to the global infimum of that function. We also obtain conditions for the coincidence of two convex functions that are equal on a dense subset of their common domain. Then, we apply these results in order to obtain some minimax results on dense sets. Also here, by an example we show that the extension of Fan's and Sion's minimax result to usual dense sets is impossible. Finally, based on our minimax results, we obtain conditions that assure the denseness of several family of functionals in the function spaces $C(K)$ and $B(K)$, respectively. This setting allows us to give an alternative proof to the famous reflexivity result of James.